Introduction to finite group theory and flavour symmetry

Morimitsu Tanimoto

Niigata University

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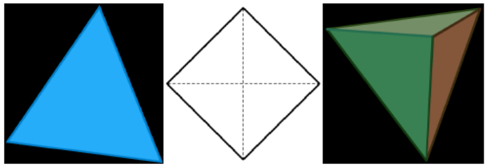
Content

1 Introduction

The discrete transformations (e.g., rotation of a regular polygon) give rise to corresponding types symmetries:



Discrete Symmetry

which is well known as the fundamental symmetry in particle physics,

C, P, T : Abelian

Non-Abelian Discrete Symmetry is expected to be also important for flavor physics of quarks and leptons.

The discrete symmetries are described by finite groups.

The classification of the finite groups has been completed in 2004,
(Gorenstein announced in 1981 that the finite simple groups had all been classified.)
about 100 years later than the case of the continuous groups.

Thompson, Gorenstein, Aschbacher ……

The classification of finite simple group

Theorem —
Every finite **simple group** is **isomorphic** to one of the following groups:

- a member of one of three infinite classes of such:
  - the **cyclic groups** of prime order, Z_n (n: prime)
  - the **alternating groups** of degree at least 5, A_n $(n>4)$
  - the **groups of Lie type** $E_6(q)$, $E_7(q)$, $E_8(q)$, ……
- one of 26 groups called the "**sporadic groups**" Mathieu groups, Monster group …
- the **Tits group** (which is sometimes considered a 27th sporadic group).

See Web: http://brauer.maths.qmul.ac.uk/Atlas/v3/

Monster group is maximal one in sporadic finite group, which may be related to the string theory.

Vertex Operator Algebra

On the other hand,
A_5 is the minimal simple finite group
except for cyclic groups.
This group is succesfully used to reproduce the lepton flavor structure.

There appears a flavor mixing angle with Golden ratio.

Platonic solids (tetrahedron, cube, octahedron, regular dodecahedron, regular icosahedron)
have symmetries of A_4, S_4 and A_5 ,
which may be related with flavor structure of leptons.

Moonshine phenomena was discovered in Monster group.

Monster group: largest sporadic finite group, of order **8×10^{53}** .
808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

McKay, Tompson, Conway, Norton (1978) observed :
strange relationship between **modular form** and an isolated **discrete group**.

q-expansion coefficients of Modular *J*-function are decomposed into a sum of dimensions of some irreducible representations of the monster group.

Moonshine phenomena

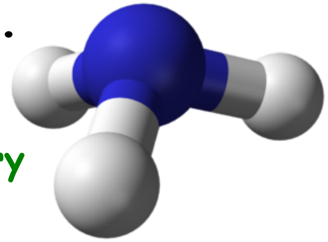
Phenomenon of monstrous moonshine has been solved mathematically in early 1990's using the technology of **vertex operator algebra in string theory.**
However, we still do not have a 'simple' explanation of this phenomenon.

"This phenomenon may possibly play an interesting role in string theory in the future."

T. Eguchi

Mathieu moonshine, Umbral moonshine ……

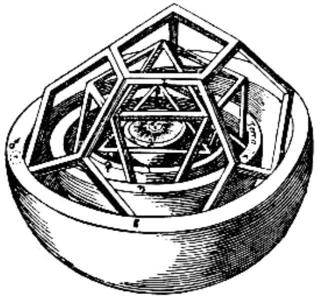
In practice, finite groups are used to classify crystal structures, regular polyhedra, and the symmetries of molecules.
The assigned point groups can then be used to determine physical properties, spectroscopic properties and to construct molecular orbitals.



molecular symmetry

Finite groups are also expected to control fundamental particle physics as well as chemistry and materials science.

Johannes Kepler



The Cosmographic Mystery

More than 400 years ago,
Kepler tried to understand cosmological structure by five Platonic solids.

Symmetry is an attractive approach when the dynamics is unknown.

People like Symmetries !

2 Basic of Finite Groups

Ishimori, Kobayashi, Ohki, Shimizu, Okada, M.T, PTP supprement, 183,2010,arXiv1003.3552,
Lect. Notes Physics (Springer) 858,2012

A group, G, is a set, where multiplication is defined such that

1. Closure: If a and b are elements of the group G, $c = ab$ is also its element.

2. Associativity: $(ab)\,c = a\,(bc)$ for $a, b, c \in G$.

3. Identity: The group G includes an identity element e, which satisfies $ae = ea = a$ for any element $a \in G$.

4. Inverse: The group G includes an inverse element a^{-1} for any element $a \in G$ such that $aa^{-1} = a^{-1}a = e$.

Finite group G

consists of a finite number of element of G.

- The order is the number of elements in G.

- The group G is called Abelian
  if all elements are commutable each other, i.e. $ab = ba$.

- The cyclic group Z_N is Abelian,
  which consists of $\{e, a, a^2, \ldots, a^{N-1}\}$, where $a^N = e$.

- If all of elements do not satisfy the commutativity,
  the group is called non-Abelian.

Subgroup

If a subset H of the group G is also a group,
H is called the subgroup of G.

The order of the subgroup H is a divisor of the order of G.
(Lagrange's theorem)

If a subgroup N of G satisfies $g^{-1}Ng = N$ for any element $g \in G$, the subgroup N is called a normal subgroup or an invariant subgroup.

The subgroup H and normal subgroup N of G satisfy $HN = NH$ and it is a subgroup of G, where HN denotes $\{h_i n_j \mid h_i \in H, n_j \in N\}$

A simple group is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

A group that is **not simple** can be broken into two smaller groups, a **normal subgroup** and the **quotient group** (factor group), and the process can be repeated.

If the group is **finite**, then eventually one arrives at uniquely determined **simple groups**.

A_5 is the minimal simple finite group except for cyclic groups with order of prime number.

A non-Abelian finite simple group has order divisible by at least three distinct primes.

$A_4 : 12=2^2\times3$ $\quad$ $A_5 : 60=2^2\times3\times5$

Elements of G are classified into

Conjugacy class

The number of irreducible representations is equal to the number of conjugacy classes.

When $a^h = e$ for an element $a \in G$,
the number h is called the order of a.

The elements $g^{-1}ag$ for $g \in G$ are called
elements conjugate to the element a.

The set including all elements
to conjugate to an element a of G,
$\{g^{-1}ag, \forall g \in G\}$, is called a conjugacy class.

All of elements in a conjugacy class have the same order
since $(gag^{-1})^h = ga\,(g^{-1}g)\,a(g^{-1}g)\,\ldots\,ag^{-1} = ga^hg^{-1} = geg^{-1} = e$.

The conjugacy class including the identity e
consists of the single element e.

Character

A representation of G is a homomorphic map of elements of G onto matrices, $D(g)$ for $g \in G$.

The representation matrices should satisfy
$D(a)D(b) = D(c)$ if $ab = c$ for $a, b, c \in G$.

Character $$\chi_D(g) = \mathrm{tr}\ D(g) = \sum_{i=1}^{d_\alpha} D(g)_{ii}.$$

The element conjugate to a has the same character because

$$\mathrm{tr}\ D(g^{-1}ag) = \mathrm{tr}\ \left(D(g^{-1})D(a)D(g)\right) = \mathrm{tr}\ D(a),$$

Suppose that there are m_n n-dimensional irreducible representations, that is, D(g) are represented by (n × n) matrices.
The identity e is always represented by the (n × n) identity matrix.

orthogonality relations

$$\sum_{g\in G} \chi_{D_\alpha}(g)^* \chi_{D_\beta}(g) = N_G \delta_{\alpha\beta}, \qquad \sum_\alpha \chi_{D_\alpha}(g_i)^* \chi_{D_\alpha}(g_j) = \frac{N_G}{n_i} \delta_{C_i C_j},$$

where N_G denotes the order of a group G, C_i denotes the conjugacy class of g_i, and n_i denotes the number of elements in the conjugacy class C_i.

Since C_1 = {e} (n_1=1) , the orthgonality relation turns to

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = N_G$$

The number of irreducible representations must be equal to the number of conjugacy classes.

$$\sum_n m_n = \text{the number of conjugacy classes},$$

Let us present a pedagogical example, S_3

smallest non-Abelian finite group

S_3 consists of all permutations among three objects, (x_1, x_2, x_3) and its order is equal to 3! = 6.
All of six elements correspond to the following transformations,

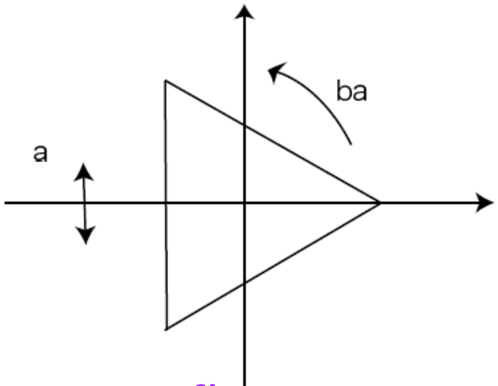
$$e: (x_1, x_2, x_3) \to (x_1, x_2, x_3) \qquad a_1: (x_1, x_2, x_3) \to (x_2, x_1, x_3)$$
$$a_2: (x_1, x_2, x_3) \to (x_3, x_2, x_1) \qquad a_3: (x_1, x_2, x_3) \to (x_1, x_3, x_2)$$
$$a_4: (x_1, x_2, x_3) \to (x_3, x_1, x_2) \qquad a_5: (x_1, x_2, x_3) \to (x_2, x_3, x_1)$$

Their multiplication forms a closed algebra, e.g.

$$a_1 a_2 = a_5 , \quad a_2 a_1 = a_4 , \quad a_4 a_2 = a_2 a_1 a_2 = a_3$$

By defining $a_1 = a$, $a_2 = b$,
all of elements are written as {e, a, b, ab, ba, bab}.

a and b are generators of S_3 group.



The S_3 group is a symmetry of an equilateral triangle.

a: reflection
ba: $2\pi/3$ rotation

Let us study irreducible representations of S_3.
The number of irreducible representations must be equal to three, because there are three conjugacy classes.

These elements are classified to three conjugacy classes,

$C_1 : \{e\}, \quad C_2 : \{ab, ba\}, \quad C_3 : \{a, b, bab\}.$

The subscript of C_n, n, denotes the number of elements in the conjugacy class C_n.
Their orders are found as

$(ab)^3 = (ba)^3 = e, \quad a^2 = b^2 = (bab)^2 = e$

Due to the orthogonal relation

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 6$$

$$\sum_n m_n = 3 \qquad m_n \geq 0$$

We obtain a solution: $(m_1, m_2) = (2, 1)$

Irreducible representations of S_3 are **two singlets 1 and 1' , one doublet 2**.

Since $(\chi_{1'}(C_2))^3 = 1,\quad (\chi_{1'}(C_3))^2 = 1$ are satisfied,
orthogonarity conditions determine the Character Table,
from which explicit representation matrices are obtained.

	h	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	-1
C_3	2	1	-1	0

C_1 : {e}, C_2 : {ab, ba}, C_3 : {a, b, bab}.

By using this table, we can construct the representation matrix for **2** .

Because of $\chi_2(C_3) = 0$, we choose $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Recalling $b^2 = e$, we can write $b = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad bab = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$$ab = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad ba = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Since the trace of elements in C_2 is equal to -1, we get $\cos\theta = -1/2$. Choosing $\theta = 4\pi/3$, we obtain the matrix representation

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad ba = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad bab = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

These are in Real representation.

By the unitary transformations,
other representations are obtained.

Kronecker products and CG coefficients

For example, each element $x_i\,y_j$ is transformed under **b** as

$$
\begin{aligned}
x_1y_1 &\to \frac{x_1y_1+3x_2y_2+\sqrt{3}(x_1y_2+x_2y_1)}{4},\\
x_1y_2 &\to \frac{\sqrt{3}x_1y_1-\sqrt{3}x_2y_2-x_1y_2+3x_2y_1}{4},\\
x_2y_1 &\to \frac{\sqrt{3}x_1y_1-\sqrt{3}x_2y_2-x_2y_1+3x_1y_2}{4},\\
x_2y_2 &\to \frac{3x_1y_1+x_2y_2-\sqrt{3}(x_1y_2+x_2y_1)}{4}.
\end{aligned}
$$

$$b(x_1y_1+x_2y_2)=(x_1y_1+x_2y_2)$$

$$b(x_1y_2-x_2y_1)=-(x_1y_2-x_2y_1)$$

Thus, it is found that $\mathbf{1}: x_1y_1+x_2y_2, \quad \mathbf{1}': x_1y_2-x_2y_1.$

It is also found

b

$$b\begin{pmatrix} x_2y_2-x_1y_1 \\ x_1y_2+x_2y_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_2y_2-x_1y_1 \\ x_1y_2+x_2y_1 \end{pmatrix}$$

$$\mathbf{2} = \begin{pmatrix} x_2y_2-x_1y_1 \\ x_1y_2+x_2y_1 \end{pmatrix}$$

Multiplying arbitrary irreducible representations r and s

$$\mathbf{r} \otimes \mathbf{s} \;=\; \sum_{\mathbf{t}} d(\mathbf{r},\mathbf{s},\mathbf{t})\; \mathbf{t}\;, \qquad d(\mathbf{r},\mathbf{s},\mathbf{t}) \;=\; \frac{1}{N}\sum_i N_i \cdot \chi_i^{[\mathbf{r}]}\chi_i^{[\mathbf{s}]}\chi_i^{[\mathbf{t}]^*}$$

Sum of i is over all classes

$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}\;,$$
$$\mathbf{1}' \otimes \mathbf{2} = \mathbf{2}\;,$$
$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}\;.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\mathbf{2}} = (x_1y_1 + x_2y_2)_{\mathbf{1}} + (x_1y_2 - x_2y_1)_{\mathbf{1}'} + \begin{pmatrix} x_1y_2 + x_2y_1 \\ x_1y_1 - x_2y_2 \end{pmatrix}_{\mathbf{2}},$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{\mathbf{2}} \otimes (y')_{\mathbf{1}'} = \begin{pmatrix} -x_2y' \\ x_1y' \end{pmatrix}_{\mathbf{2}},$$
$$(x')_{\mathbf{1}'} \otimes (y')_{\mathbf{1}'} = (x'y')_{\mathbf{1}}.$$

One can construct a lager group from more than two groups *by a certain* product.

☆ direct product.

Consider e.g. two groups G_1 and G_2. Their direct product is denoted as $G_1 \times G_2$.
Its multiplication rule is

$$(a_1,a_2)\ (b_1,b_2) = (a_1 b_1, a_2 b_2) \text{ for } a_1,b_1 \in G_1 \text{ and } a_2,b_2 \in G_2$$

☆ (innner, outer) semi-direct product

It is defined such as

$$(a_1,a_2)\ (b_1,b_2) = (a_1\ f_{a2}\ (b_1),\ a_2 b_2) \text{ for } a_1,b_1 \in G_1 \text{ and } a_2,b_2 \in G_2$$

where $f_{a2}\ (b_1)$ *denotes a* ***homomorphic map*** *from* G_2 *to* G_1.

This semi-direct product is denoted as $G_1 \rtimes_f G_2$.

We consider the group G and its **subgroup H** and **normal subgroup N**, whose elements are h_i and n_j , respectively.

When G = NH = HN and N $\cap$ H = {e}, the semi-direct product $N \rtimes_f H$ is isomorphic to G, where we use the map ***f*** *as* $f_{hi}\ (n_j) = h_i\ n_j\ (h_i)^{-1}$.

Example of semi-direct product

Let us study the semi-direct product, $Z_3 \rtimes Z_2$.
Here we denote the Z_3 and Z_2 generators by c and h, i.e., $c^3 = e$ and $h^2 = e$.
In this case, **a homomorphic map $f_{a2}(b_1)$** can be written by $h\, c\, h^{-1} = c^m$.

only the case with m = 2 is non-trivial, $h\, c\, h^{-1} = c^2$

This algebra is isomorphic to S_3, and **h** and **c** are identified as **a** and **ab**.

$$N=(e,\ ab,\ ba),\quad H=(e,\ a) \Rightarrow NH=HN \simeq S_3$$

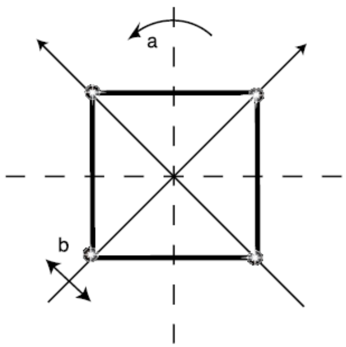
Similarly, we can consider the $Z_n \rtimes Z_m$.
When we denote the Z_n and Z_m generators by a and b, respectively,
they satisfy $a^n = b^m = e$, $bab^{-1} = a^k$,

where $k \neq 0$, although the case with k = 1 leads to the direct product $Z_n \times Z_m$.

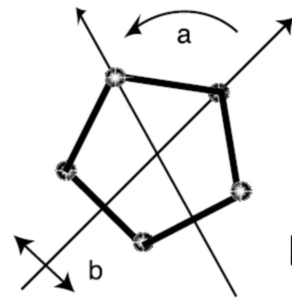
Semi-direct products generates a larger non-Abelian groups

Dihedral group $Z_N \ltimes Z_2 \simeq D_N$, $\Delta(2N)$; $a^N = e$, $b^2 = e$, $bab = a^{-1}$ **order : 2N**

D_4
square



D_5



Regular pentagon

Quasi-Dihedral group $Z_{2^{N'-1}} \ltimes Z_2$ QD_{2N} ; $a^{2N'-1} = 1$, $b^2 = 1$, $bab^{-1} = a^m$ **order : $2^{N'}$**

$m = 2^{N'-2} - 1$ ($m = -1 \Rightarrow$ Dihedral group)

$\Sigma(2N^2) \simeq (Z_N \times Z'_N) \ltimes Z_2$ $a^N = a'^N = b^2 = e$, $a\, a' = a'\, a$; , $bab = a'$

$\simeq$ $\Sigma(8)$ $D4$, $\Sigma(18)$, $\Sigma(32)$, $\Sigma(50)$ …..

$\Delta(3N^2) \simeq (Z_N \times Z'_N) \ltimes Z_3$, $a^N = a'^N = b^3 = e$, $a a' = a' a$, $bab^{-1} = a^{-1}(a')^{-1}$; $ba'b^{-1} = a$

$\Delta(27)$

$T_N \simeq Z_N \ltimes Z_3$, $a^N = e$, $b^3 = e$, $ba = a^m b$

T_7 , T_{13} , T_{19} $N = (7, 13, 19, 31, 43, 49 ....)$

$\Sigma(3N^3)$ Closed algebra of Z_N, Z'_N , Z''_N, which commute each other, and their Z_3 permutations.

$\Sigma(81)$

$\Delta(6N^2) \simeq (Z_N \times Z'_N) \ltimes S_3$,

$a^N = a'^N = b^3 = c^2 = (bc)^2 = e$, $aa' = a'a$, $bab^{-1} = a^{-1}(a')^{-1}$, $b a'b^{-1} = a$,$cac^{-1} = (a')^{-1}$, $ca'c^{-1} = a^{-1}$

$\Delta(6N^2)$ group includes the subgroup, $\Delta(3N^2)$,

$\Delta(6)=S_3$ $\Delta(24) \simeq S_4$ $\Delta(54)$ $\Delta(96)$ .....

Familiar non-Abelian finite groups

			order
S_n :	$S_2=Z_2$, S_3, S_4 …	Symmetric group	N !
A_n:	$A_3=Z_3$, $A_4=T$, A_5 …	Alternating group	(N !)/2
D_n:	$D_3=S_3$, D_4, D_5 …	Dihedral group	2N
$Q_{N(even)}$:	Q_4, Q_6 ….	Binary dihedral group	2N
$\Sigma(2N^2)$:	$\Sigma(2)=Z_2$, $\Sigma(18)$, $\Sigma(32)$, $\Sigma(50)$ …		$2N^2$
$\Delta(3N^2)$:	$\Delta(12)=A_4$, $\Delta(27)$ …		$3N^2$
$T_{N(\text{prime number})} \simeq Z_N \rtimes Z_3$:	T_7, T_{13}, T_{19}, T_{31}, T_{43}, T_{49}		3N
$\Sigma(3N^3)$:	$\Sigma(24)=Z_2 \times \Delta(12)$, $\Sigma(81)$ …		$3N^3$
$\Delta(6N^2)$:	$\Delta(6)=S_3$, $\Delta(24)=S_4$, $\Delta(54)$, $\Delta(96)$ …		$6N^2$
T' :	double covering group of $A_4=T$		24

Subgroups are important for particle physics Because symmetry breaks down to them.

Ludwig Sylow in 1872:

Theorem 1:
For every prime factor *p* with multiplicity *n* of the order of a finite group *G*, there exists a Sylow *p*-subgroup of *G*, of order p^n.

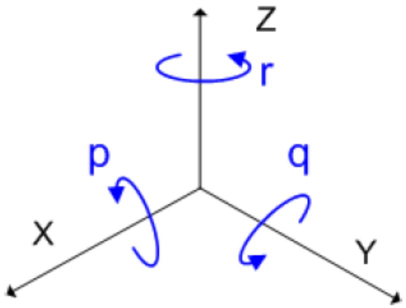
For example, A_4 has subgroups with order 4 and 3, respectively.

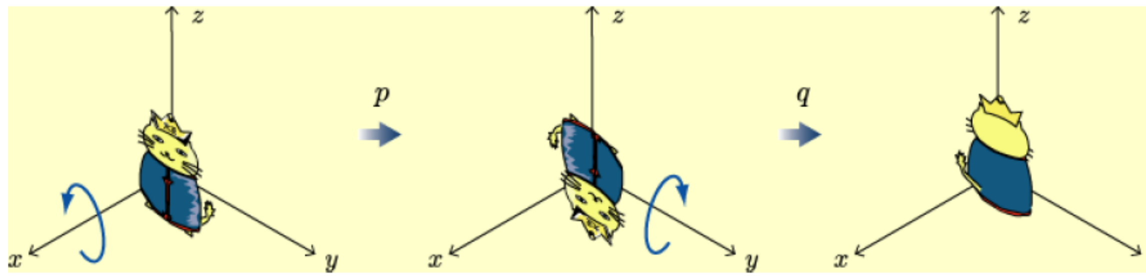
$$12 = 2^2 \times 3$$

Actually, $(Z_2 \times Z_2)$ (klein symmetry) and Z_3 are the subgroup of A_4 .

Klein four group

Felix Klein





Multiplication table

	e	p	q	r
e	e	p	q	r
p	p	e	r	q
q	q	r	e	p
r	r	q	p	e

With four elements, the Klein four group is the smallest non-cyclic group, and the cyclic group of order 4 and the Klein four-group are, up to isomorphism, the only groups of order 4. Both are abelian groups.

Normal subgroup of A_4

$Z_2 \times Z_2$ $V = \langle$ identity, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) $\rangle$

For flavour physics, we are interested in finite groups with **triplet representation**.

S_3 has two singlets and one doublet: **1, 1', 2,**
no triplet representation.

Some examples of non-Abelian Finite groups with triplet representation, which are often used in Flavour symmetry

S_4, A_4, A_5

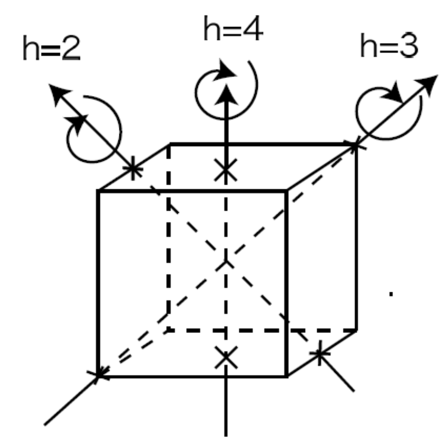
S_4 group

All permutations among four objects, 4 ! =24 elements

24 elements are generated by S, T and U:

$S^2=T^3=U^2=1, \quad ST^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$

Irreducible representations: 1, 1', 2, 3, 3'



Symmetry of a cube

For triplet 3 and 3'

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

$$U = \mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	2	3	3
C_3	2	1	1	2	-1	-1
C_6	2	1	-1	0	1	-1
$C_{6'}$	4	1	-1	0	-1	1
C_8	3	1	1	-1	0	0

Subgroups and decompositions of multiplets

S_4 group is isomorphic to $\Delta(24) = (Z_2 \times Z_2) \ltimes S_3$.

A_4 group is isomorphic to $\Delta(12) = (Z_2 \times Z_2) \ltimes Z_3$.

$S_4 \rightarrow S_3$

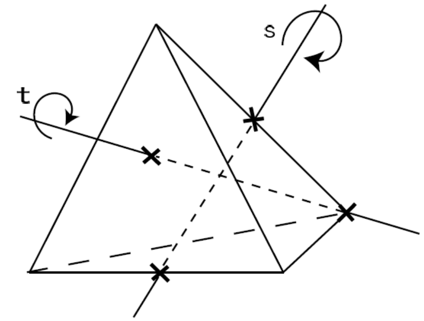
$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S_3 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{1}+\mathbf{2} & \mathbf{1}'+\mathbf{2} \end{array}$$

$S_4 \rightarrow A_4$

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & \mathbf{1} & \mathbf{1} & \mathbf{1}'+\mathbf{1}'' & \mathbf{3} & \mathbf{3} \end{array}$$

$S_4 \rightarrow (Z_2 \times Z_2) \ltimes Z_2$

A_4 group



Symmetry of tetrahedron

Even permutation group of four objects (1234)
12 elements (order 12) are generated by
S and T: $S^2=T^3=(ST)^3=1$: S=(14)(23), T=(123)

C1: 1 h=1
C3: S, T^2ST, TST^2 h=2
C4: T, ST, TS, STS h=3
C4': T^2, ST^2, T^2S, ST^2S h=3

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Irreducible representations: 1, 1', 1'', 3
The minimum group containing triplet without doublet.

For triplet

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

Subgroups and decompositions of multiplets

A_4 group is isomorphic to $\Delta(12) = (Z_2 \times Z_2) \ltimes Z_3$.

$A_4 \to Z_3$

$$\begin{array}{llll} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} & \\ & \downarrow & \downarrow & \\ Z_3 & \mathbf{1}_k & \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 & \end{array}$$

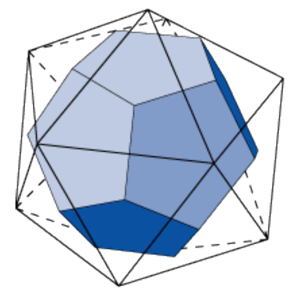
(k = 0, 1, 2)

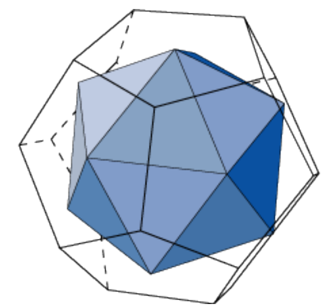
$A_4 \to Z_2 \times Z_2$

$$\begin{array}{lll} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} \\ & \downarrow & \downarrow \\ Z_2 \times Z_2 & \mathbf{1}_{0,0} & \mathbf{1}_{1,1} + \mathbf{1}_{0,1} + \mathbf{1}_{1,0} \end{array}$$

A_5 group (simple group)

The A_5 group is isomorphic to the symmetry of a regular icosahedron and a regular dodecahedron.





60 elements are generated S and T .

$$S^2 = (ST)^3 = 1 \text{ and } T^5 = 1$$

Irreducible representations:

1, 3, 3', 4, 5

	h	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{4}$	$\mathbf{5}$
C_1	1	1	3	3	4	5
C_{15}	2	1	-1	-1	0	1
C_{20}	3	1	0	0	1	-1
C_{12}	5	1	ϕ	$1-\phi$	-1	0
$C_{12'}$	5	1	$1-\phi$	ϕ	-1	0

For triplet 3

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\phi & \frac{1}{\phi} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{5}} & 0 \\ 0 & 0 & e^{\frac{8\pi i}{5}} \end{pmatrix}$$

$$\phi = \frac{1+\sqrt{5}}{2}$$ Golden Ratio

Subgroups and decompositions of multiplets

$A_5 \to A_4$

A_5	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{4}$	$\mathbf{5}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
A_4	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}+\mathbf{3}$	$\mathbf{1'}+\mathbf{1''}+\mathbf{3}$

$A_5 \to D_5$

A_5	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{4}$	$\mathbf{5}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
D_5	$\mathbf{1}_+$	$\mathbf{1}_-+\mathbf{2}_1$	$\mathbf{1}_-+\mathbf{2}_2$	$\mathbf{2}_1+\mathbf{2}_2$	$\mathbf{1}_++\mathbf{2}_1+\mathbf{2}_2$

$A_5 \to S_3 \simeq D_3$

A_5	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{4}$	$\mathbf{5}$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
D_3	$\mathbf{1}_+$	$\mathbf{1}_-+\mathbf{2}$	$\mathbf{1}_-+\mathbf{2}$	$\mathbf{1}_++\mathbf{1}_-+\mathbf{2}$	$\mathbf{1}_++\mathbf{2}+\mathbf{2}$

$A_5 \to Z_2 \times Z_2$

5 Klein four groups

$$K_1 = \{v_1, v_2, v_3, e\} \ , \quad K_2 = \{v_4, v_5, v_6, e\}$$
$$K_3 = \{v_7, v_8, v_9, e\} \ , \quad K_4 = \{v_{10}, v_{11}, v_{12}, e\} \quad \text{and} \quad K_5 = \{v_{13}, v_{14}, v_{15}, e\}$$

$$v_1 = s \ , \quad v_2 = st^2st^3st^2 \ , \quad v_3 = t^2st^3st^2 \ , \quad v_4 = t^4st \ , \quad v_5 = st^3st^2s \ ,$$
$$v_6 = t^2st^3sts \ , \quad v_7 = tst^4 \ , \quad v_8 = st^2st^3s \ , \quad v_9 = stst^3st^2 \ , \quad v_{10} = st^2st \ ,$$
$$v_{11} = t^2st^3 \ , \quad v_{12} = tst^3st^2s \ , \quad v_{13} = tst^2s \ , \quad v_{14} = t^3st^2 \ , \quad v_{15} = st^2st^3st \ .$$

Subgroup		Generators	Subgroup		Generators
$Z_2 \times Z_2$	K_1	$S,\ T^2ST^3ST^2$	Z_3	C_1	ST
	K_2	$T^4ST,\ ST^3ST^2S$		C_2	TS
	K_3	$TST^4,\ ST^2ST^3S$		C_3	TST^3
	K_4	$T^2ST^3,\ ST^2ST$		C_4	T^2ST^2
	K_5	$T^3ST^2,\ TST^2S$		C_5	T^3ST
Z_5	R_1	T		C_6	ST^3ST
	R_2	ST^2		C_7	ST^2ST^3
	R_3	T^2S		C_8	ST^3ST^2
	R_4	TST		C_9	ST^2ST^4
	R_5	TST^2		C_{10}	ST^2ST^2S
	R_6	T^2ST			

Generating elements of Subgroup of A_5

3 Toward Flavour Symmetry

3.1 General aspect

\+ Higgs sector

Standard Model = gauge sector + Yukawa sector

gauge sym | flavor sym

- abelian or non-abelian ?

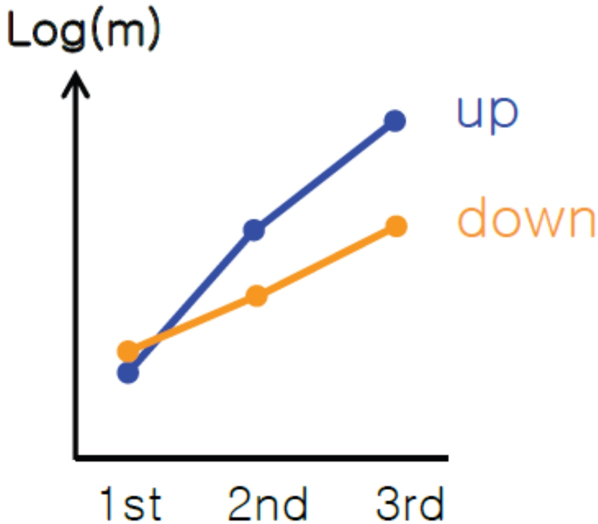
  abelian : discriminate between generations
  non-abelian : connect different generations

- continuous or discrete ?

  continuous : free rotation between generations
  discrete : definite meaning of generations

Quark sector

U(1) Symmetry ?



For example :

$$M_{\rm up} \sim \begin{pmatrix} \epsilon^8 & \epsilon^6 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & \epsilon^0 \end{pmatrix}$$

$$M_{\rm down} \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon^1 & \epsilon^0 \end{pmatrix}$$

ε=0.2

- large mass hierarchy
- small mixing

Cabibbo angle 0.225

i.e. "separate" generations

Pre-History of non-Abelian flavor symmetry

Some works challenged non-Abelian flavor symmetry in the quark sector.

There was no information of lepton flavor mixing before 1998.

Discrete Symmetry and Cabibbo Angle,
Phys. Lett. 73B (1978) 61, S.Pakvasa and H.Sugawara

S_3 symmetry is assumed for the Higgs interaction with the quarks and the leptons and for the self-coupling of the Higgs bosons.

$$\left\{\binom{p_1}{n_1}_L, \binom{p_2}{n_2}_L\right\} \quad \{p_{1R}\}, \{p_{2R}\}, \{n_{1R}, n_{2R}\}$$

one S_3 singlet $\{\phi_0\}$ and one S_3 doublet $\{\phi_1, \phi_2\}$

$$\tan\theta_c = m_d/m_s.$$

A Geometry of the generations,
Phys. Rev. Lett. 75 (1995) 3985, L.J.Hall and H.Murayama

$(S(3))^3$ flavor symmetry for quarks Q, U, D

$(S(3))^3$ flavor symmetry and p ---> K^0 e^+,
Phys. Rev.D 53 (1996) 6282, C.D.Carone, L.J.Hall and H.Murayama

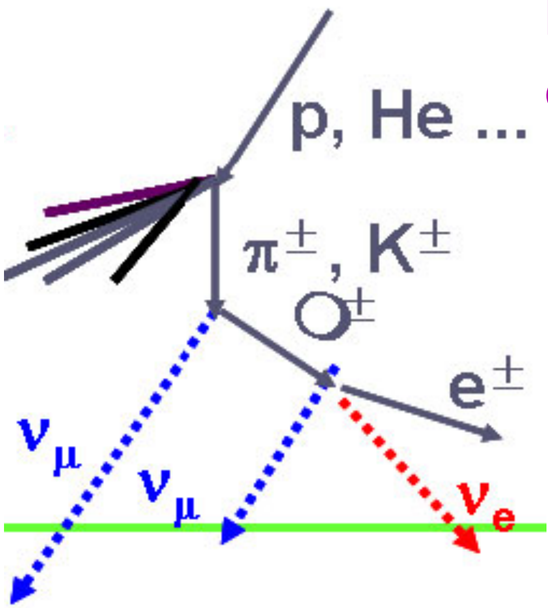
fundamental sources of flavor symmetry breaking are gauge singlet fields ϕ : flavons

Incorporating the lepton flavor based on the discrete flavor group $(S_3)^3$.

Neutrino evolution in 1998 !

Atmospheric neutrinos brought us informations of neutrino masses and flavor mixing.

$$P_{\nu_\mu \to \nu_\mu} = 1 - 4\left|U_{\mu 3}\right|^2 \left(1 - \left|U_{\mu 3}\right|^2\right) \sin^2 \frac{\Delta_{13}}{2} + 2\left|U_{\mu 2}\right|^2 \left|U_{\mu 3}\right|^2 \Delta_{12} \sin \Delta_{13} + \mathcal{O}(\Delta_{12}^2)$$



First clear evidence of neutrino oscillation was discovered in 1998

$$R = \frac{(\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)\,|_{DATA}}{(\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)\,|_{MC}} = 0.65 \pm 0.05 \pm 0.08$$

Multi-GeV

MC $(\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)\,|_{MC} \approx 2$

Talk at Takayama 1998 by T.Kajita

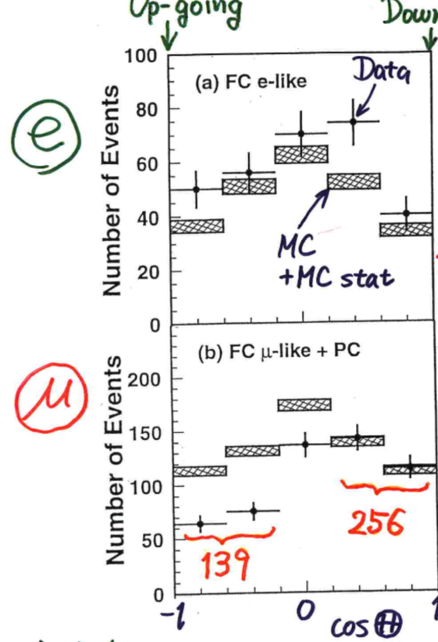
ν98, @Takayam[a]
June 1998

Atmospheric neutrino results from Super-Kamiokande & Kamiokand[e]

— Evidence for ν_μ oscillations —

T. Kajita
Kamioka observatory, Univ. of Tokyo
for the {Kamiokande / Super-Kamiokande} Collaborations

Zenith angle dependence (Multi-GeV)



χ^2(shape) $= 2.8/4$ dof

$\frac{Up}{Down} = 0.93 {+0. \atop -0.}$

χ^2(shape) $= 30/4$ dof

$\frac{Up}{Down} = 0.54 {+0. \atop -0.}$

(6.2σ !!)

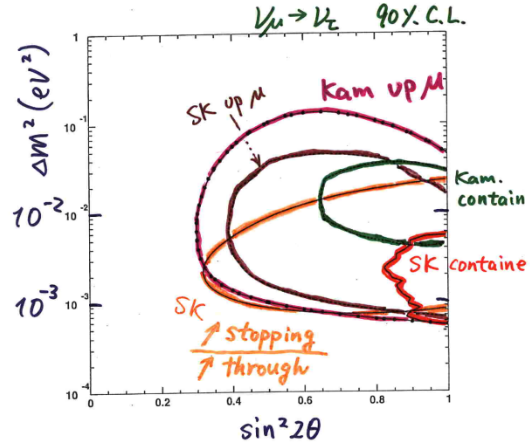
※ Up/Down syst. error for μ-like

Prediction (flux calculation …… ≲ 1% ; 1km rock above SK …. 1.5%) 1.8%

Data (Energy calib. for ↑↓ …. 0.7% ; Non ν Background …. < 2%) 2.1%

Summary

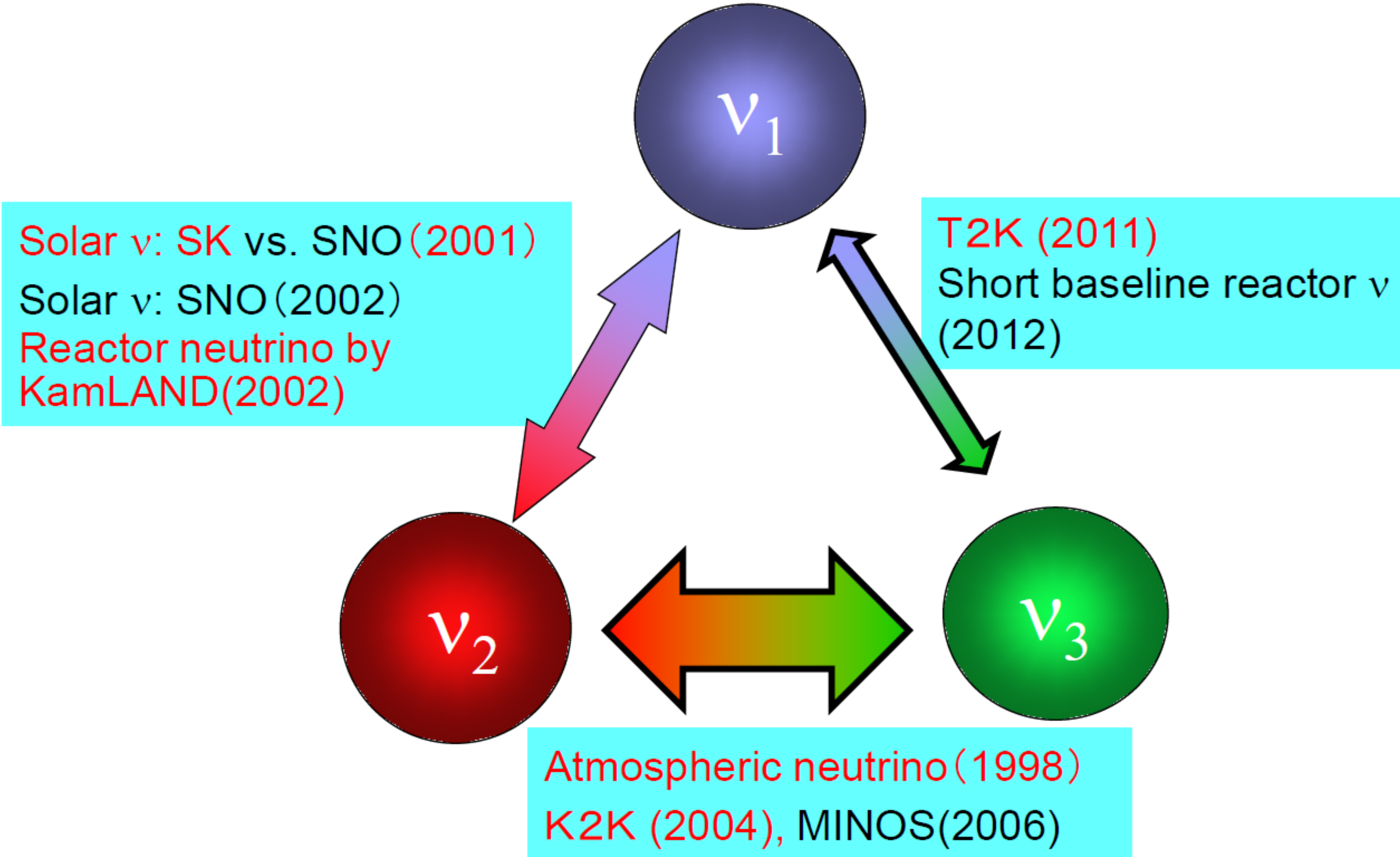
Evidence for ν_μ oscillations



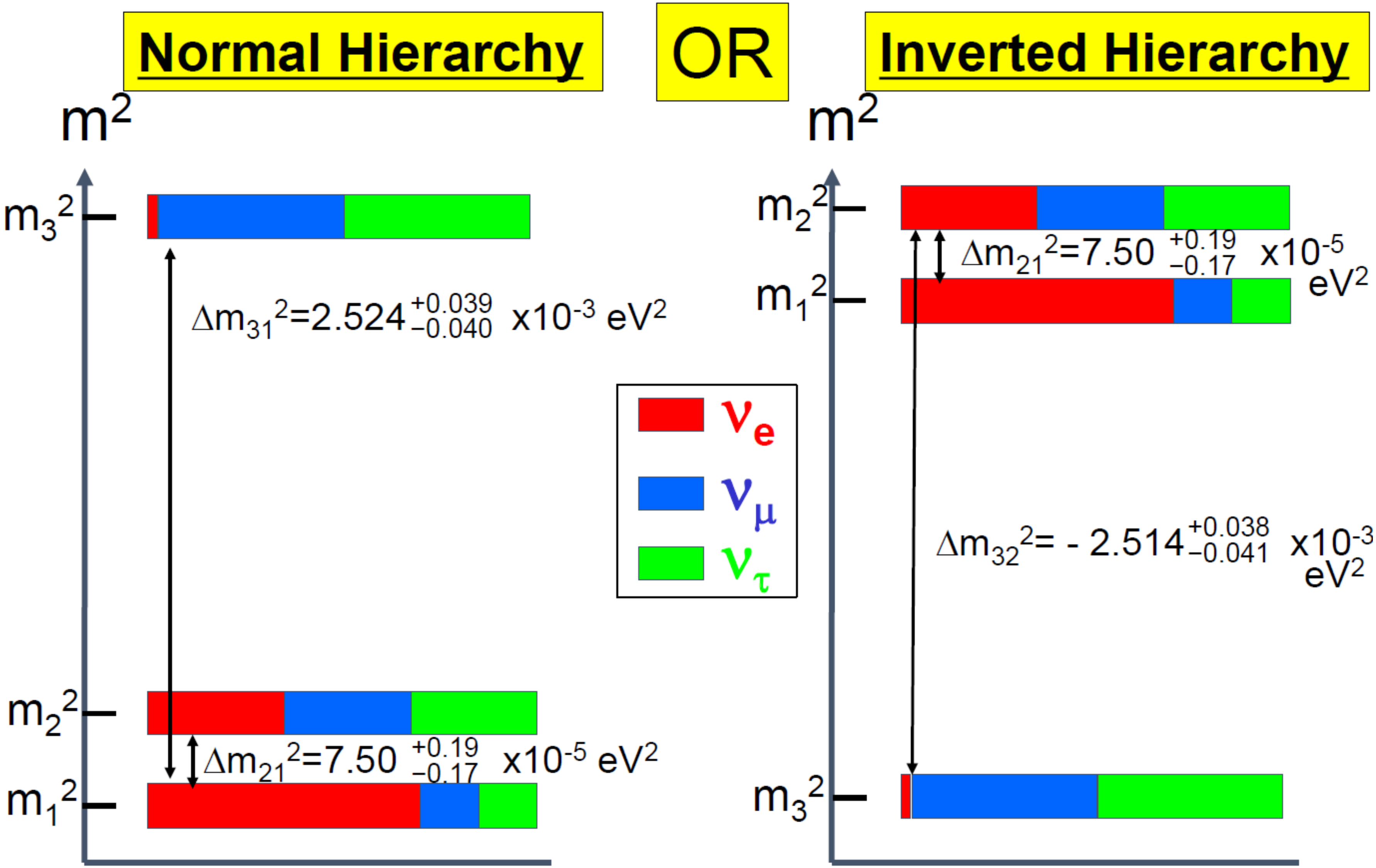
- $\begin{cases} \sin^2 2\theta > 0.8 \\ \Delta m^2 \sim 10^{-3} \sim 10^{-2} \end{cases}$

(• $\nu_\mu \to \nu_\tau$ or $\nu_\mu \to \nu_s$?)

Summary of discoveries of neutrino oscillations



Neutrino mass and mixing (what we know now)



Numbers from I. Esteban et al., JHEP 01 (2017) 087

Neutrino mixing vs. quark mixing

Neutrino mixing

(3σ C.L. range)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$\begin{pmatrix} 0.800\text{-}0.844 & 0.515\text{-}0.581 & 0.139\text{-}0.155 \\ 0.229\text{-}0.516 & 0.438\text{-}0.699 & 0.614\text{-}0.790 \\ 0.249\text{-}0.528 & 0.462\text{-}0.715 & 0.595\text{-}0.776 \end{pmatrix}$$

I. Esteban et al., JHEP 01 (2017) 087

Quark mixing (CKM matrix)

$$\begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0414 \\ 0.00875 & 0.0403 & 0.99915 \end{pmatrix}$$

Particle Data Group (2016)

They are so much different!

Before 2012 (no data for θ_{13})

Neutrino Data suggested

Tri-bimaximal Mixing of Neutrinos

Harrison, Perkins, Scott (2002)

$$\sin^2\theta_{12} = 1/3,\ \sin^2\theta_{23} = 1/2,\ \sin^2\theta_{13} = 0,$$

$$U_{\rm tri-bimaximal} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}$$

PDG

$$U_{\rm PMNS} \equiv \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}$$

$c_{ij}=\cos\theta_{ij}$
$s_{ij}=\sin\theta_{ij}$

Tri-bimaximal Mixing of Neutrinos motivates to consider non-Abelian Discrete Flavor Symmetry.

Tri-bimaximal Mixing (TBM) is realized by

$$m_{TBM} = \frac{m_1+m_3}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

in the diagonal basis of charged leptons.

Mixing angles are independent of neutrino masses.

Integer (inter-family related) matrix elements suggest Non-Abelian Discrete Flavor Symmetry.

Hint for the symmetry in TBM

$$m_{TBM} = \frac{m_1+m_3}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

A_4 symmetric

Assign A_4 triplet <u>3</u> for $(\nu_e, \nu_\mu, \nu_\tau)_L$

E. Ma and G. Rajasekaran, PRD64(2001)113012

$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{3} + \mathbf{3} + \mathbf{1} + \mathbf{1}' + \mathbf{1}''$$

$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1} = a_1 b_1 + a_2 b_3 + a_3 b_2$$

The third matrix is A_4 symmetric !
The first and second matrices are Unit matrix and Democratic matrix, respectively, which are well known matrices from S_3 symmetry.

In 2012

Reactor angle θ_{13} was measured by Daya Bay, RENO, T2K, MINOS, Double Chooz

Tri-bimaximal mixing was ruled out !

$$\theta_{13} \simeq 9^\circ \simeq \theta_c/\sqrt{2}$$

Rather large θ_{13} suggests to search for CP violation !

Challenge for Flavour symmetry and CP symmetry

3.2 Origin of Flavour symmetry

Talks of Vaudrevang, Leontaris

Is it possible to realize such discrete symmetres in string theory?
Answer is yes !

Superstring theory on a certain type of six dimensional compact space leads to stringy selection rules for allowed couplings among matter fields in four-dimensional effective field theory.

Such stringy selection rules and geometrical symmetries result in discrete flavor symmetries in superstring theory.

- Heterotic orbifold models (Kobayashi., Nilles, Ploger, Raby, Ratz, 07)

- Magnetized/Intersecting D-brane Model
  (Kitazawa, Higaki, Kobayashi,Takahashi, 06)
  (Abe, Choi, Kobayashi, Ohki, 09, 10)

Stringy origin of non-Abelian discrete flavor symmetries

T. Kobayashi, H. Niles, F. PloegerS, S. Raby, M. Ratz, hep-ph/0611020

D_4, $\Delta(54)$

Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models

H. Abe, K-S. Choi, T. Kobayashi, H. Ohki, 0904.2631

D_4, $\Delta(27)$, $\Delta(54)$

Non-Abelian Discrete Flavor Symmetry from T^2/Z_N Orbifolds

A.Adulpravitchai, A. Blum, M. Lindner, 0906.0468

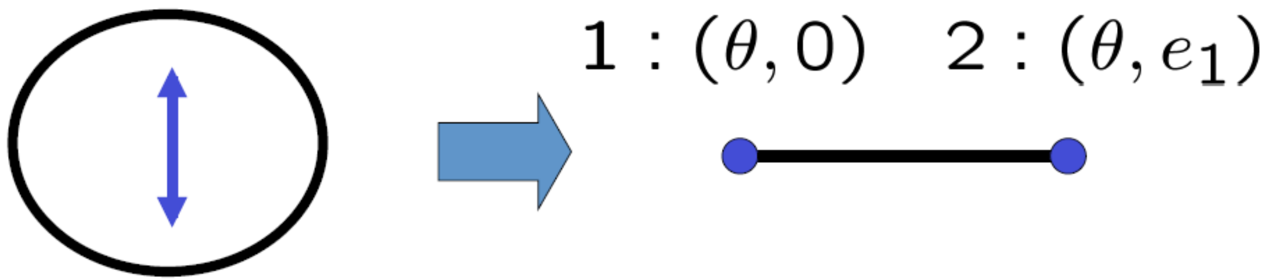
A_4, S_4, D_3, D_4, D_6

Non-Abelian Discrete Flavor Symmetries of 10D SYM theory with Magnetized extra dimensions

H. Abe, T. Kobayashi, H. Ohki, K.Sumita, Y. Tatsuta 1404.0137

S_3, $\Delta(27)$, $\Delta(54)$

S^1/Z_2 **orbifold** (Kobayashi, Nilles, Ploger, Raby, Ratz, 07)



There are two fixed point under the orbifold twist

These two fixed points can be represented by space group elements which act (θ, v)

$$(\theta, v)\alpha = \theta\alpha + v$$

e_1 : shift vector in one torus $(y \sim y + e_1)$

charge assignment of Z_2 : $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(stringy selection rule: Coupling is only allowed in matching of the string boundary conditions)

Discrete flavor symmetry from orbifold $S^1/\mathbf{Z}_2$

This effective Lagrangian also have permutation symmetry of these two fixed point (orbifold geometry).

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Closed algebra of these transformations $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

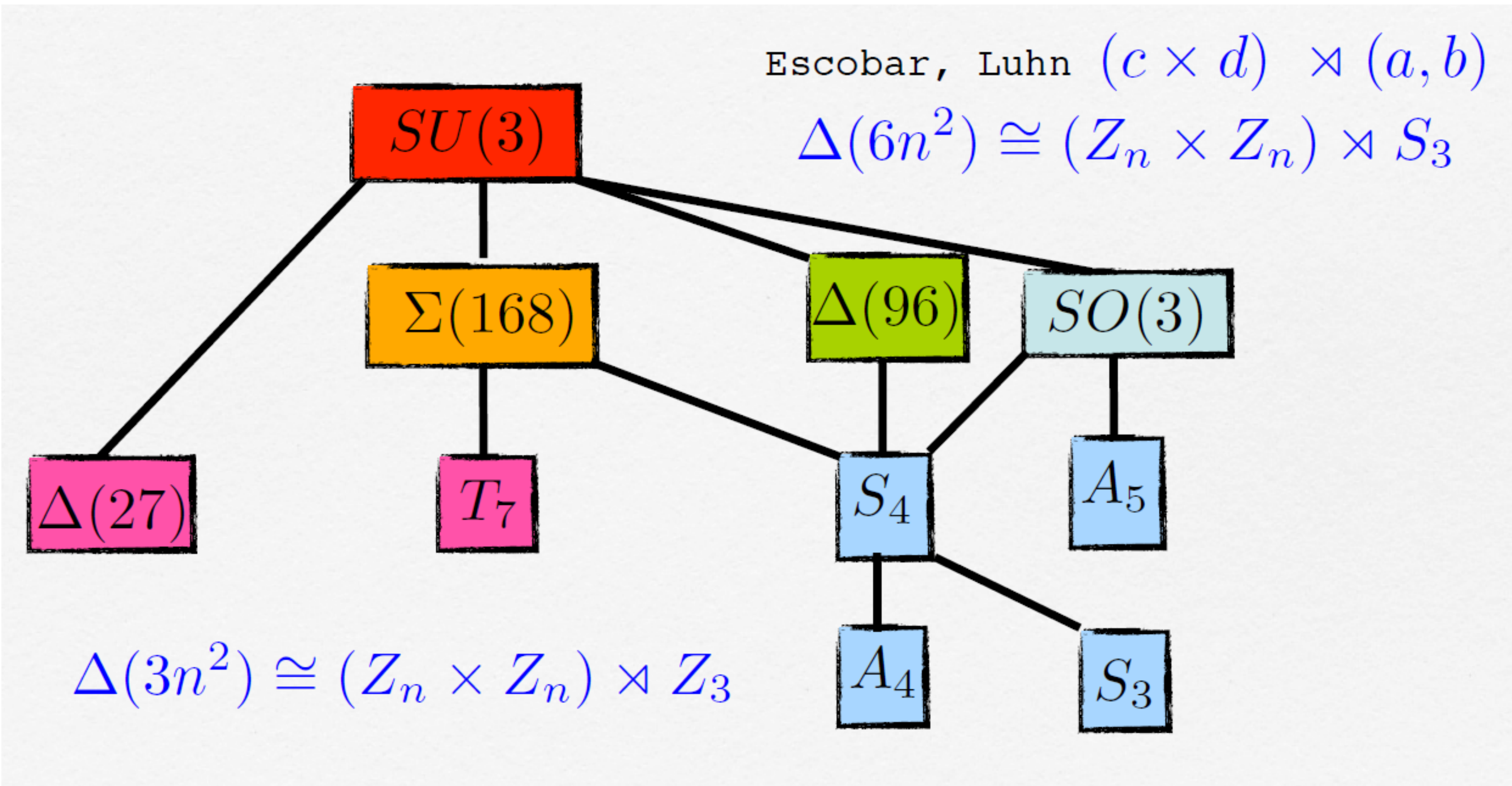
$$\Rightarrow D_4 \sim S^2 \cup (\mathbf{Z}_2 \times \mathbf{Z}_2)$$

Two field localized at two fixed points : doublet of D4 $\mathbf{2}$

Bulk mode (untwisted mode) : singlet of D4 $\mathbf{1}$

Thus full symmetry is larger than geometric symmetry

Alternatively, discrete flavor symmetries may be originated from continuous symmetries



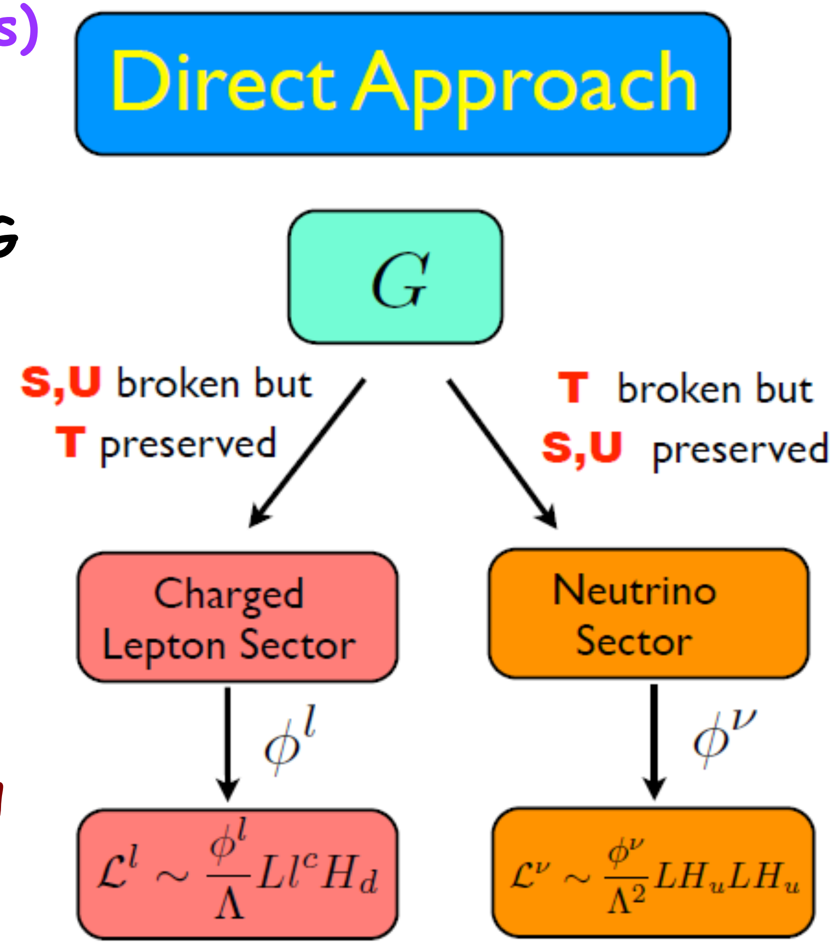
S. King

3.3 Direct and indirect approaches of Flavour Symmetry

(Talk of Ivo de Medeiros Varzieias)

Suppose Flavour symmetry group G

It breaks different subgroups which are preserved in Neutrino sector and Charged lepton sector, respectively.



S.F.King

arXiv: 1402.4271 King, Merle, Morisi, Simizu, M.T

Consider the case of S_4 flavor symmetry:

24 elements are generated by S, T and U:

$S^2=T^3=U^2=1$, $ST^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$

Irreducible representations: 1, 1', 2, 3, 3'

It has subgroups, nine Z_2, four Z_3, three Z_4, four $Z_2 \times Z_2$ (K_4)

Suppose S_4 is spontaneously broken to one of subgroups:

Neutrino sector preserves (1,S,U,SU) (K_4)

Charged lepton sector preserves (1,T,T^2) (Z_3)

For 3 and 3'

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

$$U = \mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Then, neutrinos respect S and U,
Charged leptons respect T, respectively.
Suppose neutrinos are Majorana particles.

$$S^T m^\nu_{LL} S = m^\nu_{LL}, \quad U^T m^\nu_{LL} U = m^\nu_{LL}, \qquad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

$$[S, m^\nu_{LL}] = 0, \quad [U, m^\nu_{LL}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

Mixing matrices diagonalize mass matrices also diagonalize S,U, and T, respectively !

The charged lepton mass matrix is diagonal because T is diagonal matrix.

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & /1\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Tri-bimaximal mixing $\theta_{13}=0$

C.S.Lam, PRD98(2008)
arXiv:0809.1185

which digonalizes both S and U.

Independent of mass eigenvalues !

Klein Symmetry can reproduce Tri-bimaximal mixing.

If S_4 is spontaneously broken to another subgroups,
eg. Neutrino sector preserves (1,SU) (Z_2)
Charged lepton sector preserves (1,T,T^2) (Z_3),
obtained mixing matrix is changed!

$$(SU)^T m_{LL}^{\nu} SU = m_{LL}^{\nu}, \quad T^{\dagger} Y_e Y_e^{\dagger} T = Y_e Y_e^{\dagger}$$

⬇

$$[SU, m_{LL}^{\nu}] = 0, \quad [T, Y_e Y_e^{\dagger}] = 0$$

$$V_{\nu} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -c/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

which digonalizes SU.

Eigenvalues of SU
(-1,1,1)

There is a freedom of the rotation between 2nd and 3rd column because a column corresponds to a mass eigenvalue.

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & c/\sqrt{3} & s/\sqrt{3} \\ -1/\sqrt{6} & c/\sqrt{3} - s/\sqrt{2} & -s/\sqrt{3} - c/\sqrt{2} \\ -1/\sqrt{6} & c/\sqrt{3} + s/\sqrt{2} & -s/\sqrt{3} + c/\sqrt{2} \end{pmatrix}$$

$c = \cos\theta,\ s = \sin\theta$ includes CP phase.

Tri-maximal mixing
TM1

Semi-direct model

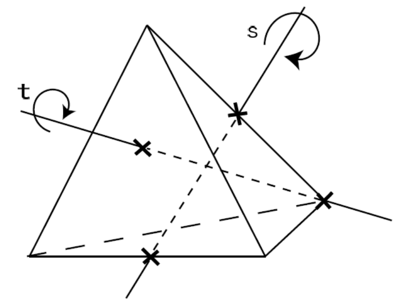
θ is not fixed by the flavor symmetry,

Mixing sum rules

$$\sin^2\theta_{23} = \frac{1}{2}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{2}\,, \quad \sin^2\theta_{12} \simeq \frac{1}{\sqrt{3}} - \frac{2\sqrt{2}}{3}\sin\theta_{13}\cos\delta_{CP} + \frac{1}{3}\sin^2\theta_{13}\cos 2\delta_{CP}$$

A_4 symmetry

Symmetry of tetrahedron



A_4 has subgroups:

three Z_2, four Z_3, one $Z_2 \times Z_2$ (klein four-group)

Z_2: {1,S}, {1,T²ST}, {1,TST²}

Z_3: {1,T,T²}, {1,ST,T²S}, {1,TS, ST²}, {1,STS,ST²S}

K_4: {1,S,T²ST,TST²}

Suppose A_4 is spontaneously broken to one of subgroups:

Neutrino sector preserves Z_2: {1,S}

Charged lepton sector preserves Z_3: {1,T,T²}

$$S^T m^\nu_{LL} S = m^\nu_{LL}, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

$$[S, m^\nu_{LL}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

Mixing matrices diagonalise m^ν_{LL}, $Y_e Y_e^\dagger$ also diagonalize S and T, respectively !

Then, we obtain PMNS matrix.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6} + s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} - c/\sqrt{2} \\ -c/\sqrt{6} - s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} + c/\sqrt{2} \end{pmatrix}$$

$c = \cos\theta, \quad s = \sin\theta$

Tri-maximal mixing : so called TM2

θ is not fixed.

Semi-direct model

In general, s is complex.
CP symmetry can predict this phase as seen later.

Another Mixing sum rules

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\,, \qquad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1 - \frac{5}{4}\sin^2\theta_{13}\right)$$

Another mixing pattern in A_5 flavor symmetry

It has subgroups, ten Z_3, six Z_5, five $Z_2 \times Z_2$ (K_4) .

Suppose A_5 is spontaneously broken to one of subgroups:
Neutrino sector preserves S and U (K_4)
Charged lepton sector preserves T (Z_5)

$$S^T m^\nu_{LL} S = m^\nu_{LL}, \quad U^T m^\nu_{LL} U = m^\nu_{LL}, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

$$\Downarrow$$

$$[S, m^\nu_{LL}] = 0, \quad [U, m^\nu_{LL}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\phi & \frac{1}{\phi} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{5}} & 0 \\ 0 & 0 & e^{\frac{8\pi i}{5}} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

F. Feruglio and Paris, JHEP 1103(2011) 101 arXiv:1101.0393

$$U_{GR} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\theta_{13}=0$

$\tan\theta_{12} = 1/\phi$: $\phi = \frac{1+\sqrt{5}}{2}$ **Golden ratio**

Neutrino mass matrix has μ-τ symmetry.

$$m_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}$$ with $z + w = x - \sqrt{2}y$

$\sin^2\theta_{12} = 2/(5+\sqrt{5}) = 0.2763\ldots$
which is rather smaller than the experimental data.

$$\sin^2\theta_{12} = 0.306 \pm 0.012$$

In order to obtain non-zero θ_{13} , A_5 should be broken to other subgroups: for example,

Neutrino sector preserves S or $T^2ST^3ST^2$ (both are K_4 generator)

Charged lepton sector preserves T (Z_5)

$$\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\tan\theta_{12} = 1/\phi \qquad \phi = \tfrac{1+\sqrt{5}}{2}$$

θ is not fixed, however, there appear testable sum rules:

$$\sin^2\theta_{12} = \frac{\sin^2\varphi}{1-\sin^2\theta_{13}} \approx \frac{0.276}{1-\sin^2\theta_{13}} \qquad \sin^2\theta_{23} \approx \frac{1}{2}\left(1 \pm (1-\sqrt{5})\sin\theta_{13}\right)$$

A.Di Iura, C.Hagedorn and D.Meloni, JHEP1508 (2015) 037

Since simple patterns predict vanishing θ_{13}, larger groups may be used to obtain non-vanishing θ_{13}.

$\Delta(96)$ group

R.de Adelhart Toorop, F.Feruglio, C.Hagedorn, Phys. Lett 703} (2011) 447
G.J.Ding, Nucl. Phys.B 862 (2012) 1
S. F.King, C.Luhn and A.J.Stuart, Nucl.Phys.B867(2013) 203
G.J.Ding and S.F.King, Phys.Rev.D89 (2014) 093020
C.Hagedorn, A.Meroni and E.Molinaro, Nucl.Phys. B 891 (2015) 499

Generator S, T and U : $S^2=(ST)^3=T^8=1$, $(ST^{-1}ST)^3=1$

Irreducible representations: 1, 1', 2, 3_1 - 3_6, 6

Subgroup : fifteen Z_2 , sixteen Z_3 , seven K_4 , twelve Z_4 , six Z_8

For triplet 3,

$$S=\frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \qquad T=\begin{pmatrix} e^{\frac{6\pi i}{4}} & 0 & 0 \\ 0 & e^{\frac{7\pi i}{4}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{4}} \end{pmatrix}$$

Neutrino sector preserves
$\{S,\ ST^4ST^4\}$ $(Z_2\times Z_2)$
Charged lepton sector preserves
ST (Z_3)

$$U_{TFH1}=\begin{pmatrix} \frac{1}{6}(3+\sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(-3+\sqrt{3}) \\ \frac{1}{6}(-3+\sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(3+\sqrt{3}) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$\Theta_{13}\sim 12^\circ$ rather large

Model building of Flavor Symmetry by flavons

indirect approach

Flavor symmetry G is (partially, completely) broken by flavon (SU_2 singlet scalors) VEV's.
Flavor symmetry controls couplings among leptons and flavons with special vacuum alignments.

Consider an example : A_4 model

	Leptons	flavons	
A_4 triplets	(L_e, L_μ, L_τ)	$\phi_\nu(\phi_{\nu 1}, \phi_{\nu 2}, \phi_{\nu 3})$	couple to neutrino sector
		$\phi_E(\phi_{E1}, \phi_{E2}, \phi_{E3})$	couple to charged lepton sector

A_4 singlets $e_R : \mathbf{1}$ $\mu_R : \mathbf{1}''$ $\tau_R : \mathbf{1}'$

$$\mathbf{3}_L \times \mathbf{3}_L \times \mathbf{3}_{flavon} \rightarrow \mathbf{1}$$

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

Flavor symmetry G is broken by VEV of flavons

$\mathbf{3_L \times 3_L \times 3_{flavon} \to 1}$

$$m_{\nu LL} \sim y \begin{pmatrix} 2\langle\phi_{\nu1}\rangle & -\langle\phi_{\nu3}\rangle & -\langle\phi_{\nu2}\rangle \\ -\langle\phi_{\nu3}\rangle & 2\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle \\ -\langle\phi_{\nu2}\rangle & -\langle\phi_{\nu1}\rangle & 2\langle\phi_{\nu3}\rangle \end{pmatrix}$$

$\mathbf{3_L \times 1_R (1_R', 1_R'') \times 3_{flavon} \to 1}$

$$m_E \sim \begin{pmatrix} y_e\langle\phi_{E1}\rangle & y_e\langle\phi_{E3}\rangle & y_e\langle\phi_{E2}\rangle \\ y_\mu\langle\phi_{E2}\rangle & y_\mu\langle\phi_{E1}\rangle & y_\mu\langle\phi_{E3}\rangle \\ y_\tau\langle\phi_{E3}\rangle & y_\tau\langle\phi_{E2}\rangle & y_\tau\langle\phi_{E1}\rangle \end{pmatrix}$$

However, specific Vacuum Alingnments preserve S and T generator.

Take $\langle\phi_{\nu1}\rangle = \langle\phi_{\nu2}\rangle = \langle\phi_{\nu3}\rangle$ and $\langle\phi_{E2}\rangle = \langle\phi_{E3}\rangle = 0$

$\Rightarrow$ $\langle\phi_\nu\rangle \sim (1,1,1)^T$, $\langle\phi_E\rangle \sim (1,0,0)^T$

$$S\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Then, $\langle\phi_\nu\rangle$ preserves S and $\langle\phi_E\rangle$ preserves T.

m_E is a diagonal matrix, on the other hand, $m_{\nu LL}$ is

$$m_{\nu LL} \sim 3y\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - y\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Rank 2

two generated masses and one massless neutrinos !
(0, 3y, 3y)
Flavor mixing is not fixed !

Adding A_4 singlet $\xi : \mathbf{1}$ in order to fix flavor mixing matrix.

$$\mathbf{3_L} \times \mathbf{3_L} \times \mathbf{1_{flavon}} \rightarrow \mathbf{1}$$

$$m_{\nu LL} \sim y_1 \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix} + y_2\langle\xi\rangle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$, which preserves S symmetry.

$$m_{\nu LL} = 3a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Flavor mixing is determined: Tri-bimaximal mixing. $\theta_{13}=0$

$$m_\nu = 3a+b,\ b,\ 3a-b \ \Rightarrow \boxed{m_{\nu_1} - m_{\nu_3} = 2m_{\nu_2}}$$

There appears a Neutrino Mass Sum Rule.

This is a minimal framework of A_4 symmetry predicting mixing angles and masses.

A_4 Model easily realizes non-vanishing θ_{13} .

Modify G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

	(l_e, l_μ, l_τ)	e^c	μ^c	τ^c	$h_{u,d}$	ϕ_l	ϕ_ν	ξ	ξ'
$SU(2)$	2	1	1	1	2	1	1	1	1
A_4	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}''$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}'$
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω

Y. Simizu, M. Tanimoto, A. Watanabe, PTP 126, 81(2011)

$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1} \quad = a_1 * b_1 + a_2 * b_3 + a_3 * b_2$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}' \quad = a_1 * b_2 + a_2 * b_1 + a_3 * b_3$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}'' \quad = a_1 * b_3 + a_2 * b_2 + a_3 * b_1$$

ξ $\qquad\qquad$ ξ'

$$\mathbf{1} \times \mathbf{1} \Rightarrow \mathbf{1} \quad , \qquad \mathbf{1}'' \times \mathbf{1}' \Rightarrow \mathbf{1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Additional Matrix

$$M_\nu = a\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + b\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + c\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + d\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$a = \frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{\Lambda}, \quad b = -\frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{3\Lambda}, \quad c = \frac{y^\nu_{\xi}\alpha_\xi v_u^2}{\Lambda}, \quad d = \frac{y^\nu_{\xi'}\alpha_{\xi'} v_u^2}{\Lambda} \qquad a = -3b$$

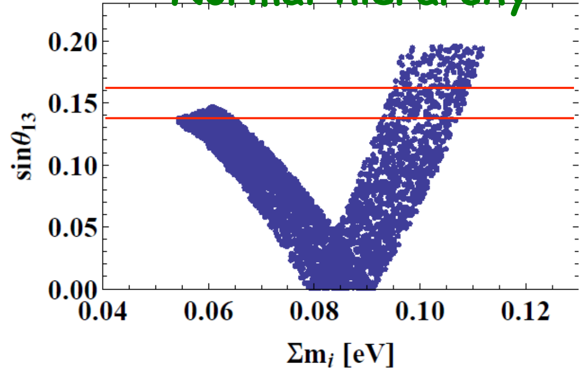
Both normal and inverted mass hierarchies are possible.

$$M_\nu = V_{\text{tri-bi}}\begin{pmatrix}a+c-\frac{d}{2} & 0 & \frac{\sqrt{3}}{2}d\\ 0 & a+3b+c+d & 0\\ \frac{\sqrt{3}}{2}d & 0 & a-c+\frac{d}{2}\end{pmatrix}V^T_{\text{tri-bi}}$$

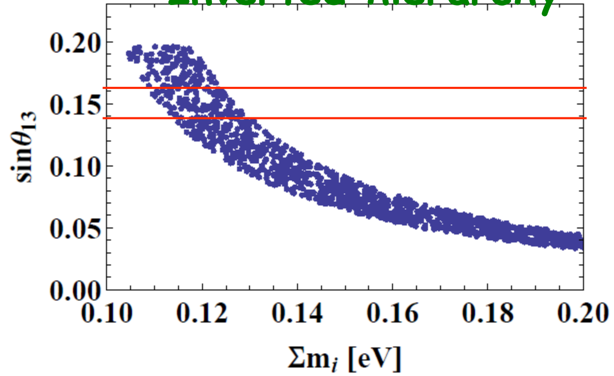
Tri-maximal mixing: TM2

$$\Delta m^2_{31} = -4a\sqrt{c^2+d^2-cd}\,, \qquad \Delta m^2_{21} = (a+3b+c+d)^2 - (a+\sqrt{c^2+d^2-cd})^2$$

Normal hierarchy



Inverted hierarchy



Mass sum rules in A_4, T', S_4, A_5, $\Delta(96)$ …

(Talk of Spinrath)   Barry, Rodejohann, NPB842(2011) arXiv:1007.5217

Different types of neutrino mass spectra correspond to the neutrino mass generation mechanism.

$$\chi\tilde{m}_2 + \xi\tilde{m}_3 = \tilde{m}_1$$ (χ=2, ξ=1) (χ=-1, ξ=1)

$$\frac{\chi}{\tilde{m}_2} + \frac{\xi}{\tilde{m}_3} = \frac{1}{\tilde{m}_1}$$ **M_R structre in See-saw**

$$\chi\sqrt{\tilde{m}_2} + \xi\sqrt{\tilde{m}_3} = \sqrt{\tilde{m}_1}$$ **M_D structre in See-saw**

$$\frac{\chi}{\sqrt{\tilde{m}_2}} + \frac{\xi}{\sqrt{\tilde{m}_3}} = \frac{1}{\sqrt{\tilde{m}_1}}$$ **M_R in inverse See-saw**

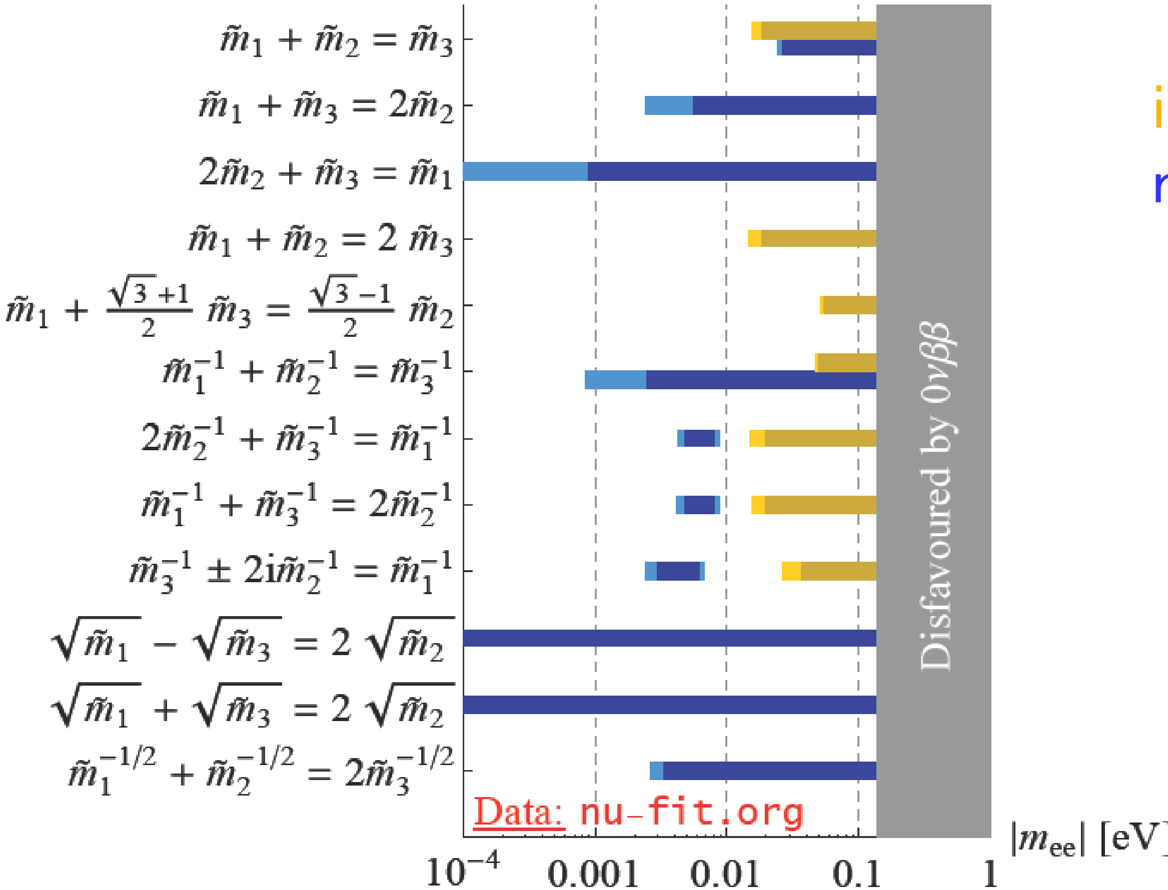
χ and ξ are model specific complex parameters

King, Merle, Stuart, JHEP 2013, arXiv:1307.2901
King, Merle, Morisi, Simizu, M.T, arXiv: 1402.4271

Sum Rule	Group	Seesaw Type	Matrix
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	A_4 [167] ([175, 178–181]); S_4 ([182]); A_5 [69]*	Weinberg	m^ν_{LL}
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	$\Delta(54)$ [183]; S_4 ([163])	Type II	M_L
$\tilde{m}_1 + 2\tilde{m}_2 = \tilde{m}_3$	S_4 [120]	Type II	M_L
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4 [165, 167] ([36, 37, 178–181, 188–194]) S_4 ([45, 124])†; T' [195, 196] ([46, 134, 197, 198]); T_7 ([199])	Weinberg	m^ν_{LL}
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4 ([200])	Type II	M_L
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	S_4 [201]‡	Dirac‡	m^D
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	$L_e - L_\mu - L_\tau$ ([202])	Type II	M_L
$\tilde{m}_1 + \frac{\sqrt{3}+1}{2}\tilde{m}_3 = \frac{\sqrt{3}-1}{2}\tilde{m}_2$	A'_5 ([203])	Weinberg	m^ν_{LL}
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	A_4 [167]; S_4 ([163, 175]); A_5 [176, 177]	Type I	M_R
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	S_4 ([163])	Type III	M_Σ
$2\tilde{m}_2^{-1} + \tilde{m}_3^{-1} = \tilde{m}_1^{-1}$	A_4 [135, 164, 165, 167, 204] ([37, 137, 145, 205–211]); T' [196]	Type I	M_R
$\tilde{m}_1^{-1} + \tilde{m}_3^{-1} = 2\tilde{m}_2^{-1}$	A_4 ([212–214]); T' [215]	Type I	M_R
$\tilde{m}_3^{-1} \pm 2i\tilde{m}_2^{-1} = \tilde{m}_1^{-1}$	$\Delta(96)$ [66]	Type I	M_R
$\tilde{m}_1^{1/2} - \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4 ([162])	Type I	m^D
$\tilde{m}_1^{1/2} + \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4 ([216])	Scotogenic	h_ν
$\tilde{m}_1^{-1/2} + \tilde{m}_2^{-1/2} = 2\tilde{m}_3^{-1/2}$	S_4 [217]	Inverse	M_{RS}

King, Merle, Stuart, JHEP 2013, arXiv:1307.2901

Restrictions by mass sum rules on $|m_{ee}|$



inverted

normal

King, Merle, Stuart, JHEP 2013, arXiv:1307.2901

Talks of Grimus, Hagedorn, Ding

3.4 CP symmetry

CP phase δ_{CP} is related with Flavour Symmetry.

A hint : under $\mu - \tau$ symmetry $|U_{\mu i}| = |U_{\tau i}| \; i = 1, 2, 3$

$$\cos\theta_{23} = \sin\theta_{23} = \frac{1}{\sqrt{2}}$$

$$\sin\theta_{13}\cos\delta = 0$$

$\delta = \pm\frac{\pi}{2}$ is predicted since we know $\theta_{13} \neq 0$

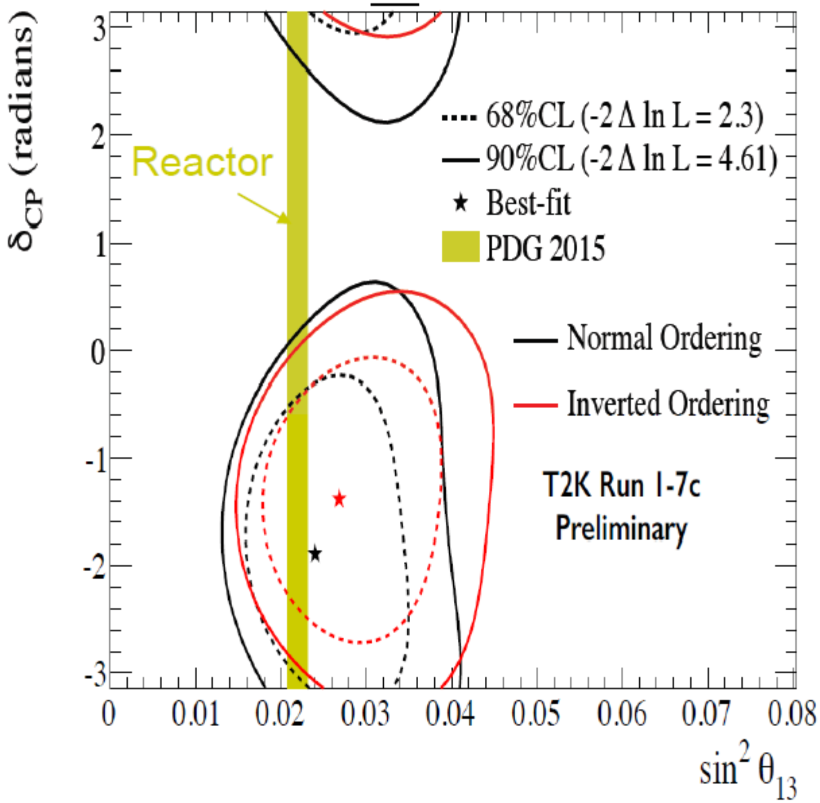
Ferreira, Grimus, Lavoura, Ludl, JHEP2012,arXiv: 1206.7072

CP violation is constrained by Flavour symmetries !

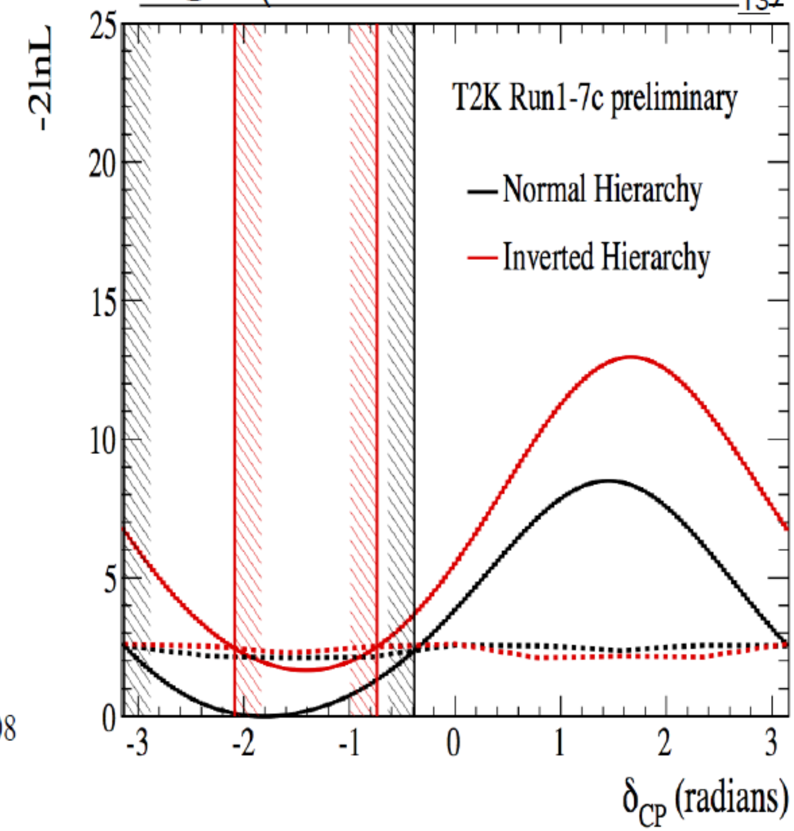
Exciting Era of Observation of CP violating phase @T2K and NOvA

T2K: Results on $\sin^2\theta_{13}$ and δCP arXiv:1701.00432 [hep-ex]

$\sin^2\theta_{13}$ vs. δCP



δCP (constrained with reactor θ_{13})

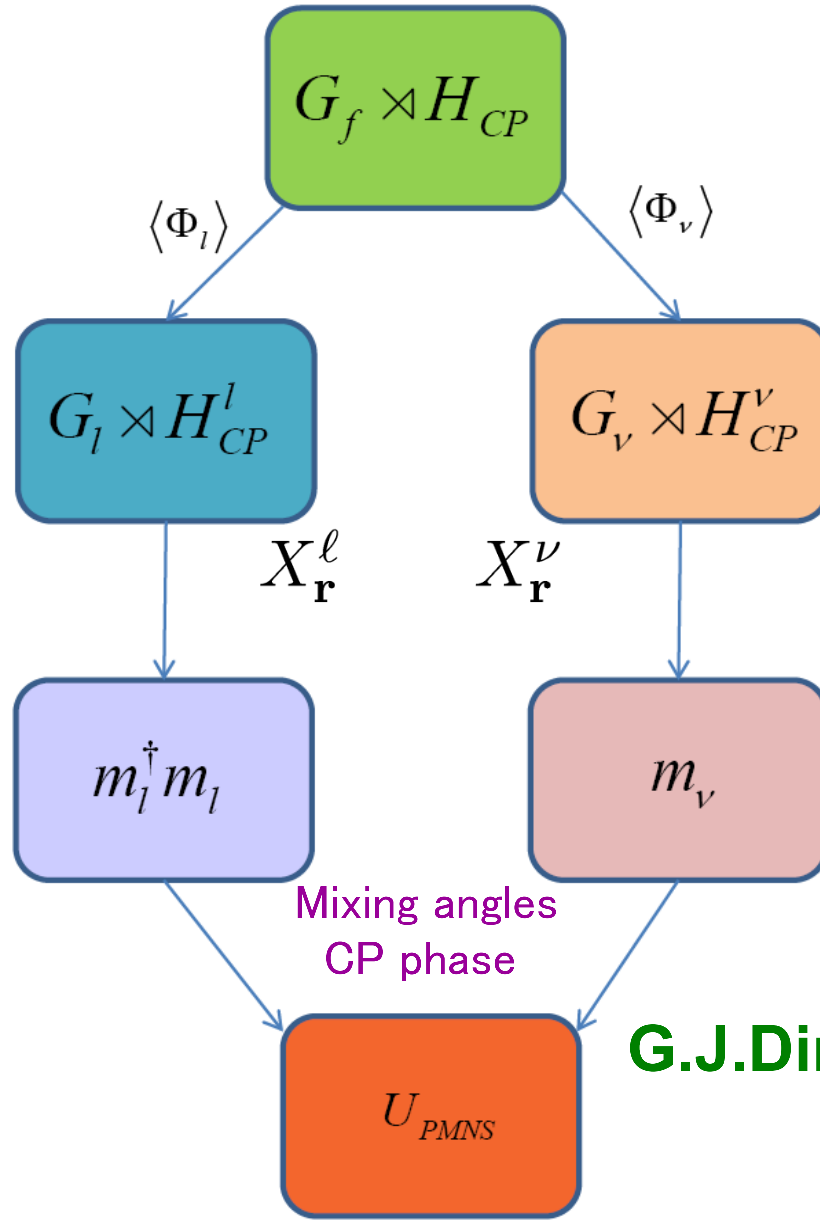


The best fit points lie near the maximally CP violating value $\delta CP=-0.5\pi$.
The CP conserving values ($\delta CP=0$ and $\delta CP=\pi$) lying outside of the T2K 90% confidence level interval.

Suppose a symmetry including FLASY and CP symmetry:

$$G_{CP} = G_f \rtimes H_{CP}$$

is broken to the subgroups in neutrino sector and charged lepton sector.



G.J.Ding

CP symmetry gives

$$X_{\mathbf{r}}^{\nu T} m_{\nu LL} X_{\mathbf{r}}^{\nu} = m_{\nu LL}^{*}$$

$$X_{\mathbf{r}}^{\ell\dagger} (m_\ell^\dagger m_\ell) X_{\mathbf{r}}^{\ell} = (m_\ell^\dagger m_\ell)^*$$

☆CP is conserved in HE theory before FLASY is broken.
☆CP is a dicrete symmetry.

Branco, Felipe, Joaquim, Rev. Mod. Physics 84(2012), arXiv: 1111.5332
Mohapatra, Nishi, PRD86, arXiv: 1208.2875
Holthhausen, Lindner, Schmidt, JHEP1304(2012), arXiv:1211.6953
Feruglio, Hagedorn, Ziegler, JHEP 1307, arXiv:1211.5560,
Eur.Phys.J.C74(2014), arXiv 1303.7178
E. Ma, PLB 723(2013), arXiv:1304.1603
Ding, King, Luhn, Stuart, JHEP1305, arXiv:1303.6180
Ding, King, Stuart, JHEP1312, arXiv:1307.4212,
Ding, King, 1403.5846
Meroni, Petcov, Spinrath, PRD86, 1205.5241
Girardi, Meroni, Petcov, Spinrath, JHEP1042(2014), arXiv:1312.1966
Li, Ding, Nucl. Phys. B881(2014), arXiv:1312.4401
Ding, Zhou, arXiv:1312.522
G.J.Ding and S.F.King, Phys.Rev.D89 (2014) 093020
P.Ballett, S.Pascoli and J.Turner, Phys. Rev. D 92 (2015) 093008
A.Di Iura, C.Hagedorn and D.Meloni, JHEP1508 (2015) 037

Ecker, Grimus, Konetschny (1981); Ecker, Grimus, Neufeld (1987); Grimus, Rebelo (1995); Branco, Lavoura, Silva (1999)

Generalized CP Symmetry

Setting with Discrete symmetries: G and CP symmetry do not commute.

CP Symmetry $\varphi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}}\varphi^*(x'), \quad x' = (t, -\mathbf{x})$

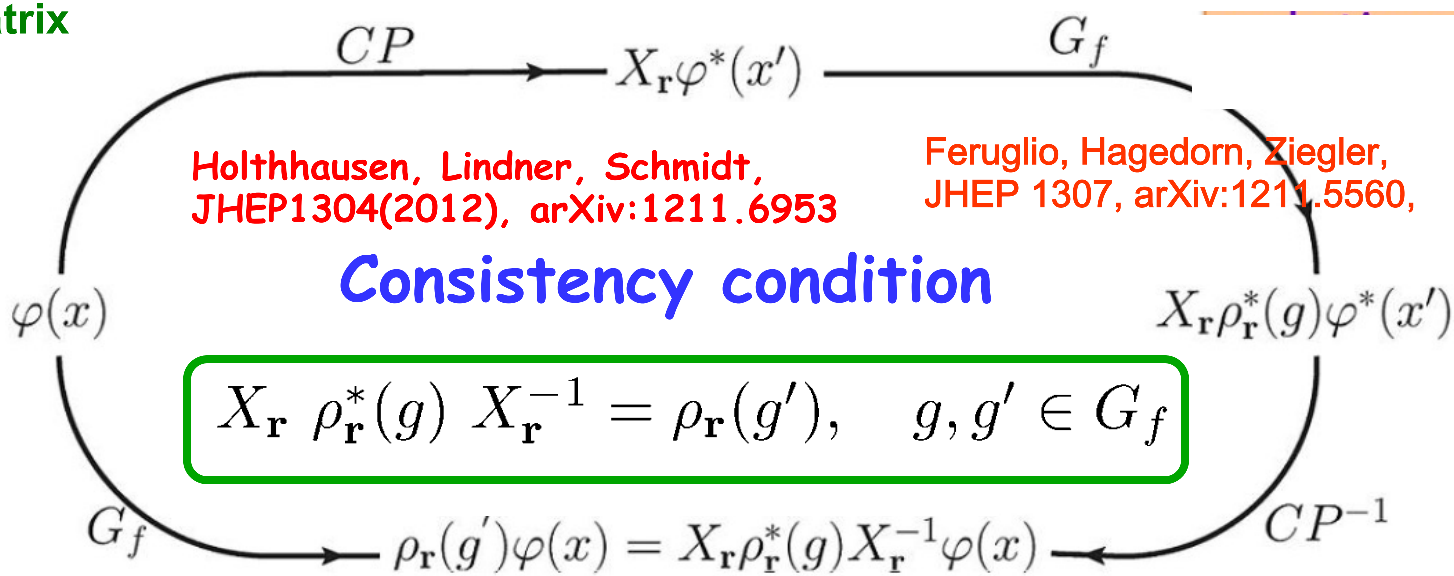
Flavour Symmetry $\varphi(x) \xrightarrow{\text{g}} \rho_{\mathbf{r}}(g)\varphi^*(x) \qquad g \in G_f$

$$X_{\mathbf{r}}^{\nu T} m_{\nu LL} X_{\mathbf{r}}^{\nu} = m_{\nu LL}^*$$

$$X_{\mathbf{r}}^{\ell\dagger}(m_\ell^\dagger m_\ell) X_{\mathbf{r}}^{\ell} = (m_\ell^\dagger m_\ell)^*$$

X_r must be consistent with Flavour Symmetry $\rho_{\mathbf{r}}(g)$

Unitary matrix



Holthhausen, Lindner, Schmidt, JHEP1304(2012), arXiv:1211.6953

Feruglio, Hagedorn, Ziegler, JHEP 1307, arXiv:1211.5560,

Consistency condition

$$X_{\mathbf{r}}\ \rho_{\mathbf{r}}^*(g)\ X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f$$

$$X_{\mathbf{r}}\ \rho^*_{\mathbf{r}}(g)\ X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f$$

Mu-Chun Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys. B883 (2014) 267–305

Condition on automorphism for physical CP transformation is discussed.

Class inverting
Involutory
automorphisms

A_5 flavour model with CP violation

A.Di Iura, C.Hagedorn and D.Meloni, JHEP1508 (2015) 037

An example of S_4 model

Ding, King, Luhn, Stuart, JHEP1305, arXiv:1303.6180

$G_\nu=\{1,S\}$ and $X_3^\nu=\{U, SU\}$, $X_3^l=\{1\}$

satisfy the consistency condition

$$X_{\mathbf{r}}\ \rho_{\mathbf{r}}^*(g)\ X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f$$

$$m_{\nu LL} = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

respects $G_\nu=\{1,S\}$

CP symmetry $X_{\mathbf{r}}^{\nu T} m_{\nu LL} X_{\mathbf{r}}^{\nu} = m_{\nu LL}^*$

⬇

α, β, γ are real, ε is imaginary.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6}+is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}-ic/\sqrt{2} \\ -c/\sqrt{6}+is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}+ic/\sqrt{2} \end{pmatrix}$$

$$c=\cos\theta,\ \ s=\sin\theta$$

⬇

$$\sin^2\theta_{13}=\frac{2}{3}\sin^2\theta,\ \ \sin^2\theta_{12}=\frac{1}{2+\cos 2\theta},\ \ \sin^2\theta_{23}=\frac{1}{2}$$

$$|\sin\delta_{CP}|=1,\quad \sin\alpha_{21}=\sin\alpha_{31}=0$$

$$\delta_{CP}=\pm\pi/2$$

The predicton of CP phase depends on
the respected Generators of FLASY and CP symmetry.
Typically, it is simple value, 0, π, ±π/2 .

A_4, T' , A_5, $\Delta(6N^2)$ …

4. Prospect

☆**How can Quarks and Leptons become reconciled in the unfied theory?** Mixing of Quark sector is rather small.

T', S_4, A_5 and $\Delta(96)$ — SU(5)
S_3, S_4, $\Delta(27)$ and $\Delta(96)$ can be embeded in SO(10) GUT.
A_4 and S_4 — PS

For example: See references S.F. King, 1701.0441

quark sector (2, 1) for SU(5) 10
lepton sector (3) for SU(5) 5

Different flavor structures of quarks and leptons appear !

Cooper, King, Luhn (2010,2012), Callen,Volkas (2012), Meroni, Petcov, Spinrath (2012)
Antusch, King, Spinrath (2013), Gehrlein, Oppermann, Schaefer, Spinrath (2014)
Gehrlein, Petcov,Spinrath (2015), Bjoreroth, Anda, Medeiros Varzielas, King (2015) …

Interesting relation: $\theta_{13} = \sin\theta_{23}\, \theta_{Cabibbo}$ @ flavor GUT

☆ How is Flavour Symmetry in Higgs sector ?

Does a Finite group control Higgs sector ?

Talks of Ivanov
Branco, Neder

2HDM, 3HDM …

an interesting question since Pakvasa and Sugawara 1978

☆ How will be Flavour Symmetry tested ?

* Mixing angle sum rules

Example: TM1

$$\sin^2\theta_{23} = \frac{1}{2}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{2}\,, \quad \sin^2\theta_{12} \simeq \frac{1}{\sqrt{3}} - \frac{2\sqrt{2}}{3}\sin\theta_{13}\cos\delta_{CP} + \frac{1}{3}\sin^2\theta_{13}\cos 2\delta_{CP}$$

TM2

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\,, \quad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1 - \frac{5}{4}\sin^2\theta_{13}\right)$$

* Neutrino mass sum rules in FLASY ⇔neutrinoless double beta decays

* Prediction of CP violating phase.

* Collider Physics Talk of Turner

We hope

Discrete Symmetry will be found
in the flavor physics in the near future.

Thank you !

Backup slides

Tests for nonsimplicity

Sylow's test:
Let *n* be a positive integer that is not prime, and let *p* be a prime divisor of *n*.
If 1 is the only divisor of *n* that is equal to 1 modulo p,
then there does not exist a simple group of order *n*.
example:
A4: $12=2^2\times3$
A5: $60=2^2\times3\times5$

Burnside:
A non-Abelian finite simple group has order divisible by at least three distinct primes.

S_3 has two singlets and one doublet: **1, 1', 2, no triplet representation.**

For flavour physics, we are interested in finite groups with **triplet representation**.

They are found among the subgroups of SU(3) :

- Groups of the type $(Z_n \times Z_m) \rtimes S_3$ **$=\Delta\ (6N^2)$**
- Groups of the type $(Z_n \times Z_m) \rtimes Z_3$ **$=\Delta\ (3N^2)$**
- The simple groups A_5 and $PSL_2(7)$ plus a few more "exceptional" groups
- The double covers of A_4, S_4 and A_5 groups

The projective special linear group $PSL_2(7)$ (isomorphic to $GL_3(2)$) is a finite simple group with order of 168.

All permutations of S_3 are represented on the reducible triplet (x_1, x_2, x_3) as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$e : (x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3)$
$a_1 : (x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3)$
$a_2 : (x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1)$
$a_3 : (x_1, x_2, x_3) \rightarrow (x_1, x_3, x_2)$
$a_4 : (x_1, x_2, x_3) \rightarrow (x_3, x_1, x_2)$
$a_5 : (x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1)$

We change the representation through the unitary transformation, $U^\dagger g U$, e.g. by using the unitary matrix U_{tribi},

$$U_{\rm tribi} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}.$$

tri-bimaximal matrix

Then, the six elements of S_3 are written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

These are completely reducible and that the (2×2) submatrices are exactly the same as those for the doublet representation. The unitary matrix U_{tribi} is called **tri-bimaximal matrix**.

We can use another unitary matrix U

$$U_w = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}.$$

magic matrix

Then, the six elements of S_3 are written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & w^2 \\ 0 & w & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & w \\ 0 & w^2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & w \end{pmatrix}$$

The (2 × 2) submatrices correspond to the doublet representation in the complex basis. This unitary matrix is called the magic matrix.

T' group

Double covering group of A_4, 24 elements

24 elements are generated by S, T and R:

$S^2 = R, \quad T^3 = R^2 = 1,$

$(ST)^3 = 1, \quad RT = TR$

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_2	$\chi_{2'}$	$\chi_{2''}$	χ_3
C_1	1	1	1	1	2	2	2	3
C'_1	2	1	1	1	-2	-2	-2	3
C_4	3	1	ω	ω^2	-1	$-\omega$	$-\omega^2$	0
C'_4	3	1	ω^2	ω	-1	$-\omega^2$	$-\omega$	0
C''_4	6	1	ω	ω^2	1	ω	ω^2	0
C'''_4	6	1	ω^2	ω	1	ω^2	ω	0
C_6	4	1	1	1	0	0	0	-1

Ireducible representations

1, 1', 1", 2, 2', 2", 3

For triplet $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega^2 & -1 & 2\omega \\ 2\omega & 2\omega^2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

Subgroups and decompositions of multiplets

T' → Z_6

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_6 & \mathbf{1}_0 & \mathbf{1}_2 & \mathbf{1}_4 & \mathbf{1}_5 + \mathbf{1}_1 & \mathbf{1}_5 + \mathbf{1}_3 & \mathbf{1}_3 + \mathbf{1}_5 & \mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_4 \end{array}$$

T' → Z_4

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Z_4 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_0 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_1 + \mathbf{1}_3 & \mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_2 \end{array}$$

T'→ Q_4

$$\begin{array}{cccccccc} T' & \mathbf{1} & \mathbf{1}' & \mathbf{1}'' & \mathbf{2} & \mathbf{2}' & \mathbf{2}'' & \mathbf{3} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ Q_4 & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{1}_{++} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{1}_{+-} + \mathbf{1}_{-+} + \mathbf{1}_{--} \end{array}$$