Plethysms and their applications in particle physics

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Plethysms

= Permutation symmetry in the contraction of fields

Suppose there is a **scalar** ϕ, **doublet under** ***SU(2)*** [and no hypercharge]

How many quartic couplings $\phi\phi\phi\phi$ can we write down?

?

Plethysms

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?

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1} + \cdots$$

So 2 couplings?

Plethysms

= Permutation symmetry in the contraction of fields

Suppose there is a **scalar** ϕ, **doublet under** $SU(2)$ [and no hypercharge]

How many quartic couplings $\phi\phi\phi\phi$ **can we write down?**

$$\mathbf{2}\times\mathbf{2}\times\mathbf{2}\times\mathbf{2}=\mathbf{1}+\mathbf{1}+\cdots$$

So 2 couplings?

But **there is only one doublet**, so we must be careful:

$$\mathbf{2}\times\mathbf{2}\times\mathbf{2}\times\mathbf{2}=(\mathbf{2}\times\mathbf{2})\times(\mathbf{2}\times\mathbf{2})=(\mathbf{1}_A+\mathbf{3}_S)\times(\mathbf{1}_A+\mathbf{3}_S)$$

Just 1 coupling then?

Not relevant since it is anti-symmetric, but we do have a singlet inside $\mathbf{3}_S\times\mathbf{3}_S$ (right?)

Plethysms

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How many quartic couplings $\phi\phi\phi\phi$ can we write down?

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Just 1 coupling then?

Not relevant since it is anti-symmetric, but we do have a singlet inside $\mathbf{3}_S\times\mathbf{3}_S$ (right?)

Aim of this talk: explore the underlying permutation symmetry in the contraction of fields

Four doublets

A very simple example

Correct answer:
0 $\phi\phi\phi\phi$ couplings

Why? What is going on?

Consider 4 different doublets $\phi_{1,2,3,4}$

There are clearly two contractions of the form $\phi_1\phi_2\phi_3\phi_4$

$$I^{(1)} \equiv \left(\phi_1^T \epsilon \phi_2\right)\left(\phi_3^T \epsilon \phi_4\right)$$

$$I^{(2)} \equiv \left(\phi_1^T \epsilon \phi_3\right)\left(\phi_2^T \epsilon \phi_4\right)$$

The 4 ϕ_i are distinct fields, but they transform in the same way, so one can make S_4 permutations of them. What happens to these two field contractions?

Note that the S_n group is generated by just the permutations $1 \leftrightarrow 2$ and and $1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1$ so it is enough to look only at these two cases

(no need to look at all 4! permutations)

Four doublets

A very simple example

$$\pi_{(12)}\begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix}$$

$$\pi_{(12\cdots n)}\begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix}$$

Two comments about these matrices

1- (Minor comment) These matrices are not unitary/orthogonal because the two invariants are not orthogonal. This can be easily fixed with redefined invariants.

[2- More importantly, **these two generator matrices of S_4 cannot be simultaneously diagonal, so the two invariants $I^{(1),(2)}$ form a doublet of S_4 under permutations of the fields**] !

[So an **important message** is that **we are not always dealing with a + or − sign** (symmetric vs anti-symmetric) – that is true only for two repeated representations, since in that case the relevant group (**S_2**) only has two 1-dimensional representations]

Four doublets

A very simple example

Let us go further then. Imagine a **model with n doublets** $\phi_{1,2,\cdots,n}$

Pragmatic questions: **How do the quartic couplings look like? For a given n, how many couplings/numbers exist?**

$$V_4 = g^{(1)}_{ijkl}\left(\phi_i\phi_j\phi_k\phi_l\right)_{(1)} + g^{(2)}_{ijkl}\left(\phi_i\phi_j\phi_k\phi_l\right)_{(2)}$$

Contraction 1

Contraction 2

How many independent entries?

We need to take a (quick) look at the permutation group and it's irreducible representations

The permutation group S_n

... and it's representations

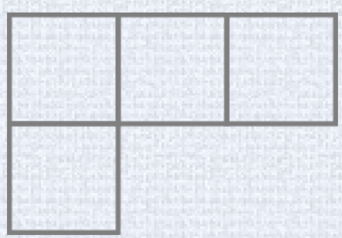
The irreducible representations (irreps) of S_n can be associated to the partitions $\{\lambda\}$ of the number n

$$\{\lambda\} = \{\lambda_1, \lambda_2, \cdots, \lambda_m\} \qquad \lambda_{i+1} \leq \lambda_i \qquad \sum \lambda_i = n \equiv |\lambda|$$

Example

For n=4 the irreps are {4}, {3,1}, {2,2}, {2,1,1,1}, {1,1,1,1}

A partition can then be associated to a **Young diagram** with λ_i boxes in row i



Example

{3,1}

(I'll refer to the irrep of $S_{|\lambda|}$ associated to the partition λ as s_λ)

The Hook Content Formula

A **standard Young tableau** is a Young diagram with the number *1,2, ..., n* strictly increasing across each row and column

1	3	4
2		

1	2	4
3		

1	2	3
4		

Example

$dim\left(s_{\{3,1\}}\right) = 3$

The well known **hook length formula** gives the **number of standard Young tableaux with a shape λ** which in turn is the **same as the size of the corresponding $S_{|\lambda|}$ irrep**

Less well known is the **hook content formula**. It gives the **number of semi-standard Young tableaux of shape λ filled with the numbers *1,2, ..., N*** (**Semi-standard Young tableau: numbers increase strictly along columns and cannot decrease along rows.**)

Example

$\lambda = \{2,2\}$
$N = 3$

1	1
2	2

1	1
2	3

1	1
3	3

1	2
2	3

1	2
3	3

2	2
3	3

6 semi-standard tableaux

For {2,2}* and a generic *N*: $\frac{1}{12}(N^2-1)N^2$ semi-standard tableaux

(***This is the 2-D irrep of S4**)

Parameter counting

Back to the contraction of 4 doublets of SU(2)

$$V_4 = g^{(1)}_{ijkl} \left(\phi_i\phi_j\phi_k\phi_l\right)_{(1)} + g^{(2)}_{ijkl} \left(\phi_i\phi_j\phi_k\phi_l\right)_{(2)}$$

Contraction 1 Contraction 2

How many independent entries?

Couplings inherit the permutation symmetry of the field contractions

More generally: couplings tensors $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with the following symmetry:

$$g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)} = \left[U_\lambda\left(\pi\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n} \qquad i_X = 1, \cdots, N$$

where U_λ are the matrices of the irrep λ of S_n

[I have no proof, but it appears that the number of independent parameters in the tensors $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ is given by the number of semi-standard tableaux with shape λ filled with the numbers *1,2,..,N*]

Then, according to the hook content formula, the quartic interactions of N doublets are encoded by ($\lambda = \{2,2\}$)

$$\frac{1}{12}(N^2-1)N^2 \text{ numbers}$$

(So, for N=1 no couplings)

Parameter counting

Back to the contraction of 4 doublets of SU(2)

$$V_4 = g^{(1)}_{ijkl} \underbrace{(\phi_i\phi_j\phi_k\phi_l)_{(1)}}_{\text{Contraction 1}} + g^{(2)}_{ijkl} \underbrace{(\phi_i\phi_j\phi_k\phi_l)_{(2)}}_{\text{Contraction 2}}$$

Couplings inherit the permutation symmetry of the field contractions

How many independent entries?

More generally: couplings tensors $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with the following symmetry:

$$g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)} = \left[U_\lambda(\pi)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n} \qquad i_X = 1, \cdots, N$$

where U_λ are the matrices of the irrep λ of S_n

Yes, I do!
(see extra slides)

[~~I have no proof~~, but it appears that the number of independent parameters in the tensors $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ is given by the number of semi-standard tableaux with shape λ filled with the numbers *1,2,..,N*]

Then, according to the hook content formula, the quartic interactions of N doublets are encoded by ($\lambda = \{2,2\}$)

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Why think about these things?

My main motivation: **given a list of fields/representations, be able to build the corresponding Lagrangian** (in a systematic way)

The discussion so far makes its clear that, in general, **it is not enough to know if there is a singlet in a product of some representations** (of the gauge group for example)

One also needs to know the symmetry under permutations of these singlets

[The LiE **program** calculates this under a function called "**plethysm**"*
Leeuwen, Cohen, Lisser 1992 (see www-math.univ-poitiers.fr/~maavl/LiE)]

For a "representation" (*i.e.*, module) R of some group G and some n,

$$\underbrace{R \times R \times \cdots \times R}_{n} = \sum_{\lambda \vdash n} \left(\sum_{i \in X(\lambda)} R_i \right) \times s_\lambda$$

You can test this online with LiE

irrep λ of $S_{|\lambda|=n}$

The plethysm function of LiE, for each λ, gives this list of R_i representations

*There is a (related) more "fundamental"/"mathematical" meaning to the word. It has to do an operation similar to composition, applied to symmetric functions.

Plethysm: an example

Example

Consider the $54 \times 54 \times 54$ and 120×120 in SO(10):

$$
\begin{aligned}
54 \times 54 \times 54 = & (1 + 54 + 54 + 660 + 770 + 1386 + 4125 + 4290 + 16380) \times s_{\{3\}} \\
& + (45 + 54 + 54 + 660 + 770 + 945 + 1386 + 1386 + 12870 + 16380 + 17920) \times s_{\{2,1\}} \\
& + (45 + 945 + 1386 + 7644 + 14784) \times s_{\{1,1,1\}}
\end{aligned}
$$

$$
\begin{aligned}
120 \times 120 = & \left(1 + 54 + 210 + 770 + 1050 + \overline{1050} + 4125\right) \times s_{\{2\}} \\
& + (45 + 210 + 945 + 5940) \times s_{\{1,1\}}
\end{aligned}
$$

So, the SO(10) singlets/invariants in $54 \times 54 \times 54 \times 120 \times 120$ transform as follows under permutations:

$$
54^3 120^2 \big|_{invs.} = \underbrace{5 s_{\{3\}} \times s_{\{2\}}}_{\text{5 invariants}} + \underbrace{3 s_{\{2,1\}} \times s_{\{2\}}}_{\text{6 invariants}} + \underbrace{2 s_{\{2,1\}} \times s_{\{1,1\}}}_{\text{4 invariants}} + \underbrace{2 s_{\{1,1,1\}} \times s_{\{1,1\}}}_{\text{2 invariants}}
$$

$54 \times 54 \times 54 \times 120 \times 120$ contains indeed 5+6+4+2=17 singlets

The Susyno and Sym2Int programs

Code related to Lie algebras and the permutation group (including all that was discussed previously) is **available in the `Susyno` package** for Mathematica

Fonseca 2012 (see renatofonseca.net/susyno/group_theory_tutorial.php)

The relevant code on the symmetry of contractions of fields is based on the algorithm described in the LiE manual.

Leeuwen, Cohen, Lisser 1992

Recently, I made the Mathematica package `Sym2Int` which uses this group theory code. Given a list of fields (*i.e.*, irreps of the Lorentz and gauge groups) it lists the allowed interactions, and counts couplings (it can also show explicitly how to contract the fields).

Fonseca 2017 (see http://renatofonseca.net/sym2int.php)

An important part of this effort is related precisely to correctly take into account what happens under the permutation of fields.

Let me show it to you with one example:
the Standard Model

The SM in Sym2Int

Input

```
gaugeGroup[SM] ^= {SU3, SU2, U1};

fld1 = {"u", {3, 1, 2 / 3}, "R", "C", 3};
fld2 = {"d", {3, 1, -1 / 3}, "R", "C", 3};
fld3 = {"Q", {3, 2, 1 / 6}, "L", "C", 3};
fld4 = {"e", {1, 1, -1}, "R", "C", 3};
fld5 = {"L", {1, 2, -1 / 2}, "L", "C", 3};
fld6 = {"H", {1, 2, 1 / 2}, "S", "C", 1};
fields[SM] ^= {fld1, fld2, fld3, fld4, fld5, fld6};

GenerateListOfCouplings[SM, MaxOrder → 4];
```

Name of the field (for bookkeeping)

Gauge representation
Minus sign for anti-representations
e.g.: -3 = anti-triplet

Lorentz representation
"S" = scalar, "R" = right-handed Weyl spinor "L" = left-handed Weyl spinor, "V" = vector, ...

Real or complex?
"R" = real, "C" = complex

Number of copies/flavours

Output

#	Operator*	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	-	{1, 1}
2	u[C] Q[R] H[R]	4	False	-	{9, 1}
3	d[C] Q[R] H[C]	4	False	-	{9, 1}
4	e[R] L[C] H[R]	4	False	-	{9, 1}
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	{{S, S}, 1, 1}

The only information which requires an explanation

*for convenience, "operator" here means just a product of fields

What's on the last column?

Are there repeated fields in the operator?

#	Operator	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	-	{1, 1}
2	u[C] Q[R] H[R]	4	False	-	{9, 1}
3	d[C] Q[R] H[C]	4	False	-	{9, 1}
4	e[R] L[C] H[R]	4	False	-	{9, 1}
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	{{S, S}, 1, 1}

If there are no repeated fields: {●, ■}

● = number of couplings needed per ind. contraction of the gauge+Lorentz indices of the fields
■ = number or independent contract of the gauge+Lorentz indices of the fields

If there are repeated fields: {▲, ●, ■}

▲ = symmetry associated to permutations of the repeated fields (*more on this in a moment*)

In other words, the last column contains the information we have been discussing

What's on the last column?

#	Operator	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	–	{1, 1}
2	u[C] Q[R] H[R]	4	False	–	{9, 1}
3	d[C] Q[R] H[C]	4	False	–	{9, 1}
4	e[R] L[C] H[R]	4	False	–	{9, 1}
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	{{S, S}, 1, 1}

Specifically: for each "operator" (i.e., each combination of fields)

$$\underbrace{\mathcal{O}_{i_{11}\cdots i_{1n_1}\cdots i_{x1}\cdots i_{xn_x}}}_{\text{flavor indices}} \equiv \underbrace{\Phi^{(1)}_{i_{11}}\cdots\Phi^{(1)}_{i_{1n_1}}}_{n_1}\underbrace{\Phi^{(2)}_{i_{21}}\cdots\Phi^{(2)}_{i_{2n_2}}}_{n_2}\cdots\underbrace{\Phi^{(x)}_{i_{x1}}\cdots\Phi^{(x)}_{i_{xn_x}}}_{n_x}$$

requires a tensor of couplings

$$g^{(\alpha)}_{i_{11}\cdots i_{1n_1}\cdots i_{x1}\cdots i_{xn_x}},\ \alpha = 1, 2, \cdots$$

which transforms as a (reducible?) representation of $S_{n_1}\times S_{n_2}\times\cdots S_{n_x}$

The last column tells the user how (A) the $g^{(\alpha)}$ decompose in irreps of this permutations group and (B) how many independent couplings (i.e., numbers) are associated to each such irrep (using the hook content formula)

What's on the last column?

#	Operator	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	–	{1, 1}
2	u[C] Q[R] H[R]	4	False	–	{9, 1}
3	d[C] Q[R] H[C]	4	False	–	{9, 1}
4	e[R] L[C] H[R]	4	False	–	{9, 1}
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	{{S, S}, 1, 1}

Examples

Row #2:

$$g_{ijk}^{(\alpha)} u_i^* Q_j H_k$$

No repeated fields, so the relevant permutation symmetry group is trivially $S_1 \times S_1 \times S_1$

"{9,1}" in the last column means ... $g_{ijk}^{(\alpha)} = \underbrace{1 s_{\{1\}} s_{\{1\}} s_{\{1\}}}_{9 \text{ couplings}}$

Row #5:

$$g_{ijkl}^{(\alpha)} H_i^* H_j^* H_k H_l$$

Relevant permutation symmetry group is $S_2 \times S_2$

"{{S,S},1,1}" means ... $g_{ijkl}^{(\alpha)} = \underbrace{1 s_{\{2\}} s_{\{2\}}}_{1 \text{ coupling}}$

What's on the last column?

Consider for a moment a model with **n Higgs doublets:**

#	Operator	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	–	$\{n^2, 1\}$
2	u[C] Q[R] H[R]	4	False	–	$\{9n, 1\}$
3	d[C] Q[R] H[C]	4	False	–	$\{9n, 1\}$
4	e[R] L[C] H[R]	4	False	–	$\{9n, 1\}$
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	$\{\{S, S\}, \frac{1}{4} n^2 (1+n)^2, 1\}$ | $\{\{A, A\}, \frac{1}{4} (-1+n)^2 n^2, 1\}$

Row #5:

$$g_{ijkl}^{(\alpha)} H_i^* H_j^* H_k H_l$$

Relevant permutation symmetry group is $S_2 \times S_2$

$$g_{ijkl}^{(\alpha)} = \underbrace{1_{S_{\{2\}}S_{\{2\}}}}_{\frac{1}{4}n^2(n+1)^2 \text{ couplings}} + \underbrace{1_{S_{\{1,1\}}S_{\{1,1\}}}}_{\frac{1}{4}n^2(n-1)^2 \text{ couplings}}$$

Now, let's get back to the Standard Model ...

SM non-renormalizable terms

No gauge bosons nor derivatives

SM up to dim 6

```
gaugeGroup[SM] ^= {SU3, SU2, U1};

fld1 = {"u", {3, 1, 2/3}, "R", "C", 3};
fld2 = {"d", {3, 1, -1/3}, "R", "C", 3};
fld3 = {"Q", {3, 2, 1/6}, "L", "C", 3};
fld4 = {"e", {1, 1, -1}, "R", "C", 3};
fld5 = {"L", {1, 2, -1/2}, "L", "C", 3};
fld6 = {"H", {1, 2, 1/2}, "S", "C", 1};
fields[SM] ^= {fld1, fld2, fld3, fld4,
    fld5, fld6};

GenerateListOfCouplings[SM, MaxOrder → 6];
```

#	Operator	Dim.	Self conj.?	Repeated fields	Symmetry and number of parameters
1	H[C] H[R]	2	True	-	{1, 1}
2	u[C] Q[R] H[R]	4	False	-	{9, 1}
3	d[C] Q[R] H[C]	4	False	-	{9, 1}
4	e[R] L[C] H[R]	4	False	-	{9, 1}
5	H[C] H[C] H[R] H[R]	4	True	{H[C], H[R]}	{{S, S}, 1, 1}
6	L[R] L[R] H[R] H[R]	5	False	{L[R], H[R]}	{{S, S}, 6, 1}
7	u[C] u[C] u[R] u[R]	6	True	{u[C], u[R]}	{{S, S}, 36, 1} | {{A, A}, 9, 1}
8	u[C] u[C] d[C] e[C]	6	False	u[C]	{S, 54, 1} | {A, 27, 1}
9	u[C] u[R] d[C] d[R]	6	True	-	{81, 2}
10	u[C] u[R] Q[C] Q[R]	6	True	-	{81, 2}
11	u[C] u[R] e[R] e[C]	6	True	-	{81, 1}
12	u[C] u[R] L[R] L[C]	6	True	-	{81, 1}
13	u[C] d[C] Q[C] L[C]	6	False	-	{81, 1}
14	u[C] d[C] Q[R] Q[R]	6	False	Q[R]	{S, 54, 2} | {A, 27, 2}
15	u[C] Q[C] Q[C] e[C]	6	False	Q[C]	{S, 54, 1}
16	u[C] Q[R] e[C] L[R]	6	False	-	{81, 2}
17	d[C] d[C] d[R] d[R]	6	True	{d[C], d[R]}	{{S, S}, 36, 1} | {{A, A}, 9, 1}
18	d[C] d[R] Q[C] Q[R]	6	True	-	{81, 2}
19	d[C] d[R] e[R] e[C]	6	True	-	{81, 1}
20	d[C] d[R] L[R] L[C]	6	True	-	{81, 1}
21	d[C] Q[R] e[R] L[C]	6	False	-	{81, 1}
22	Q[C] Q[C] Q[C] L[C]	6	False	Q[C]	{A, 3, 1} | {{2, 1}, 24, 1} | {S, 30, 1}
23	Q[C] Q[C] Q[R] Q[R]	6	True	{Q[C], Q[R]}	{{S, S}, 36, 2} | {{A, A}, 9, 2}
24	Q[C] Q[R] e[R] e[C]	6	True	-	{81, 1}
25	Q[C] Q[R] L[R] L[C]	6	True	-	{81, 2}
26	e[R] e[R] e[C] e[C]	6	True	{e[R], e[C]}	{{S, S}, 36, 1}
27	e[R] e[C] L[R] L[C]	6	True	-	{81, 1}
28	L[R] L[R] L[C] L[C]	6	True	{L[R], L[C]}	{{S, S}, 36, 1} | {{A, A}, 9, 1}
29	u[C] Q[R] H[C] H[R] H[R]	6	False	H[R]	{S, 9, 1}
30	d[C] Q[R] H[C] H[C] H[R]	6	False	H[C]	{S, 9, 1}
31	e[R] L[C] H[C] H[R] H[R]	6	False	H[R]	{S, 9, 1}
32	H[C] H[C] H[C] H[R] H[R] H[R]	6	True	{H[C], H[R]}	{{S, S}, 1, 1}

Agrees with the "Warsaw paper"

Grzadkowski, Iskrzyński, Misiak, Rosiek (2010)

Let's look at two operators...

For example

Dimension 6 terms: $L^* L^* L L$

28	L[R] L[R] L[C] L[C]	6	True	{L[R], L[C]}	{{S, S}, 36, 1} | {{A, A}, 9, 1}

Gauge symmetry: $2^* \times 2^* \times 2 \times 2 = (1_A + 3_S) \times (1_A + 3_S) = 1_{SS} + 1_{AA} + \cdots$

Lorentz symmetry: $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, 0\right) \times \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) = (0,0)_{AA} + \cdots$

Fermions anti-commute, so the **final answer** is: there are two contractions, **one SS and another AA**

$$\underbrace{(s_{SS} + s_{AA})}_{\text{Gauge}} \times \underbrace{s_{AA}}_{\text{Lorentz}} \times \underbrace{s_{AA}}_{\text{Grassmann}} = s_{SS} + s_{AA}$$

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For example

Dimension 6 terms: L^*L^*LL

28	L[R] L[R] L[C] L[C]	6	True	{L[R], L[C]}	{{S, S}, 36, 1}	{{A, A}, 9, 1}

Gauge symmetry: $2^* \times 2^* \times 2 \times 2 = (1_A + 3_S) \times (1_A + 3_S) = 1_{SS} + 1_{AA} + \cdots$

Lorentz symmetry: $\left(\frac{1}{2},0\right) \times \left(\frac{1}{2},0\right) \times \left(0,\frac{1}{2}\right) \times \left(0,\frac{1}{2}\right) = (0,0)_{AA} + \cdots$

Fermions anti-commute, so the **final answer** is: there are two contractions, **one SS and another AA**

$$\underbrace{(s_{SS} + s_{AA})}_{\text{Gauge}} \times \underbrace{s_{AA}}_{\text{Lorentz}} \times \underbrace{s_{AA}}_{\text{Grassmann}} = s_{SS} + s_{AA}$$

Question:
Does this mean we must have two terms in the Lagrangian?

$$g_{ijkl}^{(SS)} \left(L_i^* L_j^* L_k L_l\right)_{(SS)} + g_{ijkl}^{(AA)} \left(L_i^* L_j^* L_k L_l\right)_{(AA)}$$

36 par. 9 par.

For example

Dimension 6 terms: $L^* L^* L L$

28	L[R] L[R] L[C] L[C]	6	True	{L[R], L[C]}	{{S, S}, 36, 1}	{{A, A}, 9, 1}

Gauge symmetry: $2^* \times 2^* \times 2 \times 2 = (1_A + 3_S) \times (1_A + 3_S) = 1_{SS} + 1_{AA} + \cdots$

Lorentz symmetry: $\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, 0\right) \times \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) = (0,0)_{AA} + \cdots$

Fermions anti-commute, so the final answer is: there are two contractions, one SS and another AA

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Question:
Does this mean we must have two terms in the Lagrangian?

$$g_{ijkl}^{(SS)} \left(L_i^* L_j^* L_k L_l\right)_{(SS)} + g_{ijkl}^{(AA)} \left(L_i^* L_j^* L_k L_l\right)_{(AA)}$$

36 par. 9 par.

Answer:
No. One can make a linear combination of the SS and AA contractions

$$\mathcal{O}_{ijkl} \equiv \left(L_i^* L_j^* L_k L_l\right)_{(SS)} + \left(L_i^* L_j^* L_k L_l\right)_{(AA)}$$

$$g_{ijkl}^{(mix)} \mathcal{O}_{ijkl} \qquad g_{(ij)(kl)}^{(mix)} = g_{ijkl}^{(SS)} \qquad g_{[ij][kl]}^{(mix)} = g_{ijkl}^{(AA)}$$

45 par.

(So are there 2 or 1 "operators" (in the usual sense of the word) ? This counting is ambiguous)

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Dimension 6 terms: $QQQL$

(More interesting)

22	Q[C] Q[C] Q[C] L[C]	6	False	Q[C]	{A, 3, 1} | {{2, 1}, 24, 1} | {S, 30, 1}

Relevant permutation symmetry group is $S_3 \times S_1 = S_3$

Gauge symmetry:

SU(2) $\quad 2_Q \times 2_Q \times 2_Q \times 2_L = 1_{\{2,1\}} + \cdots$ ←

SU(3) $\quad 3_Q \times 3_Q \times 3_Q \times 1_L = 1_{\{1,1,1\}} + \cdots$

Note:
We are just interested in $S_3 \subset S_4$

Lorentz symmetry: $\left(\frac{1}{2},0\right)_Q \times \left(\frac{1}{2},0\right)_Q \times \left(0,\frac{1}{2}\right)_Q \times \left(0,\frac{1}{2}\right)_L = (0,0)_{\{2,1\}} + \cdots$

$$\underbrace{s_{\{2,1\}} \times s_{\{1,1,1\}}}_{\text{Gauge}} \times \underbrace{s_{\{2,1\}}}_{\text{Lorentz}} \times \underbrace{s_{\{1,1,1\}}}_{\text{Grassman}} = s_{\{3\}} + s_{\{2,1\}} + s_{\{1,1,1\}}$$

Generalizing the idea in the previous slide, we do not need 4 terms for QQQL.
One is enough:

$$\mathcal{O}_{ijkl} = c_{\{3\}} \left(Q_i Q_j Q_k L_l\right)_{\{3\}} + \left[c_{\{2,1\}} \left(Q_i Q_j Q_k L_l\right)_{\{2,1\}} + c'_{\{2,1\}} \left(Q_i Q_j Q_k L_l\right)'_{\{2,1\}}\right] + c_{\{1,1,1\}} \left(Q_i Q_j Q_k L_l\right)_{\{1,1,1\}}$$

$$\boxed{g^{(mix)}_{ijkl}\mathcal{O}_{ijkl}}$$

For any non-zero c coefficients
($c_{\{2,1\}}$, $c'_{\{2,1\}}$: one of the two can be zero)

Comparison with the literature

Recall that operator counting can give more than one answer

The **minimal number** of 4-fermion operators (in the usual sense of this word) **agrees with the "Warsaw paper"...**

Grzadkowski, Iskrzyński, Misiak, Rosiek (2010)

... since Jan 2017 when this paper was update on arXiv

Before that, I noticed that 2 operators were used in this reference, when 1 should be enough:

Q_{ll}	$(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t)$

✓ One operator is fine, as discussed

$Q_{qqq}^{(1)}$	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\varepsilon_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{qqq}^{(3)}$	$\varepsilon^{\alpha\beta\gamma}(\tau^I\varepsilon)_{jk}(\tau^I\varepsilon)_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$

One operator is enough, but it cannot be $Q^{(1)}$ nor $Q^{(3)}$:

$$Q_{qqq}^{(1)} = c_1 s_{\{1,1,1\}} + c_2 s_{\{2,1\}}$$
$$Q_{qqq}^{(3)} = c_3 s_{\{3\}} + c_4 s_{\{2,1\}}$$

arXiv update

Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$

$$Q_{qqq} = c_3 s_{\{3\}} + c_4 s_{\{2,1\}} + c_4 s_{\{3\}}$$ ✓

"If the assumption of baryon number conservation is relaxed, 5 new operators arise in the four-fermion sector." → *"... 4 new operators ..."* [arXiv update]

Another application

Another application

Counting invariants of a model with n scalars

Consider just the scalar potential $$V = Y^i_j \phi_i \phi^{j*} + Z^{ij}_{kl} \phi_i \phi_j \phi^{k*} \phi^{l*}$$

There is freedom to rotate the doublets $\phi \to V\phi,\ V \in U(n)$ → **... or just *SU(n)* since *V* is *U(1)* invariant**

To get rid of this freedom one can focus on combinations of *Y* and *Z* which are invariant under these basis changes

$$Y^a_a \qquad Z^{ab}_{ab} \qquad Y^a_b Y^b_f Z^{cd}_{ca} Z^{ef}_{ed}$$ **... and so on**

Jarlskog (1985) | Bernabeu, Branco, Gronau (1986) | Davidson, Haber (2005) | Gunion, Haber (2005) | Varzielas, King, Luhn, Neder (2016) | ...
(*more references can be found in there papers*)

How many invariants are there with $n_Y Y's + n_Z Z's$?

Approach #1

More details: Thomas Neder tomorrow

Count all distinct contractions of the form

$$Y^{a_1}_{\pi(a_1)} Y^{a_2}_{\pi(a_2)} \cdots Y^{a_{n_Y}}_{\pi(a_{n_Y})} Z^{b_1 c_1}_{\pi(b_1)\pi(c_1)} Z^{b_2 c_2}_{\pi(b_2)\pi(c_2)} \cdots Z^{b_{n_Z} c_{n_Z}}_{\pi(b_{n_Z})\pi(c_{n_Z})}$$

for different permutations π, and taking into account the relevant symmetries:

1 $Z^{ij}_{kl} = Z^{ji}_{kl} = Z^{ij}_{kl}$ and **2** freedom to relabel the a's, b's and c's.

The counting strategy [very briefly]

Build all matrices* with the following characteristics:

1- Dimension $(n_Y + n_Z) \times (n_Y + n_Z)$ and entries 0, 1 or 2.
2- The sum of the entries of each of the first n_Y rows/columns should add up to 1. For the remaining n_Z rows/columns, it should add up to 2.
3- If two matrices are equal up to a simultaneous permutation of rows and columns, keep just one of them.

1 matrix = 1 invariant

***Adjacency matrix which describes a graph with directed edges**

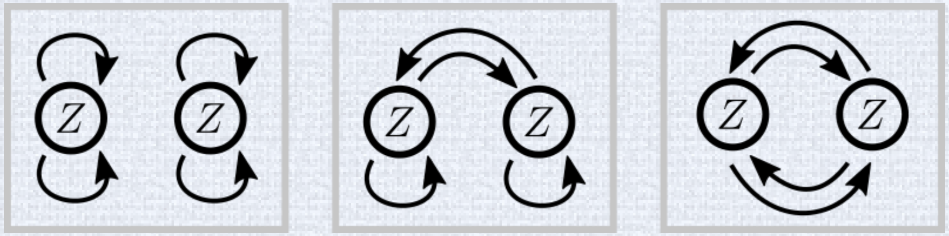
Approach #1

Number of invariants

The result:

$n_Y \backslash n_Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	521
3	3	9	47	265	1742
4	5	17	104	673	?

For example **2 Z's:**



Some of the invariants are clearly the **product of "smaller" ones**

Let's ignore this issue.

One can still ask: (A) **is the list of invariants complete?** (B) **For a given number of Y's and Z's are all invariants found linearly independent?**

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Completeness, independence of invariants

Why would the list of invariants be conceivably incomplete?

Approach #1 to the counting of invariants assumes that the only way to obtain $SU(n)$ invariants is through δ **contractions**

What about **contractions with the Levi-Civita tensor with n indices**? Are the invariants obtained in this way linearly dependent on the other?

No invariant unaccounted for (probably)

Are all the invariants obtained with δ contractions linearly independent?

All invariants seem to be independent

Let us count them in an alternative way...

Approach #2

Counting invariants with the help of $SU(n)$ theory and plethysms

Under basis transformations the tensors Y and Z transform as ...

$$\left[\begin{array}{l} Y^i_j \rightarrow F \times \overline{F} \equiv R^Y_1 + R^Y_2 \\ Z^{ij}_{kl} \rightarrow (F \times F)_S \times \left(\overline{F} \times \overline{F}\right)_S \equiv R^Z_1 + R^Z_2 + R^Z_3 \end{array}\right]$$

Dictionary

$$\begin{aligned} F &= \text{Fundamental of } SU(n) = \{1,0,\cdots,0\} \\ \overline{F} &= \text{Anti-fundamental of } SU(n) = \{0,\cdots,0,1\} \\ R^Y_1 &= \{0,\cdots,0\} \text{ (singlet)} \\ R^Y_2 &= \{1,0,\cdots,0,1\} \text{ (adjoint)} \\ R^Z_1 &= \{0,\cdots,0\} \text{ (singlet)} \\ R^Z_2 &= \{1,0,\cdots,0,1\} \text{ (adjoint)} \\ R^Z_3 &= \{2,0,\cdots,0,2\} \end{aligned}$$

So the symmetry of Z is being taken care of

 # Independent components

$Y^i_j : n^2$ ind. comp.

$Z^{ij}_{kl} : \left[n(n+1)/2\right]^2$ ind. comp.

same →

$$\begin{aligned} \#R^{Y,Z}_1 &= 1 \\ \#R^{Y,Z}_2 &= n^2 - 1 \\ \#R^Z_3 &= n^2(n-1)(3+n)/4 \end{aligned}$$

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Approach #2

Counting invariants with the help of $SU(n)$ theory and plethysms

$$\begin{bmatrix} Y^i_j \rightarrow F \times \overline{F} \equiv R^Y_1 + R^Y_2 \\ Z^{ij}_{kl} \rightarrow (F \times F)_S \times \left(\overline{F} \times \overline{F}\right)_S \equiv R^Z_1 + R^Z_2 + R^Z_3 \end{bmatrix}$$

How many $SU(n)$ singlets/invariants are there in the product

$$\left(R^Y_1 + R^Y_2\right)^{n_Y} \left(R^Z_1 + R^Z_2 + R^Z_3\right)^{n_Z}$$

?

The answer depends on n_Y, n_Z and n now

For example n=8:

$n_Y \backslash n_Z$	0	1	2	3	4
0	0	1	3	21	282
1	1	2	11	120	2202
2	2	7	58	861	20100
3	6	33	378	7227	207324
4	24	192	2892	68868	2372256

But these are not the correct numbers since we have only one R^Y_1, one R^Y_2, one R^Z_1, one R^Z_2 and one R^Z_3.

Approach #2

Counting invariants with the help of $SU(n)$ theory and plethysms

$$\left(R_1^Y + R_2^Y\right)^{n_Y}\left(R_1^Z + R_2^Z + R_3^Z\right)^{n_Z}\Big|_{Invs} =$$

$$\left[\sum_{i=0}^{n_Y}\frac{n_Y!}{i!\,(n_Y-i)!}\left(R_1^Y\right)^i\left(R_2^Y\right)^{n_Y-i}\right]\left[\sum_{i+j+k=n_Z}\frac{n_Z!}{i!j!k!}\left(R_1^Z\right)^i\left(R_2^Z\right)^j\left(R_3^Z\right)^k\right]\Bigg|_{Invs}$$

Correct approach: 1- keep only completely symmetric contractions
2- remove repeated invariants

(One can get rid of the singlets as well)

$$\left[\sum_{i=0}^{n_Y}\left(R_2^Y\right)^{n_Y-i}_{Sym}\right]\left[\sum_{i+j+k=n_Z}\left(R_2^Z\right)^j_{Sym}\left(R_3^Z\right)^k_{Sym}\right]\Bigg|_{Invs}$$

✓

This will give the number of linearly independent invariants for a fixed n_Y and n_Z

Invariant counting: results

n=2

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	5	9
1	1	2	5	11	20
2	2	4	12	24	48
3	2	5	15	36	72
4	3	7	23	53	113

n=3

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	20
1	1	2	7	23	82
2	2	5	20	74	284
3	3	8	37	164	682
4	4	13	67	314	1414

n=4

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	110
2	2	5	22	93	476
3	3	9	45	243	1438
4	5	16	94	563	3744

n=5

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	516
3	3	9	47	262	1697
4	5	17	102	651	4886

n=6

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	521
3	3	9	47	265	1737
4	5	17	104	670	5162

n=7

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	521
3	3	9	47	265	1742
4	5	17	104	673	5202

n=8

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	521
3	3	9	47	265	1742
4	5	17	104	673	5207

n=9 does not change

Approach #1

$n_Y \backslash Z$	0	1	2	3	4
0	0	1	3	8	25
1	1	2	7	26	115
2	2	5	22	96	521
3	3	9	47	265	1742
4	5	17	104	673	?

Summary

Summary

1

The permutation symmetries of field contractions (plethysms) are very important.

2

It is not just a matter of $+$ or $-$ signs.

3

It is worth investing time to understand the complications associated to this unavoidable discrete symmetry.

4

It allows us to tackle various problems in a systematic way

The `Susyno` and `Sym2Int` packages might be useful in this regard

Thank you

Extra slides

Symmetric tensors:

Counting the number of independent entries in the general case

Recall:

Field contractions induce a permutation symmetry in coupling tensors $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$.

The flavor indices

$$g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)} = \left[U_\lambda\left(\pi\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n} \qquad i_a = 1, \cdots, N$$

where U_λ are the matrices of the irrep λ of S_n

$n = |\lambda|$ (means that λ is a partition of n)

There is a total of $\dim\left(s_\lambda\right) N^n$ entries in the tensor.

How many are independent?*

The cases $\lambda = \{n\}$ and $\lambda = \{1, 1, \cdots, 1\}$ where s_λ is 1-dimensional and U_λ is just a + or – sign are easy ...

$$\lambda = \{n\} : \frac{(N+n-1)!}{(N-1)!n!} \qquad \lambda = \{1, \cdots, 1\} : \frac{N!}{(N-n)!n!}$$

*Disclaimer: with all likelihood, this answer is well known to mathematicians. Since I could not find it anywhere, I'll try to derive it.

Symmetric tensors:

Counting the number of independent entries in the general case

Consider the sorted indices

$$i_1 i_2 \cdots i_n = \underbrace{aa\cdots a}_{m_1}\underbrace{bb\cdots b}_{m_2}\cdots \qquad a < b < c < \cdots \qquad \sum m_i = n$$

All entries $g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)}$ depend just on the $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with $\alpha = 1, \cdots, dim\,(s_\lambda)$

So we can consider just sorted indices. $\left(\, i_1 i_2 \cdots i_n = babc \cdots \right)$

Still, if there are repeated indices ($m_i \neq 1$) then not all $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with the indices sorted are independent because

$$\widetilde{\pi}\,(i_1 i_2 \cdots i_n) = (i_1 i_2 \cdots i_n) \qquad \widetilde{\pi} \in S_{m_1} \times S_{m_2} \times \cdots \subset S_n$$

so ...

$$g^{(\alpha)}_{i_1 i_2 \cdots i_n} = \left[U_\lambda\left(\widetilde{\pi}\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n}$$

How many constraints are there in these equations? **How many of the $dim\,(s_\lambda)$ entries of $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ (for fixed i_X) remain independent after we inforce these constraints?**

Symmetric tensors:

Counting the number of independent entries in the general case

Consider the sorted indices

$$i_1 i_2 \cdots i_n = \underbrace{aa\cdots a}_{m_1}\underbrace{bb\cdots b}_{m_2}\cdots \qquad a<b<c<\cdots \qquad \sum m_i = n$$

All entries $g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)}$ depend just on the $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with $\alpha = 1, \cdots, dim\,(s_\lambda)$

So we can consider just sorted indices. $\left(i_1 i_2 \cdots i_n = \text{~~bac~~}\cdots \right)$

Still, if there are repeated indices ($m_i \neq 1$) then not all $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ with the indices sorted are independent because

$$\widetilde{\pi}\left(i_1 i_2 \cdots i_n\right) = \left(i_1 i_2 \cdots i_n\right) \qquad \widetilde{\pi} \in S_{m_1} \times S_{m_2} \times \cdots \subset S_n$$

so ...

$$g^{(\alpha)}_{i_1 i_2 \cdots i_n} = \left[U_\lambda\left(\widetilde{\pi}\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n}$$

How many constraints are there in these equations? **How many of the $dim\,(s_\lambda)$ entries of $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ (for fixed i_X) remain independent after we inforce these constraints?**

Symmetric tensors:

Counting the number of independent entries in the general case

$$g^{(\alpha)}_{i_1 i_2 \cdots i_n} = \left[U_\lambda\left(\widetilde{\pi}\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n} \qquad \widetilde{\pi} \in S_{m_1} \times S_{m_2} \times \cdots \subset S_n$$

This question has a very nice and obvious interpretation (just by looking at the equation): if we restrict the full permutation group S_n to $S_{m_1} \times S_{m_2} \times \cdots \subset S_n$, how many trivial representations of the subgroup are there in the original representation λ?

That is the answer we seek

We can get an answer with the (famous) Littlewood-Richardson coefficients:

$$s_{\lambda_1} \times s_{\lambda_2} = \sum_\lambda c^{\lambda}_{\lambda_1 \lambda_2} s_\lambda \qquad |\lambda_1| + |\lambda_2| = |\lambda|$$

This expression has many important interpretations. The one we seek is the following: The representation $s_{\lambda_1} \times s_{\lambda_2}$ of $S_{|\lambda_1|} \times S_{|\lambda_2|}$ is contained $c^{\lambda}_{\lambda_1 \lambda_2}$ times in the representation $s_{|\lambda|}$ of $S_{|\lambda|}$.

Symmetric tensors:

Counting the number of independent entries in the general case

Example: $s_{\{2,1\}} \times s_{\{2\}} = 1 s_{\{4,1\}} + 1 s_{\{3,2\}} + 1 s_{\{3,1,1\}} + 1 s_{\{2,2,1\}}$

This means that $s_{\{2,1\}} \times s_{\{2\}}$ is contained once in each of the four representations of S_5 on the right.

Since we have many pairs of repeated indices, we may have to do such a product many times.

Recall: $i_1 i_2 \cdots i_n = \underbrace{aa\cdots a}_{m_1}\underbrace{bb\cdots b}_{m_2}\cdots \qquad \sum m_i = n$

In particular, we are interested in the product of trivial/fully-symmetric representations:

$$s_{\{m_1\}} \times s_{\{m_2\}} \times \cdots = \cdots + X s_\lambda + \cdots$$

We want this X. That's our final answer.

Symmetric tensors:

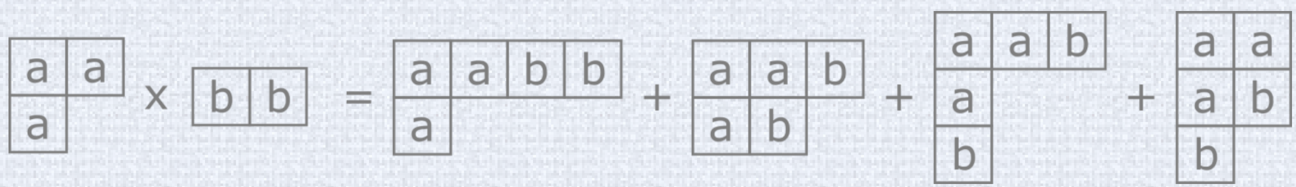
Counting the number of independent entries in the general case

Now, the Littlewood-Richardson rule gives us the Littlewood-Richardson coefficients, so by repeatedly application of this rule we can get X.

It turns out that for “products” of a representation s_{λ_1} with the trivial one $s_{\{m\}}$, the rule is quite simple (Pieri’s formula):

[Write the Young diagram of λ_1 and add m boxes in all possible ways, no two in the same column.]

Example: $s_{\{2,1\}} \times s_{\{2\}} = 1s_{\{4,1\}} + 1s_{\{3,2\}} + 1s_{\{3,1,1\}} + 1s_{\{2,2,1\}}$



{2,1} {2} {4,1} {3,2} {3,1,1} {2,2,1}

We can do this recursively (adding letters c,d,e,f,...). But then, we notice that the number of different ways of arriving at a given shape λ (i.e, the answer we seek) is the same as the number of possible Young diagrams where the letters are non-decreasing along each row, and increasing along each columns

!

Symmetric tensors:

Counting the number of independent entries in the general case

In other words, the number we seek is the number of semi-standard Young tableaux (SSYT) that can be made with the indices $i_1 i_2 \cdots i_n = \underbrace{aa\cdots a}_{m_1}\underbrace{bb\cdots b}_{m_2}\cdots$

The number of independent entries of the tensor $g^{(\alpha)}_{i_1 i_2 \cdots i_n}$ such that

$$g^{(\alpha)}_{\pi(i_1 i_2 \cdots i_n)} = \left[U_\lambda\left(\pi\right)\right]_{\alpha\beta} g^{(\beta)}_{i_1 i_2 \cdots i_n} \qquad i_X = 1, \cdots, N$$

where U_λ are the matrices of the irrep λ of S_n

is the same as the number SSYT(N, λ) of semi-standard Young tableaux of shape λ filled with the numbers 1,2,...,N

In turn, the number SSYT(N, λ) is given by the **Hook content formula**

$$SSYT\left(N,\lambda\right) = \prod_{x_{ij}\in\lambda} \frac{N+i-j}{|h_{ij}|}$$

$|h_{ij}|$ is the well known hook length of the (ij) entry of the diagram

$|h_{ij}|$

7	3	1
4	1	
2		
1		

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