Algebras and Colour-Kinematics Duality

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Amplitudes 2012, 8 March 2012

Colour-kinematics, BCJ & all that

- Bern, Carrasco & Johansson found a new structure in Yang-Mills

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

  - Colour factors made from f^{abc}
  - Sum over cubic diagrams only

$$c_i \pm c_j \pm c_k = 0 \Rightarrow n_i \pm n_j \pm n_k = 0$$

- Kinematics numerators mirror colour factors ("duality")

Two consequences

1. BCJ relations for amplitudes

$$s_{35}A_5(1,2,4,3,5) - (s_{13}+s_{23})A_5(1,2,3,4,5) - s_{13}A_5(1,3,2,4,5) = 0$$

- Basis of independent colour-ordered *n*-point amplitudes *(n - 3)!*

2. New expression for gravity amplitude (the double-copy formula)

$$\mathcal{M} = \sum_{i \in \text{cubic}} \frac{n_i \tilde{n}_i}{D_i}$$

Generalised gauge invariance

- Numerators are not unique

$$n_i \to n_i' = n_i + \Delta_i$$

- Amplitude invariant provided

$$\sum_{i \in \text{cubic}} \frac{\Delta_i c_i}{D_i} = 0$$

- Large space of numerators

Recent progress

- BCJ relations proven by Bjerrum-Bohr, Damgaard & Vanhove, and by Stieberger (string methods)
  - Later BCFW-based proof by Feng, Huang & Jia
- Double-copy formula proven by Bern, Dennen, Huang & Kiermaier
  - Existence of appropriate numerators was assumed
- Specific set of on-shell numerators numerators found by Mafra, Schlotterer & Stieberger

Questions

- Are the numerators associated with some kind of Lie algebra?
- What is the physical origin of these Jacobi relations?
- Can we better understand the redundancy of the numerators?
- Do the numerators exist off shell? What about loops?

The Self-dual Sector

Yang-Mills

- Light cone coordinates $u = t - z, v = t + z, w = x + iy$
- Yang-Mills : choose gauge $A_u = 0$, find $A_w = 0$
- Remaining components of A expressed in terms of a scalar Φ
- Equation of motion

$$\partial^2 \Phi + ig[\partial_w \Phi, \partial_u \Phi] = 0$$

Leznov & Mukhtarov
Parkes
Bardeen
Chalmers & Siegel
Cangemi

Gravity

- Similar story
  - Express metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$
  - Gauge choices $h_{ui} = 0, i \in (u, v, w, \bar{w})$
- Non-vanishing components of h expressed in terms of ϕ
- Exact equation of motion

$$\partial^2 \phi + \kappa \{\partial_w \phi, \partial_u \phi\} = 0$$

Plebanski

- Poisson algebra $\{f, g\} = \partial_w f \, \partial_u g - \partial_u f \, \partial_w g$

Fourier transform

- Self-dual gauge theory equation of motion becomes

$$k^2\Phi^a(k) = \frac{1}{2}g\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}F_{p_1p_2}{}^k f^{b_1b_2a}\Phi^{b_1}(p_1)\Phi^{b_2}(p_2)$$

- with

$$F_{p_1p_2}{}^k = (2\pi)^4\delta^4(p_1+p_2-k)X(p_1,p_2)$$

$$X(p_1,p_2) = p_{1w}p_{2u} - p_{1u}p_{2w}$$

- Raise and lower indices $\delta_{pq} = (2\pi)^4\delta^4(p+q)$

Structure constants

- F has properties of structure constants...

$$F_{p_1 p_2 p_3} = (2\pi)^4 \delta^4(p_1 + p_2 + p_3) X(p_1, p_2)$$

$$F_{p_2 p_1 p_3} = -F_{p_1 p_2 p_3}$$

$$F_{p_2 p_3 p_1} = F_{p_1 p_2 p_3}$$

- ... including Jacobi relation

$$F_{p_1 p_2}{}^q F_{q p_3 p_4} + F_{p_2 p_3}{}^q F_{q p_1 p_4} + F_{p_3 p_1}{}^q F_{q p_2 p_4} = 0$$

- Lie algebra of area-preserving diffeomorphisms in u, w plane

Gravity and gauge theory

- Gravity equation of motion is

$$k^2\phi(k) = \frac{1}{2}\kappa \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} F_{p_1p_2}{}^k X(p_1,p_2)\phi(p_1)\phi(p_2)$$

- Recall gauge theory case

$$k^2\Phi^a(k) = \frac{1}{2}g \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} F_{p_1p_2}{}^k f^{b_1b_2a}\Phi^{b_1}(p_1)\Phi^{b_2}(p_2)$$

- Colour-kinematics duality manifest

$$f^{b_1b_2a} \to F^{p_1p_2k} = X(p_1,p_2)\delta(p_1+p_2+k)$$

  - ... but omit the overall delta function

Diffeomorphisms

- Self-dual gravity field equation

$$\partial^2\phi + \kappa\{\partial_w\phi, \partial_u\phi\} = 0$$

- Poisson algebra

$$\begin{aligned}\{e^{-ip_1\cdot x}, e^{-ip_2\cdot x}\} &= -X(p_1,p_2)\, e^{-i(p_1+p_2)\cdot x} \\ &= -F_{p_1p_2}{}^k e^{-ik\cdot x}\end{aligned}$$

- Infinitesimal diffeomorphisms in u, w plane

$$w \to w'(w,u), \quad u \to u'(w,u)$$

- Area preserving: unit Jacobian to leave bracket invariant

Kinematic algebra

- Three point function from gravity equation

$$\partial^2\phi + \kappa\{\partial_w\phi, \partial_u\phi\} = 0 \quad \Rightarrow \mathcal{M}_3 = X(p_1, p_2)^2\delta(p_1 + p_2 + p_3)$$

- "Reverse KLT"

$$\mathcal{A}_3 = X(p_1, p_2)\delta(p_1 + p_2 + p_3)f^{abc} = F_{p_1p_2p_3}f^{abc}$$

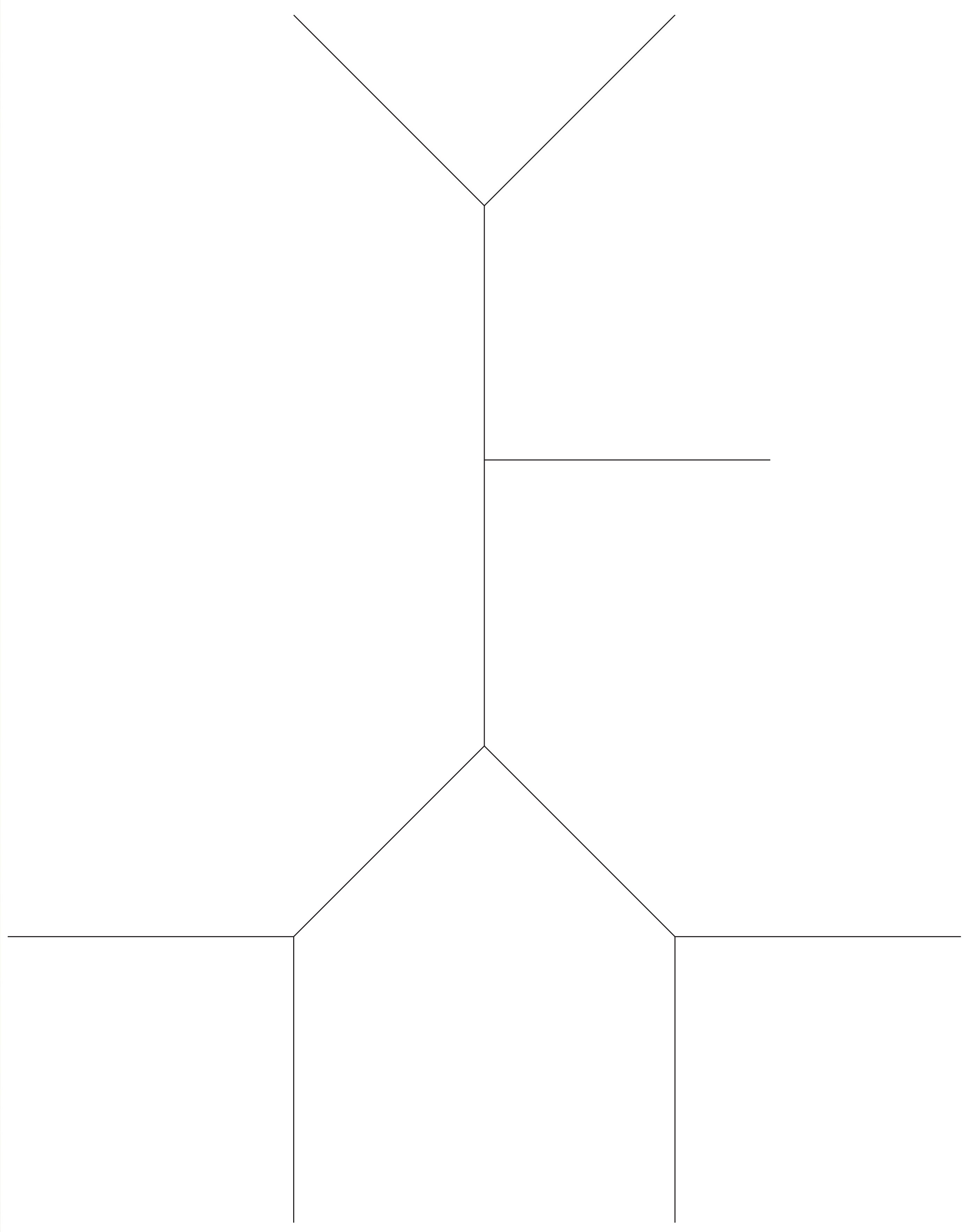
  - So same algebra appears in YM

- Physics of colour-kinematics is gravitational

MHV amplitudes

Feynman diagrams

- Diagrams easy in self-dual
  - Always cubic
  - Vertex just $f\,F$



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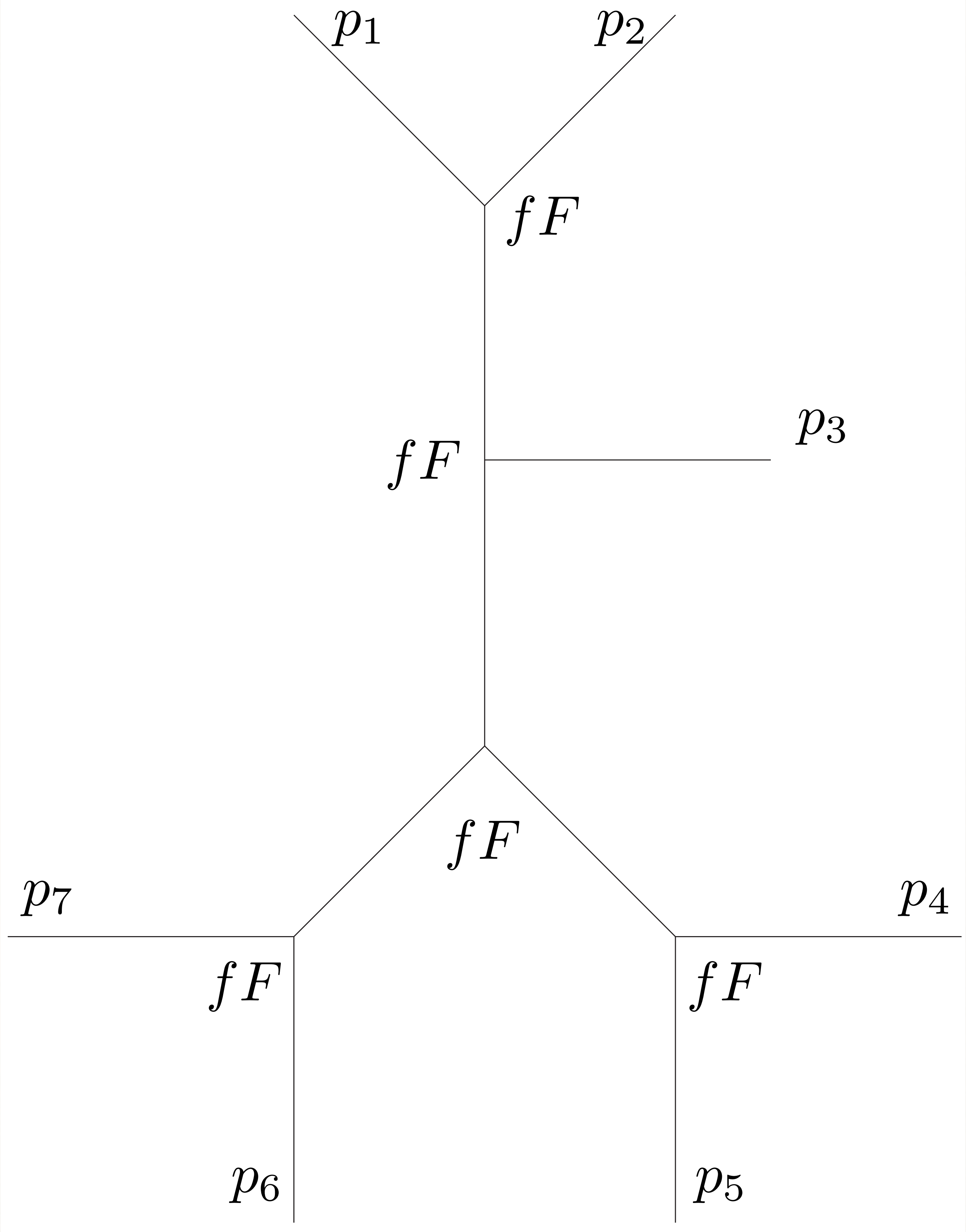

Feynman diagrams

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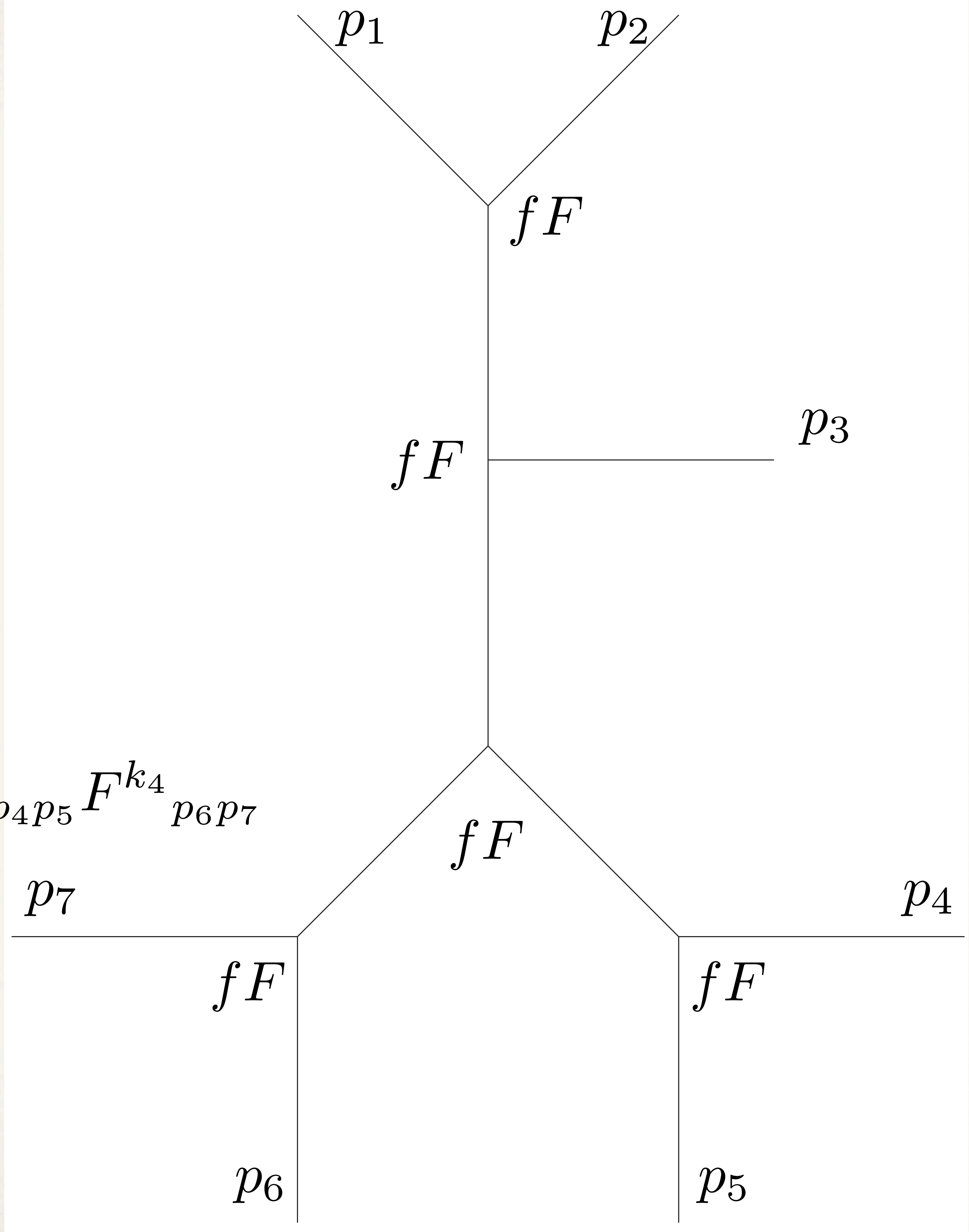
$$n = F_{p_1 p_2}{}^{k_1} F_{k_1 p_3}{}^{k_2} F_{k_2 k_3 k_4} F^{k_3}{}_{p_4 p_5} F^{k_4}{}_{p_6 p_7}$$



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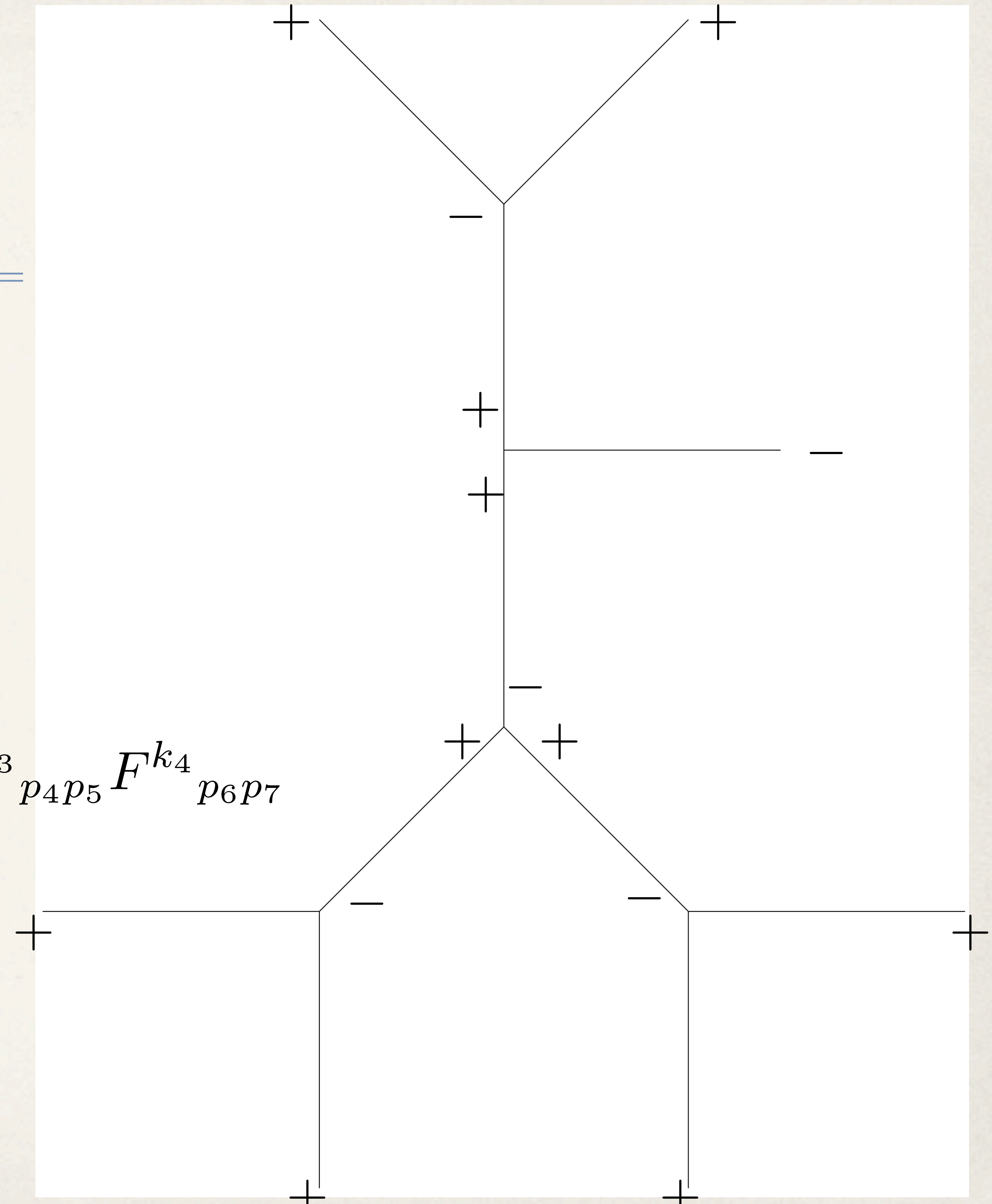
- But all amplitudes vanish!
- Need one more minus



Light-cone Yang-Mills

- Lagrangian is

Self-dual sector

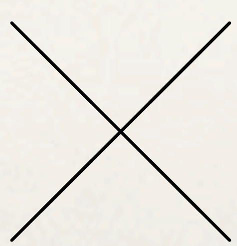
$$\mathcal{L} = \mathrm{tr}\Big\{ \frac{1}{4}\bar{A}\Box A - ig\Big(\frac{\partial_w}{\partial_u}A\Big)[A, \partial_u \bar{A}]$$
$$- ig\Big(\frac{\partial_{\bar{w}}}{\partial_u}\bar{A}\Big)[\bar{A}, \partial_u A] - g^2[A, \partial_u \bar{A}]\frac{1}{\partial_u^2}[\bar{A}, \partial_u A] \Big\}$$

- Now another cubic vertex and a (bad) quartic vertex
  - Cubic vertex: makes another algebra

$$\bar{F}_{p_1 p_2 p_3} = (2\pi)^4 \delta^4(p_1 + p_2 + p_3)\bar{X}(p_1, p_2),$$
$$\bar{X}(p_1, p_2) = p_{1\bar{w}} p_{2u} - p_{1u} p_{2\bar{w}}$$

Remaining freedom

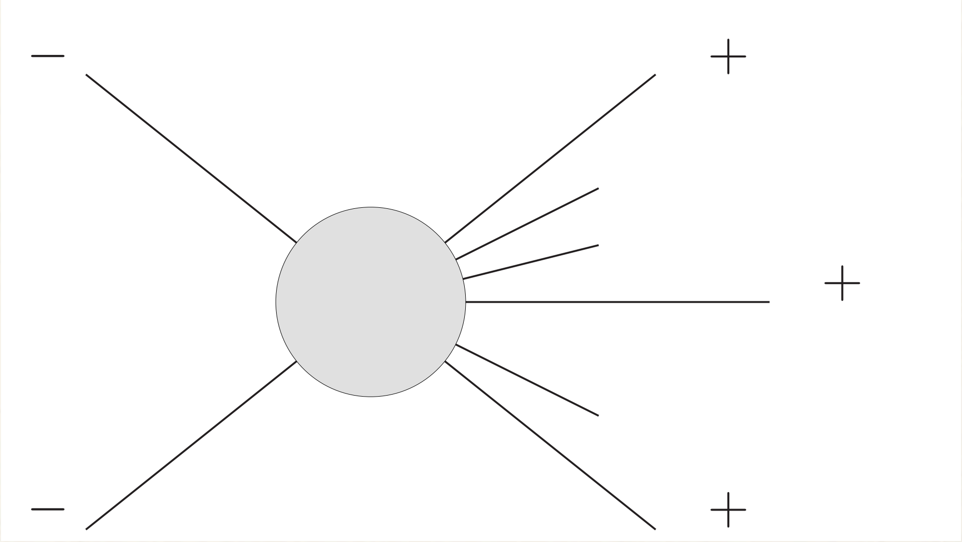
- May coordinate our gauge choice with momentum of a single particle
  - Choice of reference leg — Chalmers & Siegel
- Pick a frame in which reference particle has $p_u = 0$
  - Since $\epsilon^- \propto p_u$, negative helicity can only attach to vertices with a pole in p_u
  - - - + vertex has a pole



$$= g^2 f^{abe} f^{cde} \frac{p_{1u}p_{2u} - p_{3u}p_{4u}}{(p_{2u}+p_{4u})^2} \to 0$$

MHV topologies

- MHV: two negative helicites
  - Can constrain one to be on - - + three point vertex
- Count helcities appearing in diagram



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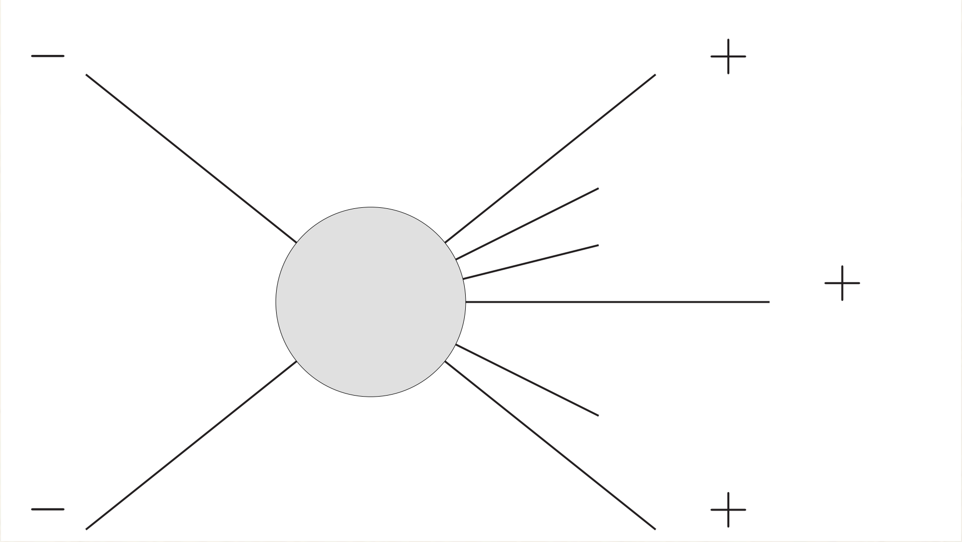
$$
\begin{aligned}
2 + I &= n_+ + 2n_- + 2n_4 \\
n - 2 + I &= 2n_+ + n_- + 2n_4 \\
n - 2 &= n_+ + n_- + 2n_4
\end{aligned}
$$



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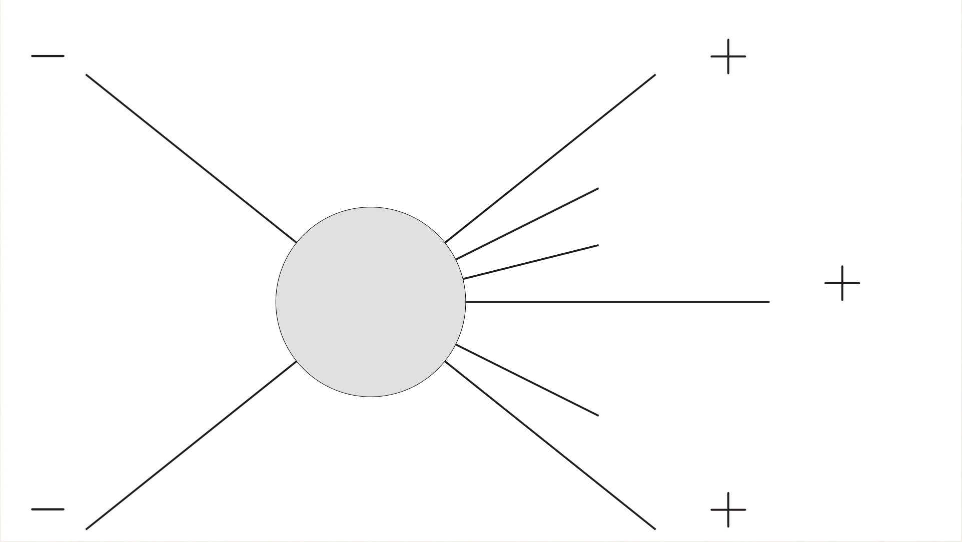
$$2 + I = n_+ + 2n_- + 2n_4$$
$$n - 2 + I = 2n_+ + n_- + 2n_4$$
$$n - 2 = n_+ + n_- + 2n_4$$

$$I = n_+$$
$$1 = n_- + n_4$$



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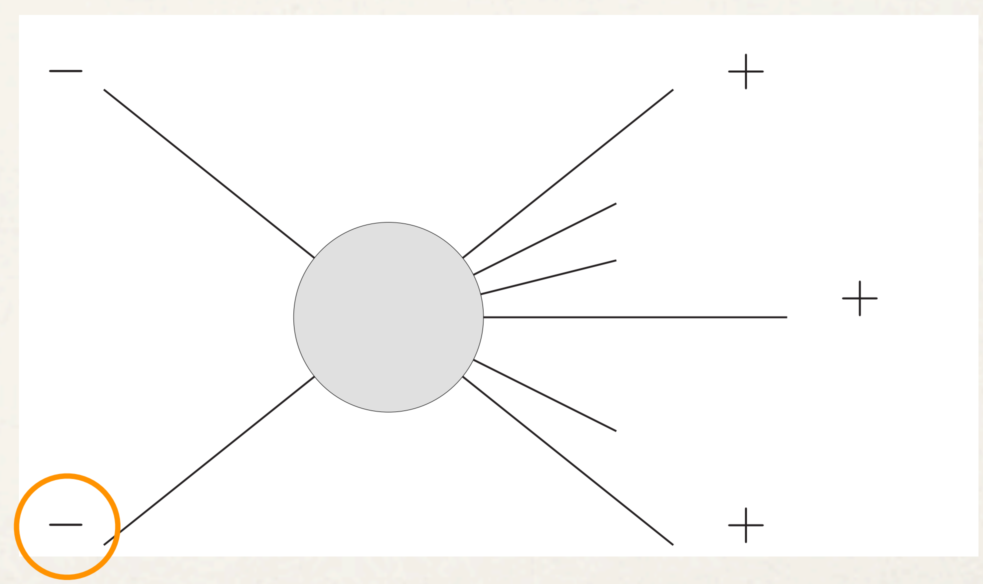


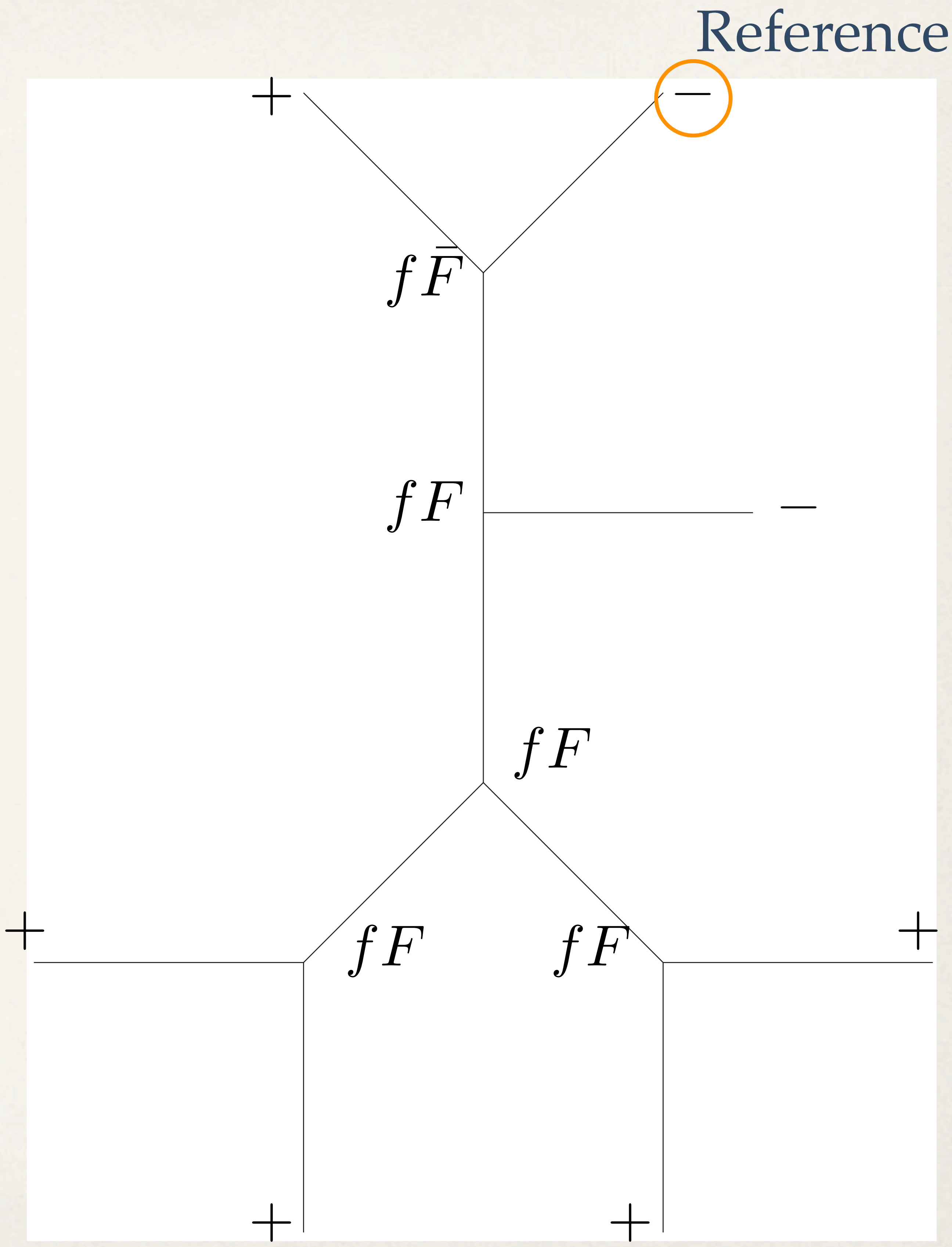
Reference $\Rightarrow n_- = 1, \quad n_4 = 0$

MHV Jacobi

- Only cubic vertices
- One $f\bar{F}$ vertex attached to reference
    - But $f\bar{F} \propto fF$ since $p_u = 0$!

$$\bar{X}(p_1, p_2) = p_{1\bar{w}}p_{2u} - p_{1u}p_{2\bar{w}},$$
$$X(p_1, p_2) = p_{1w}p_{2u} - p_{1u}p_{2w}$$

- So Jacobi relations assured



A basis for BCJ

A linear space

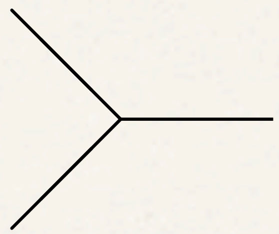
- YM amplitudes satisfy KK, BCJ relations
- These are linear equations
  - There is a linear operator acting on *(n - 1)!* space of amplitudes

$$\sum_{\beta} \tilde{S}[\alpha_{2,n}|\beta_{2,n}]A(1,\beta_{2,n}) = 0$$

- Kernel of $\tilde{S}$ has dimensions *(n - 3)!*

Scalar theories

- YM is not only theory with KK, BCJ relations
- Consider cubic scalar theory with two different structure constants at each vertex



$$= g f^{abc} f'^{a'b'c'}$$

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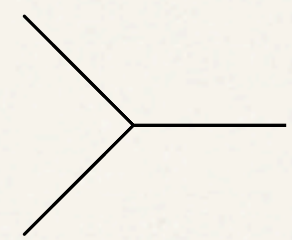

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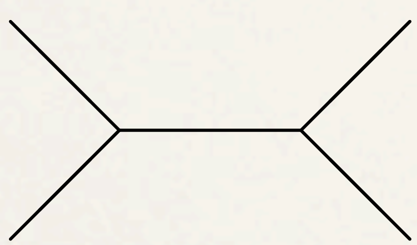

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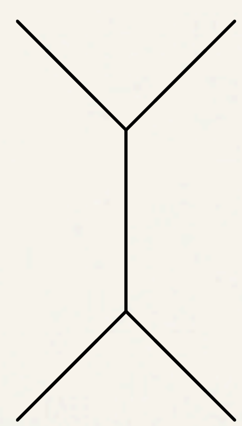

$$= \quad A(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}$$

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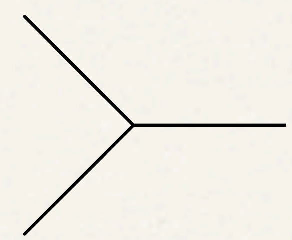
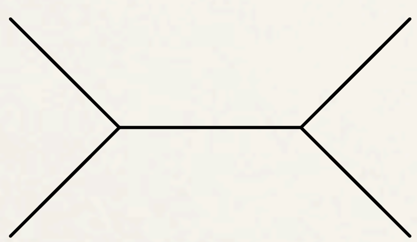


$= g f^{abc} f'^{a'b'c'}$



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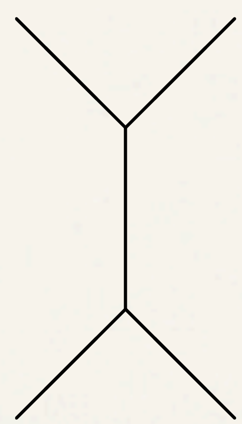

$= \quad A(1,2,3,4) = \frac{c_s}{s} - \frac{c_t}{t}$

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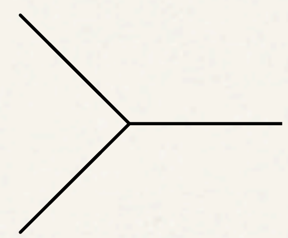
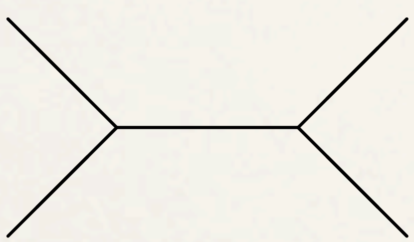


$$= \ g f^{abc} f'^{a'b'c'}$$



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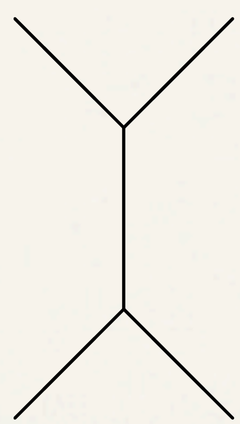

$$= \qquad A(1,2,3,4) = \frac{c_s}{s} - \frac{c_t}{t}$$

  - KK true
  - BCJ true (explicit proof by Du, Feng and Fu)

A basis

- YM, scalar theories all in kernel of $\tilde{S}$
- Pick *(n-3)!* independent cubic scalar theories; YM a linear combination

*J*th colour-ordered amplitude
of *I*th scalar theory

$$A_{(J)} = \sum_{I=1}^{(n-3)!} \alpha_I \, \theta_{IJ}$$

- When θ is non-singular, unique expansion coefficients α exist

$$\mathcal{A} = \sum_{I=1}^{(n-3)!} \alpha_I \, \Theta_I$$

Colour dressed, so
permutation symmetric

Manifest colour-kinematics duality

- Scalar basis has cubic numerators with manifest Jacobi

$$\Theta_I = \sum_k \frac{\tilde{c}_{I,k}\, c_k}{D_k}$$

- So YM numerators represented as

$$n_k^{YM} = \sum_{I=1}^{(n-3)!} \alpha_I\, \tilde{c}_{I,k}$$

- Jacobi relations follow by linearity

- Basis change: generalized gauge invariance

Colour-dual amplitudes

- To get colour-ordered amplitudes, expand colour factors as traces
- Colour-dual - expand numerators — Bern and Dennen
  - Easy with basis, numerators made from auxiliary colour factors

$$\mathcal{A} = g^{n-2} \sum_{\sigma} \tau_{(1,\sigma_2,\ldots,\sigma_n)} \, A^{\text{dual}}(1, \sigma_2, \ldots, \sigma_n)$$

Made with scalar theory with YM gauge group

- In terms of basis $\quad \tau_{(1,2,\ldots,n)} = \sum_{I=1}^{(n-3)!} \alpha_I \, \mathrm{Tr}[T^{\hat{a}_1^I} T^{\hat{a}_2^I} \ldots T^{\hat{a}_n^I}]$

Gravity amplitudes

- Double copy and colour dual form lead to

$$\mathcal{M} = i\left(\frac{\kappa}{2}\right)^{n-2} \sum_{\sigma} \tau_{(1,\sigma_2,\ldots,\sigma_n)}\, A_{YM}(1,\sigma_2,\ldots,\sigma_n)$$ Bern, Dennen

- Inserting our expression for τ, $\tau_{(1,2,\ldots,n)} = \sum_{I=1}^{(n-3)!} \alpha_I \, \mathrm{Tr}[T^{\hat{a}_1^I} T^{\hat{a}_2^I} \cdots T^{\hat{a}_n^I}]$
    - Recognise colour-dressed YM amplitude

$$\mathcal{M} = i\left(\frac{\kappa}{2}\right)^{n-2} \sum_{I} \alpha_I \mathcal{A}_{YM}(1, a_1^I; 2, a_2^I; \ldots ; n, a_n^I)$$

Conclusions

What did we learn?

- Self-dual sector: manifest colour-kinematics
  - Kinematic algebra is an area-preserving diffeomorphism algebra
- MHV sector: inherits kinematic algebra from self-dual
- Beyond MHV: described manifestly dual bases
  - Different choice of basis: generalised gauge invariance
  - Gravity amplitudes: linear combinations of colour-dressed YM
- The physics of colour-kinematics duality is gravitational