The Yangian and the S-Matrix

Bullimore, DS: 1112.1056
Caron-Huot, He: 1112.1060

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[Drummond,Henn,Korchemsky,Sokatchev;
Beisert,Ricci,Tseytlin,Wolf; Berkovits,Maldacena; Drummond,Henn,Plefka]

Tree level amplitudes are highly constrained by the requirement that they be Yangian

Original Superconformal Symmetry	+	Dual Superconformal Symmetry	=	Infinite Dimensional Yangian

$$j_a^{(0)} = \sum_{i=1}^{n} j_{ia}^{(0)} \qquad j_a^{(1)} = f_a{}^{bc} \sum_{i<k} j_{ib}^{(0)} j_{kc}^{(0)}$$

- Reflection of integrability of planar $\mathcal{N} = 4$ SYM
- Yangian invariant if both superconformally invariant and dual superconformally invariant

Much of this symmetry is broken at loop level. Broken symmetries are still powerful, so long as we understand the structure of the breaking

[Drummond,Henn,Korchemsky,Sokatchev]

$$\mathrm{W}_n = \mathrm{Z}_n \mathrm{F}_n$$

e.g.

$$[K^\mu, \mathrm{F}_n] = \frac{\Gamma_{\mathrm{cusp}}}{2} \sum_{i=1}^{n} x^\mu_{i,i+1} \log \frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}$$

- Fixes BDS ansatz, and remainder function must be dual conformally invariant

Can we constrain all anomalies in $\mathcal{Y}\,[\mathrm{PSU}(2,2|4)]$?

- Non-chiral superloop [Caron-Huot; Beisert, Vergu]
- Collinear limits [Bargheer, Beisert, Galleas, Loebbert, McLoughlin; Beisert, Henn, McLoughlin, Plefka; Sever, Vieira]

Which dual supercharges are broken?

Amplitudes in planar $\mathcal{N}=4$ can be written as

$$\mathcal{M} = \sum \begin{pmatrix} \text{algebraic} \\ \text{Yangian invariant} \end{pmatrix} \times \begin{pmatrix} \text{bosonic} \\ \text{loop integral} \end{pmatrix}$$

Annihilated by $Q, \bar{S} \sim Z\partial/\partial\chi$, but not by $\bar{Q}, S \sim \chi\partial/\partial Z$

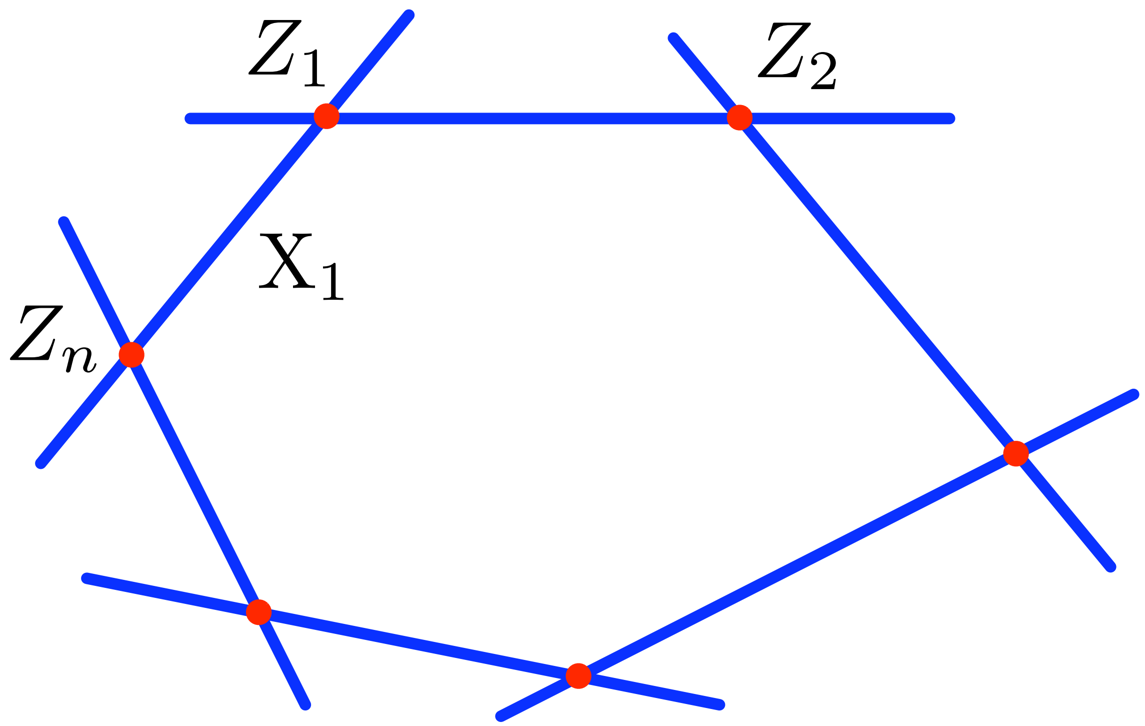
- These (dual) supersymmetries also fail for finite quantities such as the remainder or ratio functions
- Parity conjugate statement in dual momentum twistor space

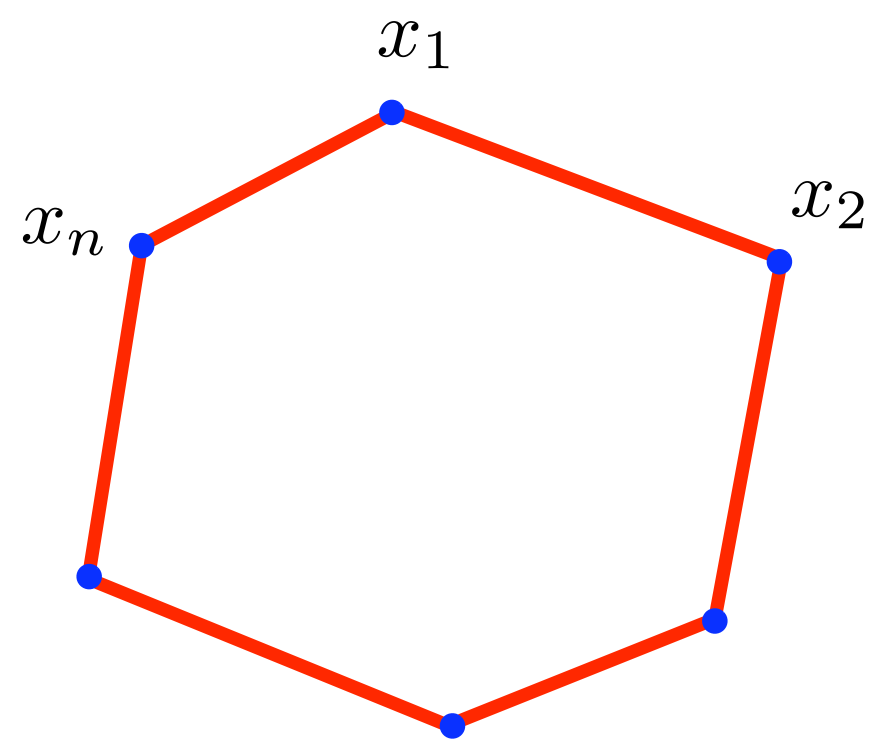
Suggests the failure is more to do with the chiral representation rather than any genuine anomaly

[Mason,DS]

Twistor lines $\longleftrightarrow$ Space-time vertices

Twistor vertices $\longleftrightarrow$ Null edges of polygon





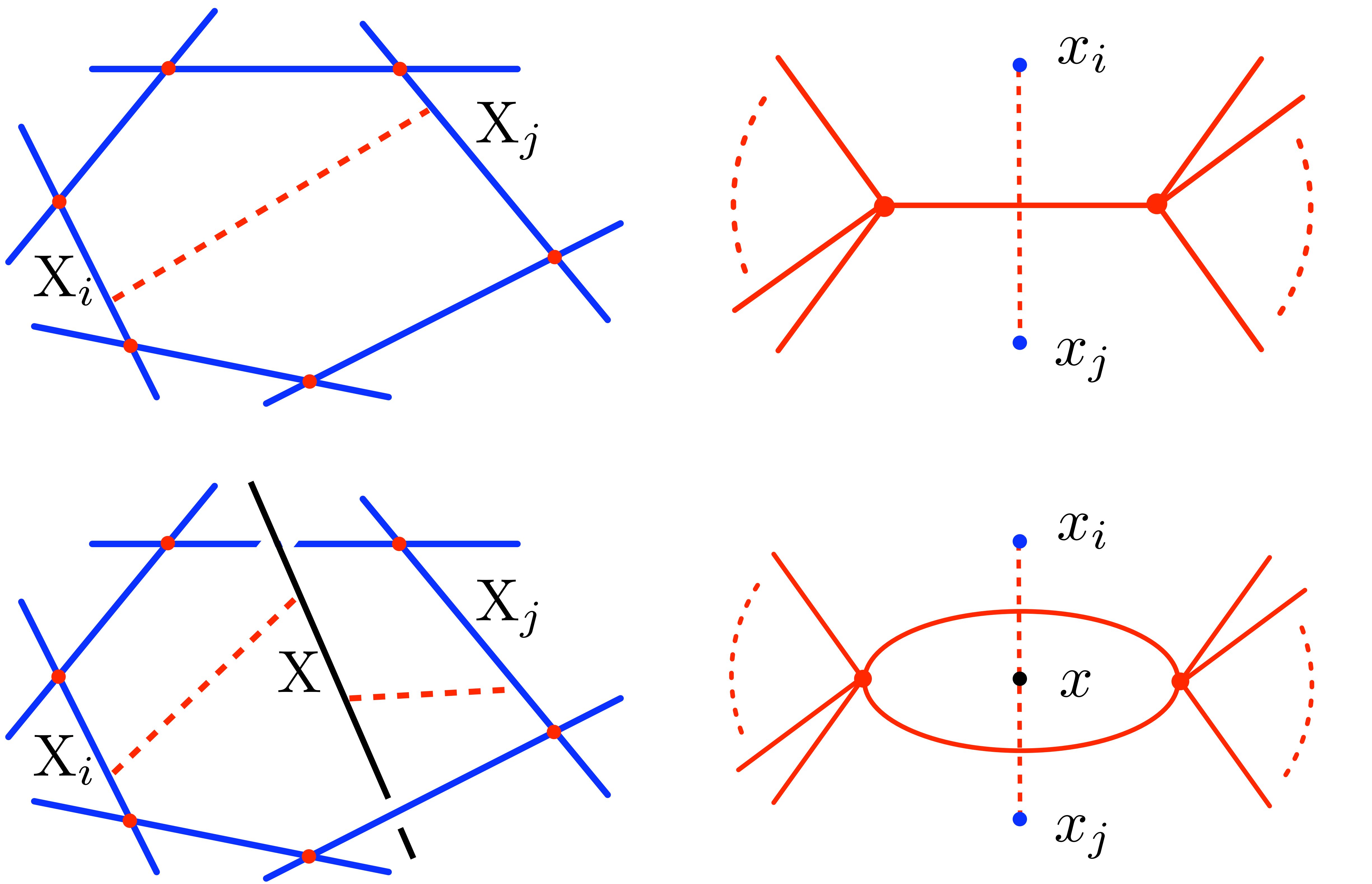
The duality extends to all helicities if one constructs a supersymmetric extension of the Wilson Loop

$$(\bar{\partial} + \mathcal{A})\big|_{\mathrm{X}}\, \mathrm{U}(\sigma_1, \sigma_2) = 0 \qquad \mathrm{U}(\sigma_1, \sigma_1) = \mathrm{id}$$

$\mathcal{N} = 4$ twistor superfield

Superloop in sd $\mathcal{N} = 4$ $\longleftrightarrow$ Tree superamplitude

Superloop in full $\mathcal{N} = 4$ $\longleftrightarrow$ Planar superamplitude



The full and self-dual theories both enjoy $\mathcal{N}=4$ superconformal symmetry, but this symmetry is represented differently on the space of fields

- The differences occur precisely for $\bar{Q}$ and S
- The geometric action on (momentum) twistor space corresponds to the self-dual transformations,

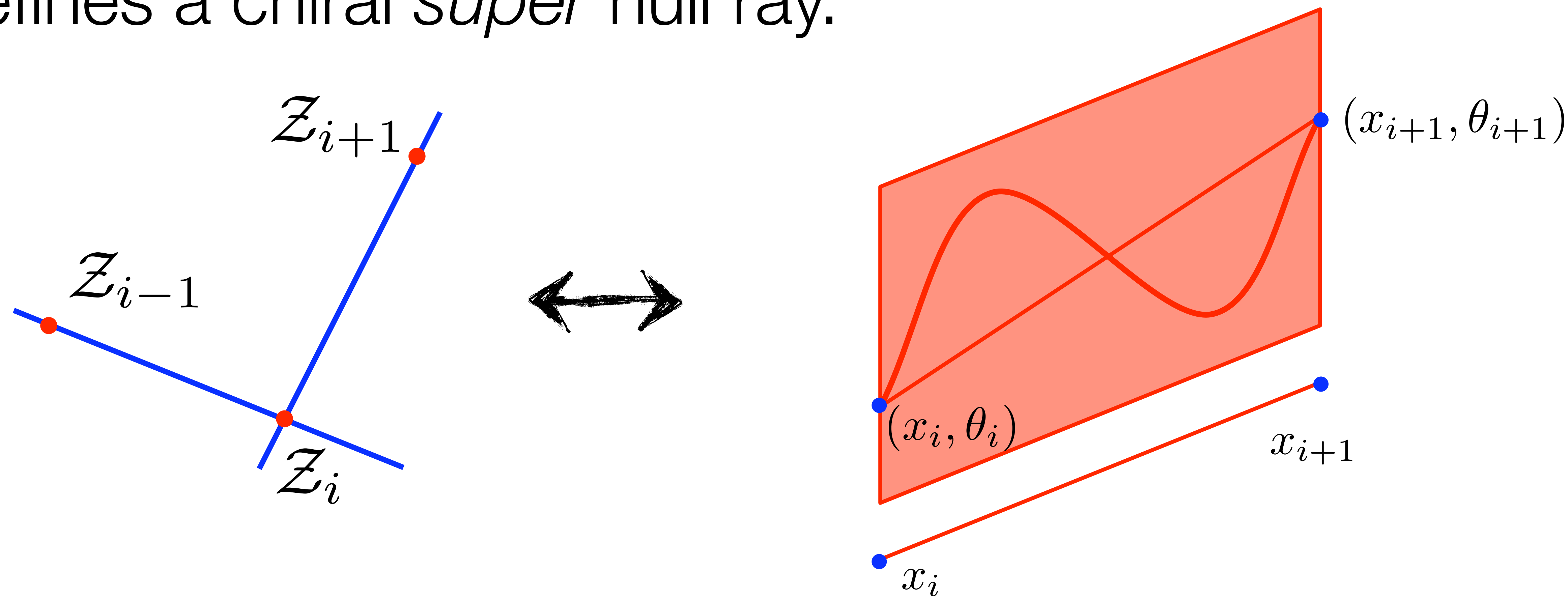
e.g. $$(\delta_{\bar{Q}}\mathcal{A})_{\mathrm{sd}} = \chi\,\partial\mathcal{A}/\partial\mu$$

Self-dual susy *vs* full susy for superloop	↔	Tree dual susy *vs* loop dual susy for amplitude

In twistor *super*space, the incidence relations

$\mu^{\dot\alpha} = \mathrm{i}x^{\alpha\dot\alpha}\lambda_\alpha \quad \chi^a = \theta^{\alpha a}\lambda_\alpha$ mean that $\mathcal{Z} \in \mathbb{CP}^{3|4}$

defines a chiral *super* null ray.



$$\frac{1}{N}\mathrm{Tr}\,\mathrm{P}\exp\left(\mathrm{i}\oint \mathbb{A}\right) \qquad \mathbb{A} = A_{\alpha\dot\alpha}\mathrm{d}x^{\alpha\dot\alpha} + \Gamma_{\alpha a}\mathrm{d}\theta^{\alpha a}$$

is the space-time superloop, where

$$\lambda^\alpha\lambda^\beta\left[\nabla^{\text{bos}}_{\alpha\dot\alpha}, \nabla^{\text{ferm}}_{\beta b}\right] = 0 \qquad \lambda^\alpha\lambda^\beta\left\{\nabla^{\text{ferm}}_{\alpha a}, \nabla^{\text{ferm}}_{\beta b}\right\} = 0$$

Any susy transformation deforms the superloop

$$\delta_{\bar{Q}}(\mathrm{W}_n) = \frac{\mathrm{i}}{N}\left\langle \oint \mathrm{Tr}\,\mathrm{P}\left(\delta x\,\mathbb{F}(x,\theta)\,\mathrm{Hol}[\mathbb{A}]\right)\right\rangle$$

$$= \frac{\mathrm{i}}{N}\left\langle \oint \mathrm{Tr}\,\mathrm{P}\left(\bar{\epsilon}\theta\left(\mathcal{F}^{+} + \mathcal{F}^{-} + \Psi\right)\mathrm{Hol}[\mathbb{A}]\right)\right\rangle$$

fermion-boson curvature

- In the self-dual theory, up to field equations and gauge transformations, the insertion is just $[\bar{Q}_{\mathrm{sd}}, \mathbb{A}]$

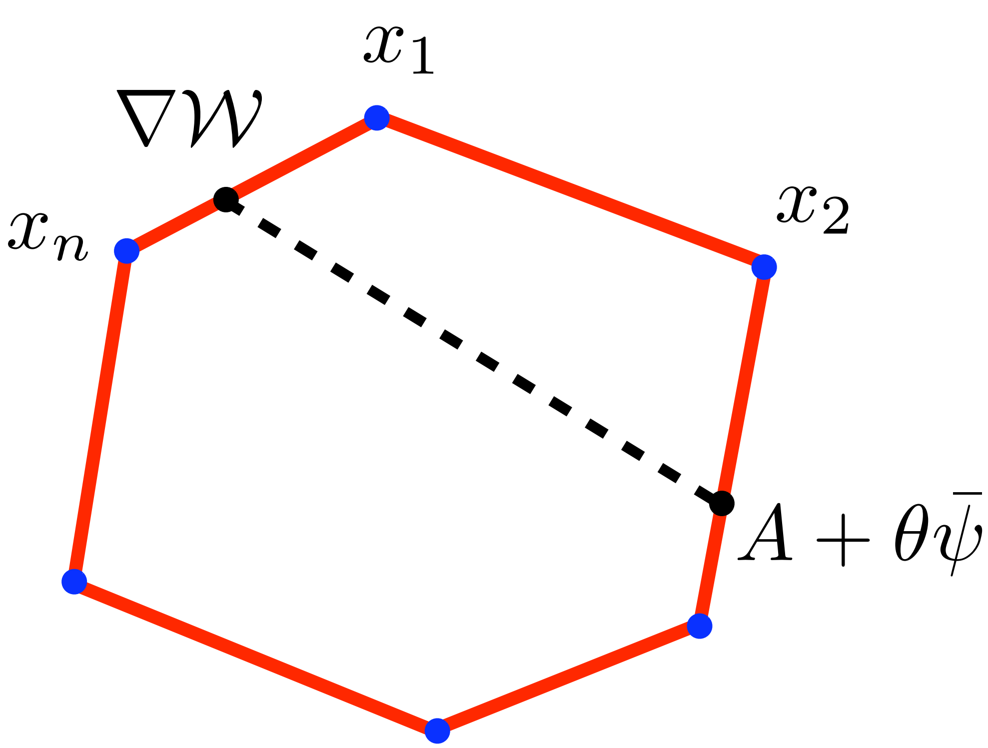
$$\delta_{\bar{Q}}(W_n) = \frac{1}{N}\left\langle\left[\bar{Q}_{\mathrm{sd}}, \mathrm{Tr}\,\mathrm{P}\exp\left(\mathrm{i}\oint\mathbb{A}\right)\right]\right\rangle = 0$$

- In the full theory, both the field equations and susy transformations are different, so there is a mismatch

Explicitly, the remaining piece is

$$\delta_{\bar{Q}}(\mathrm{W}_n) = \frac{\mathrm{i}g^2\varepsilon^{abcd}}{3!N}\left\langle \oint \mathrm{Tr}\,\mathrm{P}[\bar{\epsilon}_a|\mathrm{d}x|\nabla_b\mathcal{W}_{cd}\rangle\,\mathrm{Hol}[\mathbb{A}]\right\rangle$$

fermion-fermion supercurvature



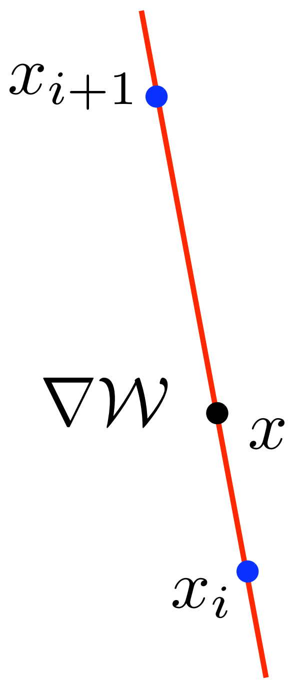
- In Abelian case

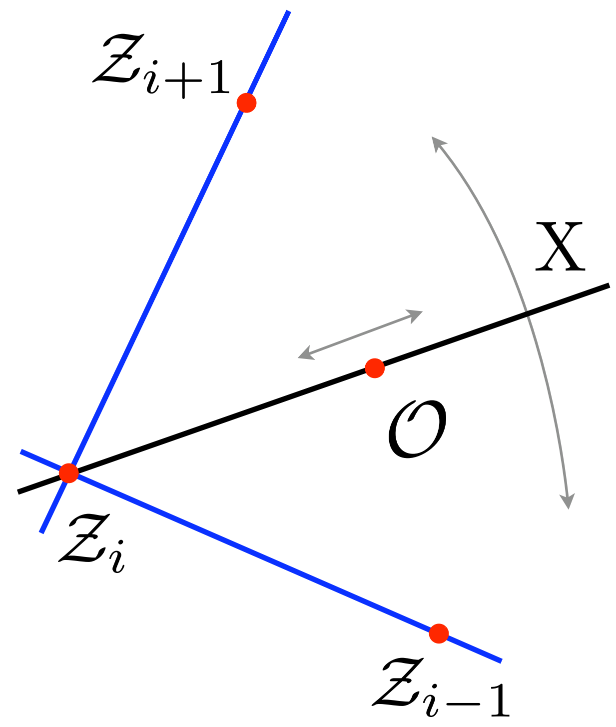
$$\varepsilon^{abcd}\nabla_{\beta b}\mathcal{W}_{cd} = \psi^a_\beta + \theta^{\alpha a}G_{\alpha\beta}$$

and easy to compute directly

$$\sum_{i,j}\left(\bar{Q}\log(i-1,i,i+1,j)\right)\log\frac{x^2_{i,j+1}x^2_{i+1,j}}{x^2_{i,j}x^2_{i+1,j+1}}$$ as expected

- Requires regularization, but not sensitive to scheme
- Appears at order $\theta\bar{\theta}$ in non-chiral superloop[Caron-Huot]

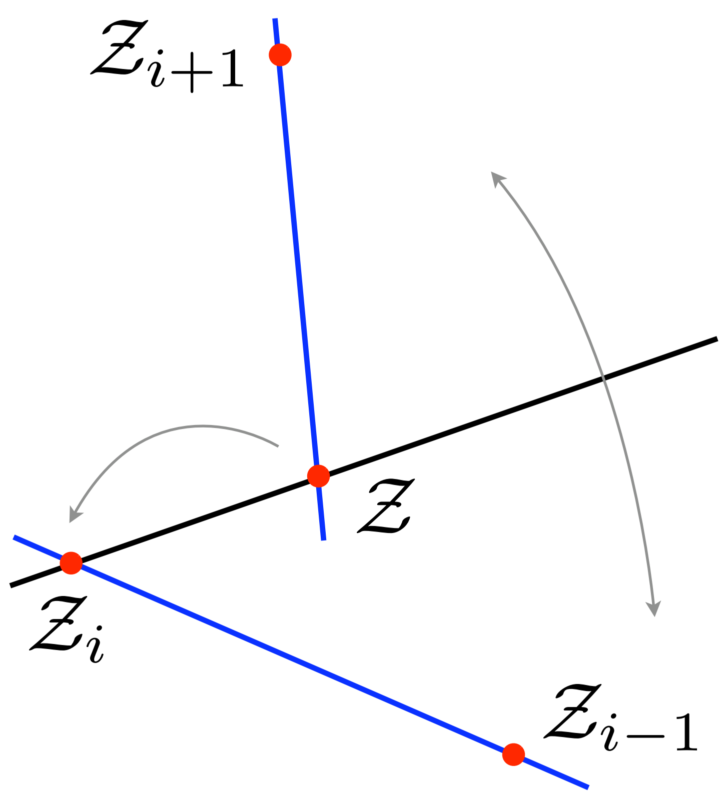




$$\mathcal{Z}_i \in \mathrm{X}$$

$$\mathrm{X} \subset (i{-}1, i, i{+}1)$$

$$\delta_{\bar{Q}} \mathrm{W}_n = a \sum_i \int V \lrcorner\, \Omega\, \mathrm{W}_{n+1}(\ldots, \mathcal{Z}_i, \mathcal{Z}, \mathcal{Z}_{i+1}, \ldots)$$



$$V \lrcorner\, \Omega \equiv \langle \lambda \mathrm{d}\lambda \rangle \wedge [\bar{\epsilon}^a\, \mathrm{d}\mu]\, \varepsilon_{abcd} \mathrm{d}\chi^b \mathrm{d}\chi^c \mathrm{d}\chi^d$$

$$= \bar{\epsilon} \cdot \chi \frac{\partial}{\partial \mu} \lrcorner\, \mathrm{D}^{3|4}\mathcal{Z}$$

Contour takes residue as $\mathcal{Z} \to \mathcal{Z}_i$

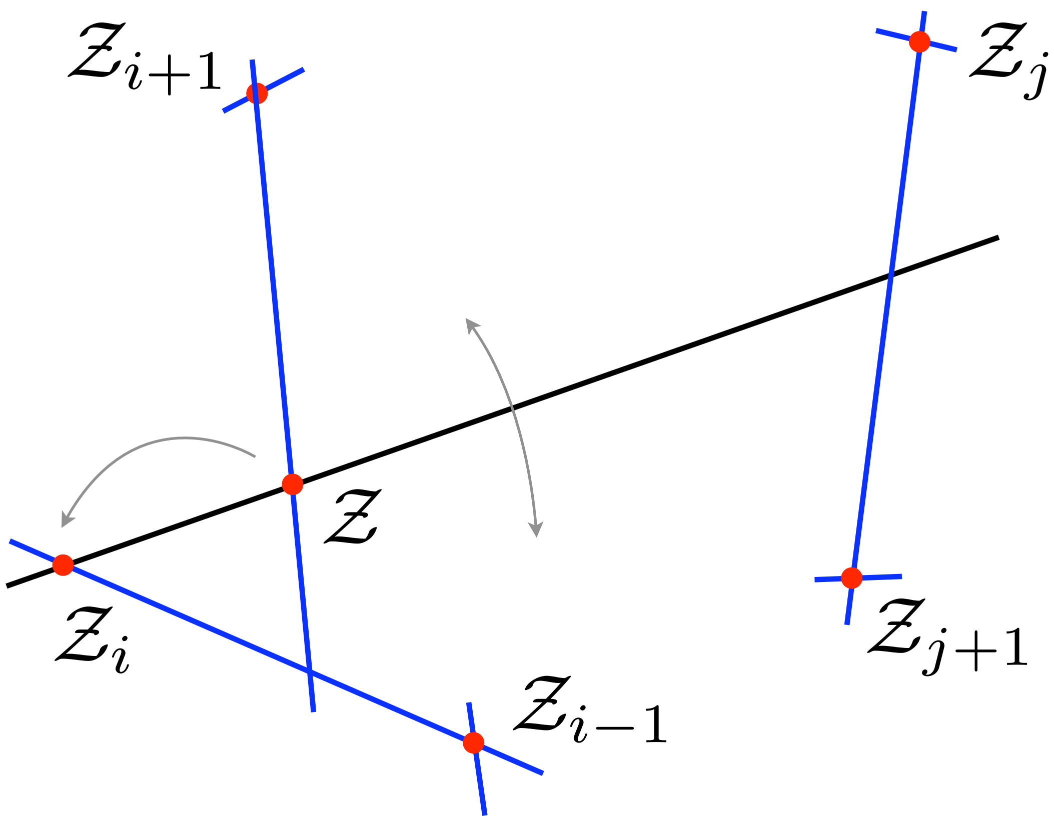
$a \to \dfrac{\Gamma_{\mathrm{cusp}}(a)}{2}$ to all loops [Caron-Huot,He]

Under the BCFW deformation $\mathcal{Z} \to \mathcal{Z}(r) = \mathcal{Z} + r\mathcal{Z}_{i+1}$

$$\mathrm{W}(\ldots, i, \mathcal{Z}, i{+}1, \ldots) = \mathrm{W}(\ldots, i, i{+}1, \ldots)$$

$$+ \sum_{j=i+2}^{i-2} [i, \mathcal{Z}, i{+}1, j, j{+}1]\, \mathrm{W}(j{+}1, \ldots, i, \mathcal{Z}_j^{\times})\, \mathrm{W}(\mathcal{Z}_j^{\times}, \mathcal{Z}_j^{\sharp}, i{+}1, \ldots, j)$$

- Only inhomogeneous terms have required poles



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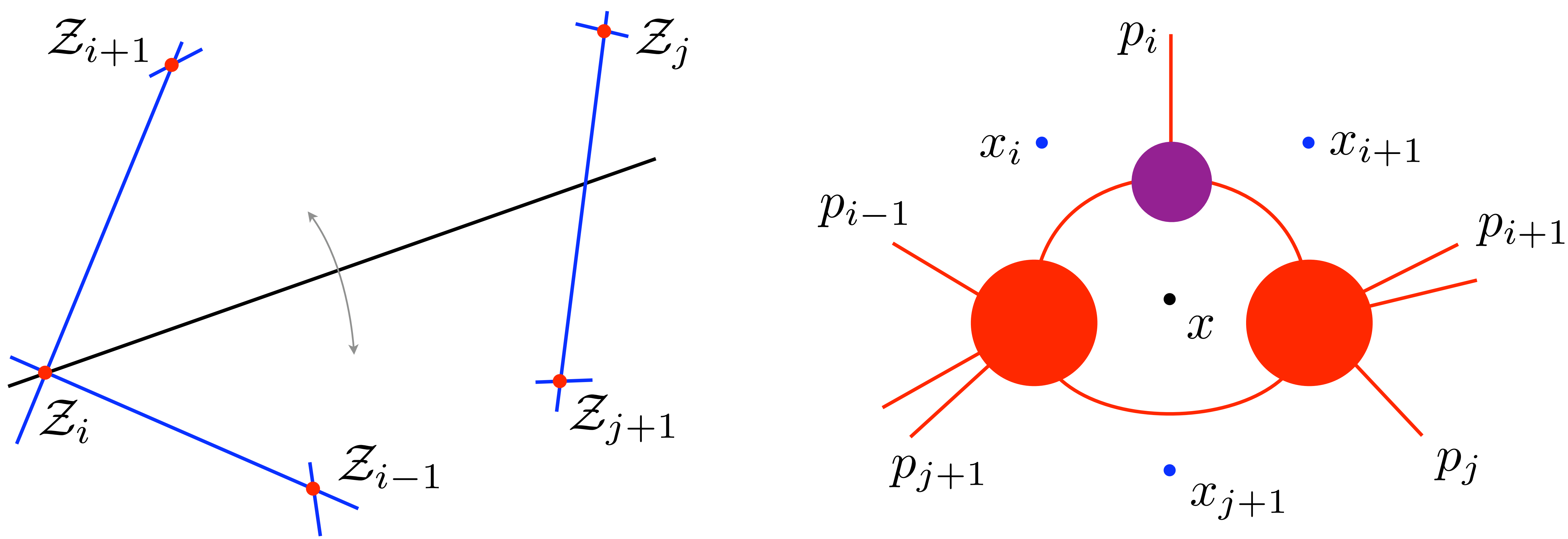

- On residue, $\mathcal{Z}_j^{\sharp} \to \mathcal{Z}_i$ and subloops share $\mathrm{X} \equiv (i\mathcal{Z}_j^{\times})$
- Dispersion integral over momentum fraction remains

[Beisert,Henn,Loebbert,McLoughlin,Plefka]

The anomaly for remainder function

$$R_n \equiv \frac{\mathrm{W}_n}{(\mathrm{W}_n^{\mathrm{Ab}})^{\Gamma_{\mathrm{cusp}}/\mathrm{g}^2}\big|_{\chi_i=0}} = \mathrm{e}^{-\Gamma_{\mathrm{cusp}}\mathcal{M}^{(1)}_{\mathrm{MHV}}}\,\langle \mathrm{W}_n\rangle$$

follows from Leibnitz rule & collinear behaviour of BDS

$$\bar{Q}(R_n) = \Gamma_{\mathrm{cusp}}\int V_{\lrcorner}\,\Omega\left[\frac{\mathrm{W}_{n+1}(\mathcal{Z})}{(\mathrm{W}_n^{\mathrm{Ab}})^{\Gamma_{\mathrm{cusp}}/\mathrm{g}^2}\big|_{\chi_i=0}} - R_n\mathrm{W}^{\mathrm{tree}}_{\mathrm{NMHV}}(\mathcal{Z})\right]$$

$$= \Gamma_{\mathrm{cusp}}\int V_{\lrcorner}\,\Omega\left[R_{n+1}(\mathcal{Z}) - R_n\mathrm{W}^{\mathrm{tree}}_{\mathrm{NMHV}}(\mathcal{Z})\right]$$

- Together with parity conjugate, fixes all Yangian anomalies

The apparent violation of Yangian invariance for finite quantities is entirely due to the difference between supersymmetries in the full and sd theories

The fact that the 'anomaly' is given recursively by a lower ℓ, but higher k, amplitude means there really is no anomaly for the complete planar super S-matrix

- *c.f.* κ-symmetry in Type IIB [Beisert,Ricci,Tseytlin,Wolf]

This recursion relation provides a powerful constraint on multi-loop amplitudes ⟶ *Song He's talk*

- Also seems likely to be related to differential equation approach [Drummond,Henn]