Evaluating five-loop Konishi in $\mathcal{N} = 4$ SYM

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Its anomalous dimension

$$\Delta_{\mathcal{K}} \,=\, 2 + \gamma_{\mathcal{K}}(a) = 2 + \sum_{\ell=1}^{\infty} a^\ell \gamma_{\mathcal{K}}^{(\ell)}$$

with $a = g^2 N_c/(4\pi^2)$

$$\gamma_K(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

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one loop

[L. Andrianopoli & S. Ferrara'96]

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five loops: a prediction based on integrability in AdS/CFT

[Z. Bajnok & R. A. Janik'08; Z. Bajnok, A. Hegedus, R. A. Janik & T. Lukowski'09; G. Arutyunov, S. Frolov & R. Suzuki'10; J. Balog & A. Hegedus'10]

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Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

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- $\gamma_{\mathcal{K}}^{(5)}$ in terms of four-loop integrals (using infrared rearrangement)
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of 24 master integrals (using IBP reduction)
- evaluating unknown master integrals (using gluing)

Our tool to evaluate it is the OPE of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM, with the superconformal primary state

$$\mathcal{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \frac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K \Phi^K\right) \, .$$

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It proves convenient to introduce auxiliary $SO(6)$ harmonic variables Y_I, defined as a (complex) null vector, $Y^2 \equiv Y_I Y_I = 0$, and project the indices of $\mathcal{O}^{IJ}$ as

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Its scaling dimension is protected.

The four-point correlation function is the first one to receive perturbative corrections:

$$G_4 = \langle \mathcal{O}(x_1, y_1) \mathcal{O}(x_2, y_2) \mathcal{O}(x_3, y_3) \mathcal{O}(x_4, y_4) \rangle = \sum_{\ell=0}^{\infty} a^{\ell} \, G_4^{(\ell)}(1, 2, 3, 4) \, ,$$

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where (for $\ell \geq 1$)

$$G_4^{(\ell)}(1,2,3,4) = \frac{2\,(N_c^2-1)}{(4\pi^2)^4} R(1,2,3,4) F^{(\ell)}(x_i)$$

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

$$F^{(\ell)}(x_i) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(-4\pi^2)^\ell} \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1, \ldots, x_{4+\ell})\,,$$

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where $x_{ij}^2 = (x_i - x_j)^2$,

$$f^{(\ell)}(x_1, \ldots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \ldots, x_{4+\ell})}{\prod_{1\leq i<j\leq 4+\ell} x_{ij}^2}$$

and $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of degree $(\ell-1)(\ell+4)/2$.
It is symmetric under the exchange of any pair of points x_i and x_j (both external and internal).

$F^{(\ell)}(x_i)$ up to six loops in the planar sector

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'12]

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For example,

$$P^{(1)} = 1\,, \qquad P^{(2)} = \frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6} = x^2_{12}x^2_{34}x^2_{56} + \ldots$$

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OPE ($x_2 \to x_1$)

$$\mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2) = c_{\mathcal{I}}\frac{(Y_1\cdot Y_2)^2}{x^4_{12}}\mathcal{I} + c_{\mathcal{K}}(a)\frac{(Y_1\cdot Y_2)^2}{(x^2_{12})^{1-\gamma_{\mathcal{K}}/2}}\mathcal{K}(x_2) + c_{\mathcal{O}}\frac{(Y_1\cdot Y_2)}{x^2_{12}}\mathcal{O}^{IJ}_{\mathbf{20'}}(x_2) + \ldots$$

The operators $\mathcal{I}$ and $\mathcal{O}_{\mathbf{20'}}$ are protected:

$$c_{\mathcal{I}} = (N_c^2 - 1)/(32\pi^4)\,, \qquad c_{\mathcal{O}} = 1/(2\pi^2)$$

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Apply OPE to the first and the second pairs of the operators (in the limit $x_1 \to x_2$, $x_3 \to x_4$) using

$$\begin{aligned} \langle \mathcal{K}(x_2)\, \mathcal{K}(x_4) \rangle &= \frac{d_{\mathcal{K}}}{(x_{24}^2)^{2+\gamma_{\mathcal{K}}}}\,, \\ \langle \mathcal{O}^{IJ}_{\mathbf{20'}}(x_2)\, \mathcal{O}^{KL}_{\mathbf{20'}}(x_4) \rangle &= \frac{c_{\mathcal{I}}}{2x_{24}^4} \left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK} - \frac{1}{3}\delta^{IJ}\delta^{KL} \right) \end{aligned}$$

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with the normalization choice

$$d_{\mathcal{K}} = 3\,\frac{N_c^2 - 1}{(4\pi^2)^2}\,, \qquad c_{\mathcal{K}}(a) = \frac{1}{12\pi^2} + O(a)$$

to obtain

$$G_4 \xrightarrow[x_4 \to x_3]{x_2 \to x_1} \frac{(N_c^2-1)^2}{4(4\pi^2)^4} \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2-1}{(4\pi^2)^4} \left[\frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4} \right.$$

$$\left. + \frac{1}{3} \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4} \left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1 \right) \right] + \ldots \,,$$

to obtain

$$G_4 \overset{x_2\to x_1}{\underset{x_4\to x_3}{\longrightarrow}} \frac{(N_c^2-1)^2}{4(4\pi^2)^4}\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2-1}{(4\pi^2)^4}\left[\frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4}\right.$$

$$\left. + \frac{1}{3}\frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4}\left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1\right)\right] + \ldots \,,$$

where $y_{ij}^2 = (Y_i \cdot Y_j)$ and

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,,$$

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$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$

so that $u \to 0$, $v \to 1$ in the limit $x_2 \to x_1\,, x_4 \to x_3$.

Therefore,

$$\sum_{\ell \geq 1} a^\ell F^{(\ell)}(x_i) \overset{\substack{x_2 \to x_1 \\ x_4 \to x_3}}{\longrightarrow} \frac{1}{6x_{13}^4} \left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1 \right) \left[1 + O(u, 1-v) \right]$$

Therefore,

$$\sum_{\ell\geq 1} a^\ell F^{(\ell)}(x_i) \xrightarrow[x_4\to x_3]{x_2\to x_1} \frac{1}{6x_{13}^4}\left(c_{\mathcal{K}}^2(a)u^{\gamma_{\mathcal{K}}(a)/2}-1\right)\left[1+O(u,1-v)\right]$$

and

$$\ln\left(1+6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right) \xrightarrow[v\to 1]{u\to 0} \frac{1}{2}\gamma_{\mathcal{K}}(a)\ln u + \ln\left(c_{\mathcal{K}}^2(a)\right) + O(u,1-v)$$

Therefore,

$$\sum_{\ell\geq 1} a^\ell F^{(\ell)}(x_i) \xrightarrow[]{\substack{x_2\to x_1\\ x_4\to x_3}} \frac{1}{6x_{13}^4}\left(c_{\mathcal{K}}^2(a)u^{\gamma_{\mathcal{K}}(a)/2}-1\right)\left[1+O(u,1-v)\right]$$

and

$$\ln\left(1+6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right) \xrightarrow[]{\substack{u\to 0\\ v\to 1}} \frac{1}{2}\gamma_{\mathcal{K}}(a)\ln u+\ln\left(c_{\mathcal{K}}^2(a)\right)+O(u,1-v)$$

where $x_{13}^4 F^{(\ell)}(x_i) \xrightarrow[]{\substack{x_2\to x_1\\ x_4\to x_3}} \widehat{F}(x_i)$ and

$$\gamma_{\mathcal{K}}(a)=\sum_{\ell\geq 1} a^\ell \gamma_{\mathcal{K}}^{(\ell)}\,, \qquad (c_{\mathcal{K}}(a))^2 = 1+3\sum_{\ell\geq 1} a^\ell \alpha^{(\ell)}$$

The $\ell-$loop correction to the logarithm of the correlation function is given by an $\ell-$folded integral over the internal coordinates $x_5, \ldots, x_{4+\ell}$:

$$\ln\left(1+6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right)$$

$$=\sum_{\ell\geq 1} a^\ell \int d^4x_5 \ldots d^4x_{4+\ell}\, \mathcal{I}_\ell(x_1,\ldots,x_4|x_5,\ldots,x_{4+\ell})\,,$$

$$=\sum_{\ell\geq 1} a^\ell\, I^{(\ell)}$$

where $\mathcal{I}_\ell$ is symmetric under the $S_4 \times S_\ell$ permutations of the four external coordinates, $x_1, \ldots, x_4$ and the ℓ internal coordinates $x_5, \ldots, x_{4+\ell}$.

Up to five loops

$$
\begin{aligned}
I^{(1)} &= 6\,\widehat{F}^{(1)}\,, \\
I^{(2)} &= 6\Big[\widehat{F}^{(2)} - 3(\widehat{F}^{(1)})^2\Big]\,, \\
I^{(3)} &= 6\Big[\widehat{F}^{(3)} - 6\widehat{F}^{(1)}\widehat{F}^{(2)} + 12(\widehat{F}^{(1)})^3\Big]\,, \\
I^{(4)} &= 6\Big[\widehat{F}^{(4)} - 6\widehat{F}^{(1)}\widehat{F}^{(3)} - 3(\widehat{F}^{(2)})^2 + 36\widehat{F}^{(2)}(\widehat{F}^{(1)})^2 - 54(\widehat{F}^{(1)})^4\Big]\,, \\
I^{(5)} &= 6\Big[\widehat{F}^{(5)} - 6\widehat{F}^{(1)}\widehat{F}^{(4)} - 6\widehat{F}^{(3)}\widehat{F}^{(2)} + 36\widehat{F}^{(3)}(\widehat{F}^{(1)})^2 \\
&\quad +36\widehat{F}^{(1)}(\widehat{F}^{(2)})^2 - 216\widehat{F}^{(2)}(\widehat{F}^{(1)})^3 + \tfrac{1296}{5}(\widehat{F}^{(1)})^5\Big]\,.
\end{aligned}
$$

$l = 5$, the planar limit:

$$I^{(5)} = \int d^4x_5 \ldots d^4x_9 \, \mathcal{I}_5(x_1, \ldots, x_4 | x_5, \ldots, x_9)$$

with

$$\mathcal{I}_5 = -\frac{6}{5!(4\pi^2)^5} \frac{x_{13}^4}{\prod_{i=5}^9 x_{1i}^4 x_{3i}^4} \left[\frac{1}{5!} \frac{\widehat{P}_{5,6,7,8,9}}{x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{68}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \right.$$

$$-\frac{1}{4} x_{12}^4 \frac{\widehat{P}_{5,6,7,8}}{x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{68}^2 x_{78}^2} - \frac{1}{2} x_{13}^4 \frac{\widehat{P}_{5,6,7}}{x_{56}^2 x_{57}^2 x_{67}^2}$$

$$\frac{\widehat{P}_{8,9}}{x_{89}^2} + 6(x_{13}^4)^2 \frac{\widehat{P}_{5,6,7}}{x_{56}^2 x_{57}^2 x_{67}^2} + 9(x_{13}^4)^2 \frac{\widehat{P}_{5,6}}{x_{56}^2} \frac{\widehat{P}_{7,8}}{x_{78}^2}$$

$$\left. -108(x_{13}^4)^3 \frac{\widehat{P}_{5,6}}{x_{56}^2} + \frac{1296}{5} (x_{13}^4)^4 \right] + S_5 \text{ permutations}$$

where $\widehat{P}_{5,6,7,8,9} = P^{(5)} \Big|_{x_2 = x_1, x_4 = x_3}$ etc., and

$$
\begin{aligned}
P^{(5)} = & -\tfrac{1}{2}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^4x_{26}^2x_{29}^2x_{37}^2x_{38}^2x_{39}^2x_{47}^2x_{48}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2\,, \\
& +\tfrac{1}{4}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^4x_{26}^2x_{29}^2x_{37}^4x_{39}^2x_{48}^4x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\
& +\tfrac{1}{4}x_{13}^4x_{17}^2x_{19}^2x_{24}^2x_{26}^2x_{27}^2x_{29}^2x_{36}^2x_{39}^2x_{48}^6x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\
& +\tfrac{1}{6}x_{13}^2x_{16}^2x_{19}^4x_{24}^4x_{28}^2x_{29}^2x_{37}^4x_{38}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{68}^2 \\
& -\tfrac{1}{8}x_{13}^4x_{16}^2x_{18}^2x_{24}^4x_{28}^2x_{29}^2x_{37}^2x_{39}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{69}^2x_{78}^2 \\
& +\tfrac{1}{28}x_{13}^2x_{17}^2x_{18}^2x_{19}^2x_{24}^8x_{36}^2x_{38}^2x_{39}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2x_{69}^2x_{78}^2 \\
& +\tfrac{1}{12}x_{13}^2x_{16}^2x_{17}^2x_{19}^2x_{26}^2x_{27}^2x_{28}^2x_{29}^2x_{35}^2x_{38}^2x_{39}^2x_{45}^2x_{46}^2x_{47}^2x_{49}^2x_{57}^2x_{58}^2x_{68}^2 \\
& +S_9 \text{ permutations}
\end{aligned}
$$

To evaluate

$$\gamma_K(a) = 2\frac{d}{d\ln u}\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right)$$

we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.

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we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.
Put $x_1 = x_2$ and $x_3 = x_4$ and introduce dimensional regularization (in coordinate space) with $D = 4 - 2\epsilon$

$$\mu^{l\epsilon} \int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_\ell(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9) \,.$$

To evaluate

$$\gamma_{\mathcal{K}}(a) = 2\frac{d}{d\ln u}\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right)$$

we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.
Put $x_1 = x_2$ and $x_3 = x_4$ and introduce dimensional regularization (in coordinate space) with $D = 4 - 2\epsilon$

$$\mu^{l\epsilon}\int d^D x_5 \ldots d^D x_9\, \mathcal{I}_\ell(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)\,.$$

The integral has a simple pole in $\epsilon = (4-D)/2$.

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IRR (infrared rearrangement) [A.A. Vladimirov'80]

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If the pole part of a given (incompletely renormalized) diagram is independent of momenta and masses, try to set to zero as many momenta and masses as possible without generating IR divergences.

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A four-loop example:

$$I(x_1, x_3) = \frac{1}{\pi^{2D}} \int \frac{(x_{13}^2)^4 \, d^D x_5 \ldots d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{38}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}$$

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There is an UV simple pole in ϵ

$$I(x_1, x_3) = (x_{13}^2)^{-4\epsilon} \left[\frac{C}{\epsilon} + O(\epsilon^0) \right]$$

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The integrand is unintegrable in a vicinity of the two external points, x_1 and x_3
In a vicinity of x_1 ($x_3 \to \infty$):

$$F(x_1, x_5, \ldots, x_8) = \frac{1}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}$$

Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \ldots, x_8) = \frac{C}{2\epsilon}\delta(x_1 - x_5)\ldots\delta(x_1 - x_8)\,,$$

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The pole part of $I(x_1, x_3)$ is C/ϵ.
We are not going to momentum space via Fourier transform because

- we would obtain four-loop integrals,
- exponents of the propagators would depend on ϵ.

Apply IRR in the coordinate space: treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points.
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This propagator integral is three-loop:

$$F(x_1,x_5) = f(\epsilon)\frac{1}{(x_{15}^2)^{2+3\epsilon}}\,.$$

IRR

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$$\mathcal{F}\left[\frac{1}{(x^2)^\lambda}\right] = \frac{1}{\pi^{D/2}} \int d^D x \; \mathrm{e}^{ipx} \, \frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda}}{\Gamma(\lambda)} \frac{\Gamma(D/2-\lambda)}{(p^2)^{D/2-\lambda}} \, .$$

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For $\lambda = 2 + 3\epsilon$ and for $x_5 = 0$:

$$\mathcal{F}\left[F(x_1, 0)\right] = f(\epsilon) \frac{4^{-4\epsilon}\Gamma(-4\epsilon)}{\Gamma(2+3\epsilon)} \frac{1}{(p^2)^{-4\epsilon}} = -\frac{f(0)}{4\epsilon} + O(\epsilon^0) \, .$$

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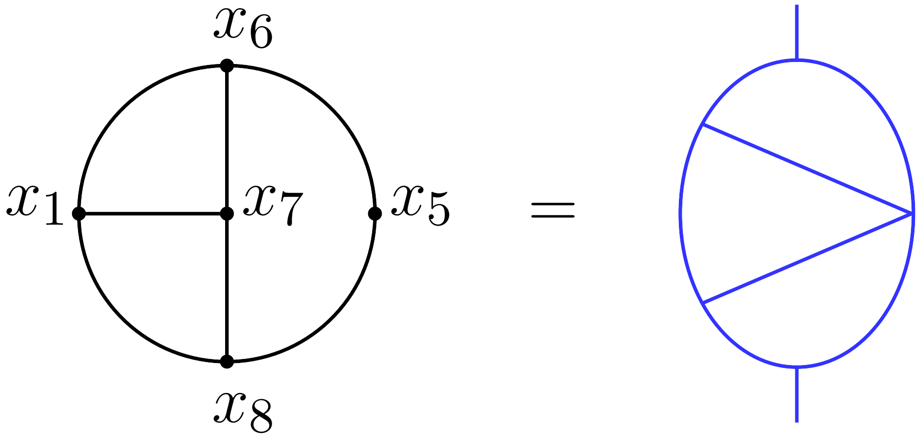
We obtain

$$C = -\frac{1}{2} f(0) = -\frac{1}{2} F(x_1, 0)\Big|_{x_1^2=1, D=4}$$

The integral $F(x_1, x_5)$ corresponds to a planar graph.



Using a known result for the corresponding dual integral at $d = 4$ leads to

$$C = -10\,\zeta(5)$$

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

To evaluate the pole part (a simple pole) apply IRR.

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

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Apply IRR:

consider x_1 and x_5 external and x_6, x_7, x_8, x_9 internal.

The problem reduces to the evaluation of the residue of

$$\frac{1}{(x_{15}^2)^{2+4\epsilon}} \int d^D x_6 d^D x_7 d^D x_8 d^D x_9 \, \hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) \,.$$

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The pole comes from $\frac{1}{(x_{15}^2)^{2+4\epsilon}}$ so that we need to evaluate

$$\int d^D x_6 d^D x_7 d^D x_8 d^D x_9 \, \hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9)\,.$$

at $x_{15}^2 = 1$.

Around 17000 four-loop two-point Feynman integrals contributing to this integral and belonging to the family

$$G(a_1,\ldots,a_{14}) = \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2)^{a_1}(x_{17}^2)^{a_2}(x_{18}^2)^{a_3}(x_{19}^2)^{a_4}(x_6^2)^{a_5}(x_7^2)^{a_6}(x_8^2)^{a_7}}$$

$$\times \frac{1}{(x_9^2)^{a_8}(x_{67}^2)^{a_9}(x_{68}^2)^{a_{10}}(x_{69}^2)^{a_{11}}(x_{78}^2)^{a_{12}}(x_{79}^2)^{a_{13}}(x_{89}^2)^{a_{14}}}\,.$$

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An IBP reduction to master integrals.

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An IBP reduction to master integrals.

See Chapter 5 of

Evaluating Feynman integrals (STMP 211, Springer 2004)

Feynman Integrals Calculus (Springer 2006)

Analytic Tools for Feynman Integrals (STMP, Springer 2013(?))

IBP

An old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Integral calculus:

$$\int_a^b u v' \mathrm{d}x = uv|_a^b - \int_a^b u' v \mathrm{d}x$$

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Feynman integral calculus:
Use IBP and neglect surface terms

[Chetyrkin & Tkachov'81]

$$\int \ldots \int \left[\left(q_i \cdot \frac{\partial}{\partial k_j} \right) \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0 \,;$$

$$\int \ldots \int \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0 \,.$$

An example

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Taking derivatives:

$$\frac{\partial}{\partial k} \cdot k = \frac{\partial}{\partial k_\mu} \cdot k_\mu = d$$

IBP

$$
\begin{aligned}
k \cdot \frac{\partial}{\partial k} \frac{1}{(k^2 - m^2)^a} &= -a \frac{2k^2}{(k^2 - m^2)^{a+1}} \\
&= -2a \left[\frac{1}{(k^2 - m^2)^a} + \frac{m^2}{(k^2 - m^2)^{a+1}} \right]
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Its solution

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1)$$

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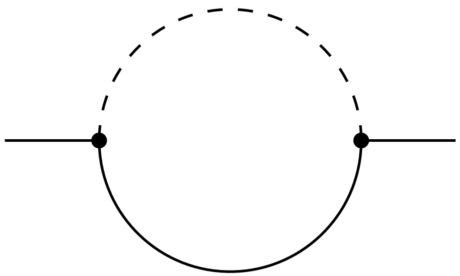
Explicitly,

$$F(a) = \frac{(-1)^a \, (1 - d/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1 \ ,$$

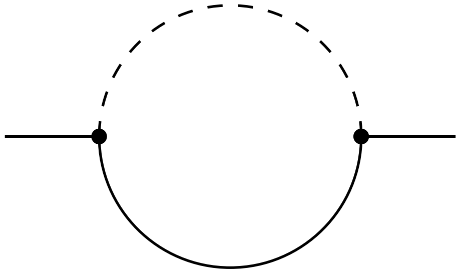
where $(x)_a$ is the Pochhammer symbol

One more example



$$F_\Gamma(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

One more example



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Apply IBP

$$\int \frac{\partial}{\partial k}\cdot k \left(\frac{1}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}} \right) \mathrm{d}^d k \;=\; 0\,,$$

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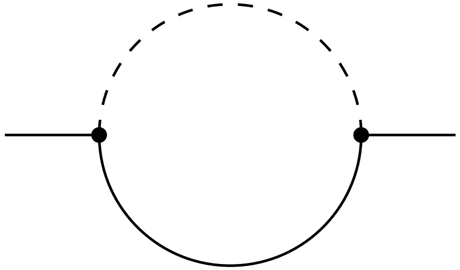

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use $2k\cdot(k-q) \to (k-q)^2 + (k^2-m^2) - q^2 + m^2$ to obtain

IBP

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \quad (A)$$

$$a_2 - a_1 - a_1 \mathbf{1}^+ (q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+ (\mathbf{1}^- - q^2 + m^2) = 0 \quad (B)$$

where, e.g., $\mathbf{1}^+ \mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1)$.

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A manual solution.

1. Apply $(q^2+m^2)(A) - 2m^2(B)$,

$$(q^2-m^2)^2 a_2 \mathbf{2}^+ = (q^2-m^2) a_2 \mathbf{1}^- \mathbf{2}^+$$
$$-(d-2a_1-a_2)q^2 - (d-3a_2)m^2 + 2m^2 a_1 \mathbf{1}^+ \mathbf{2}^-$$

to reduce a_2 to 1 or 0.

```
F[a1_, a2_ /; a2 > 1] :=
1/(a2 - 1)/(qq - mm)^2 (
(a2 - 1) (qq - mm) F[a1 - 1, a2]
- ((d - 2 a1 - a2 + 1) qq
+ (d - 3 a2 + 3) mm) F[a1, a2 - 1]
+ 2 mm a1 F[a1 + 1, a2 - 2]);
```

2. Suppose that $a_2 = 1$. Apply $(A) - (B)$, i.e.

$$(q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^-$$

to reduce a_1 to 1 or a_2 to 0.

```
F[a1_ /; a1 > 1, 1] :=
1/(a1 - 1)/(qq - mm) ((a1 - 1) F[a1, 0]
-(d - a1 - 1)  F[a1 - 1, 1]);
```

Therefore, any $F(a_1, a_2)$ can be reduced to $I_1 = F(1,1)$ and integrals with $a_2 \leq 0$ (which can be evaluated in terms of gamma functions for general d).

3. Let $a_2 \leq 0$. Apply (A) to reduce a_1 to one.

```
F[a1_ /; a1 > 1, a2_ /; a2 <= 0] :=
1/(a1 - 1)/2/mm ((d - 2 a1 - a2 + 2) F[a1 - 1, a2]
-a2 F[a1 - 2, a2 + 1] + a2 (qq - mm) F[a1 - 1, a2 + 1]);
```

4. Let $a_1 = 1$. Apply the following corollary of (A) and (B)

$$(d - a_2 - 1)\mathbf{2}^- = (q^2 - m^2)^2 a_2 \mathbf{2}^+ + (q^2 + m^2)(d - 2a_2 - 1)$$

to increase a_2 to zero or one starting from negative values.

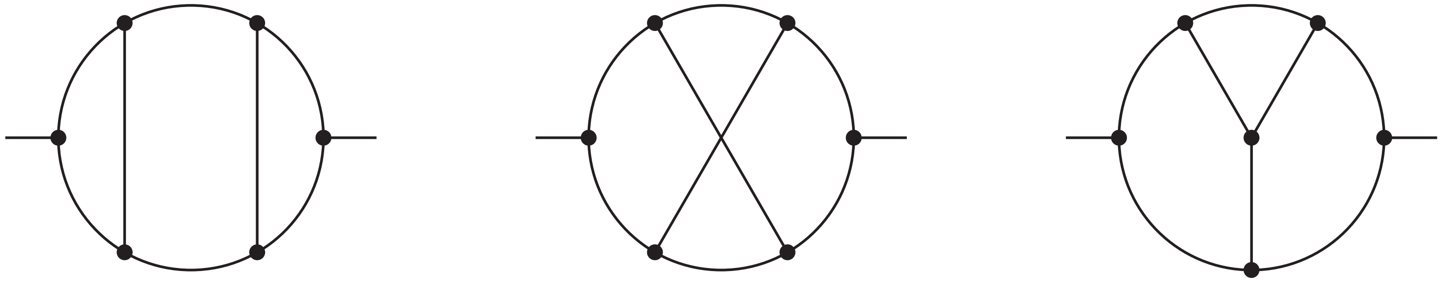
```
F[1, a2_ /; a2 < 0] := 1/(d - a2 - 2) (
(a2 + 1) (qq - mm)^2 F[1, a2 + 2] +
(qq + mm) (d - 2 a2 - 3) F[1, a2 + 1] );
```

Any $F(a_1, a_2)$ is a linear combination of the two master integrals $I_1 = F(1,1)$ and $I_2 = F(1,0)$.

For example,

```
F[3, 2]=
(-(((-5 + d) (-3 + d) (-4 mm + d mm - 8 qq + d qq))/(
2 (mm - qq)^4)) I1
+ ((-2 + d) (96 mm^2 - 39 d mm^2 + 4 d^2 mm^2
+ 28 mm qq - 6 d mm qq - 4 qq^2 + d qq^2))/
(8 mm^2 (mm - qq)^4) I2)
```

A manual solution of IBP relations for massless three-loop propagator diagrams [K.G. Chetyrkin & F.V. Tkachov'81]



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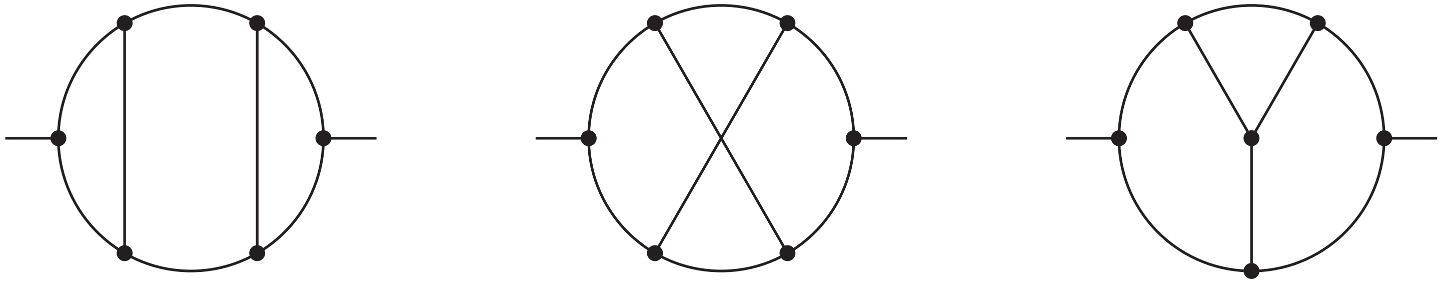

MINCER:

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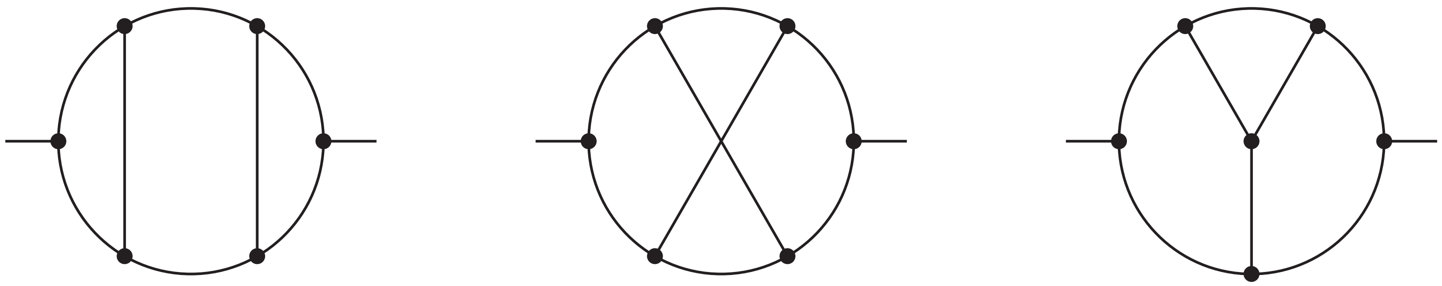

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Solving IBP relations algorithmically:

- Laporta's algorithm

  [Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

  Use IBP relations written at points $(a_1, \ldots, a_L)$ with $\sum |a_i| \leq N$ and solve them for the Feynman integrals involved.

  (A Gauss elimination)

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  (A Gauss elimination)

  Feynman integrals on the right-hand sides of such solutions are master integrals.

  When N increases, the situation stabilizes, in the sense that the number of the master integrals becomes stable starting from sufficiently large N.

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(for Feynman integrals with usual propagators)

The same example

$$F_\Gamma(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

The left-hand sides of the two primary IBP relations:

```
ibp1[a1_, a2_] := (d - 2 a1 - a2) F[a1, a2]
- 2 mm a1 F[a1 + 1, a2] - a2 (F[a1 - 1, a2 + 1]
+ (mm - qq)  F[a1, a2 + 1]);
ibp2[a1_, a2_] := (a2 - a1) F[a1, a2] -
a1 ((qq + mm) F[a1 + 1, a2] - F[a1 + 1, a2 - 1]) -
a2 (F[a1 - 1, a2 + 1] + (mm - qq)  F[a1, a2 + 1]);
```

Let us consider the sector $a_1 > 0, a_2 \leq 0$
Use IBP at various (a_1, a_2) with $a_1 + |a_2| \leq N$
Solve the corresponding linear system of equation with respect to $F(a_1, a_2)$ involved.
Increase N.

$N = 1$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0},
{F[2, 0], F[2, -1]}]


{F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
 F[2, 0] -> ((-2 + d) F[1, 0])/(2 mm)}
```

IBP

$N = 2$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0,
ibp1[2, 0] == 0, ibp2[2, 0] == 0,
ibp1[1, -1] == 0,  ibp2[1, -1] == 0 },
{F[2, 0], F[3, 0], F[1, -1],
F[2, -1], F[3, -1],  F[2, -2]}]

{F[2, -2] -> (((2 + d) mm^2 + 2 (2 + d) mm qq
+ (-2 + d) qq^2) F[1,0])/(2 mm),
F[3, -1] -> ((-2 + d) (-4 qq + d (mm + qq))
F[1, 0])/(8 mm^2),
F[3, 0] -> ((-4 + d) (-2 + d) F[1, 0])/(8 mm^2),
F[1, -1] -> (mm + qq) F[1, 0],
F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
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```

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Private versions

[Gehrmann & Remiddi, Laporta, Czakon, Schröder, Pak, Sturm, Marquard & Seidel, Velizhanin, . . .]

Solving reduction problems algorithmically in other ways:

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- Baikov's method [P.A. Baikov'96-. . .]

  [V.A. Smirnov & M. Steinhauser'03]

  An Ansatz for coefficient functions at master integrals

$$\int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{a_1} \ldots x_N^{a_N}} \left[P(\underline{x'})\right]^{(d-h-1)/2}$$

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- Lee's approach (based on Lie algebras) [R.N. Lee'08]

C++ version of FIRE $\rightarrow$

$$C_4 = w_{44}M_{44} + w_{61}M_{61} + w_{36}M_{36} + w_{31}M_{31} + w_{35}M_{35}$$
$$+ w_{22}M_{22} + w_{32}M_{32} + w_{33}M_{33} + w_{34}M_{34} + w_{25}M_{25} + w_{23}M_{23}$$
$$+ w_{27}M_{27} + w_{24}M_{24} + w_{26}M_{26} + w_{01}M_{01}$$
$$+ w_{21}M_{21} + w_{12}M_{12} + w_{11}M_{11} + w_{14}M_{14} + w_{13}M_{13} + w_1 I_1 + w_2 I_2$$

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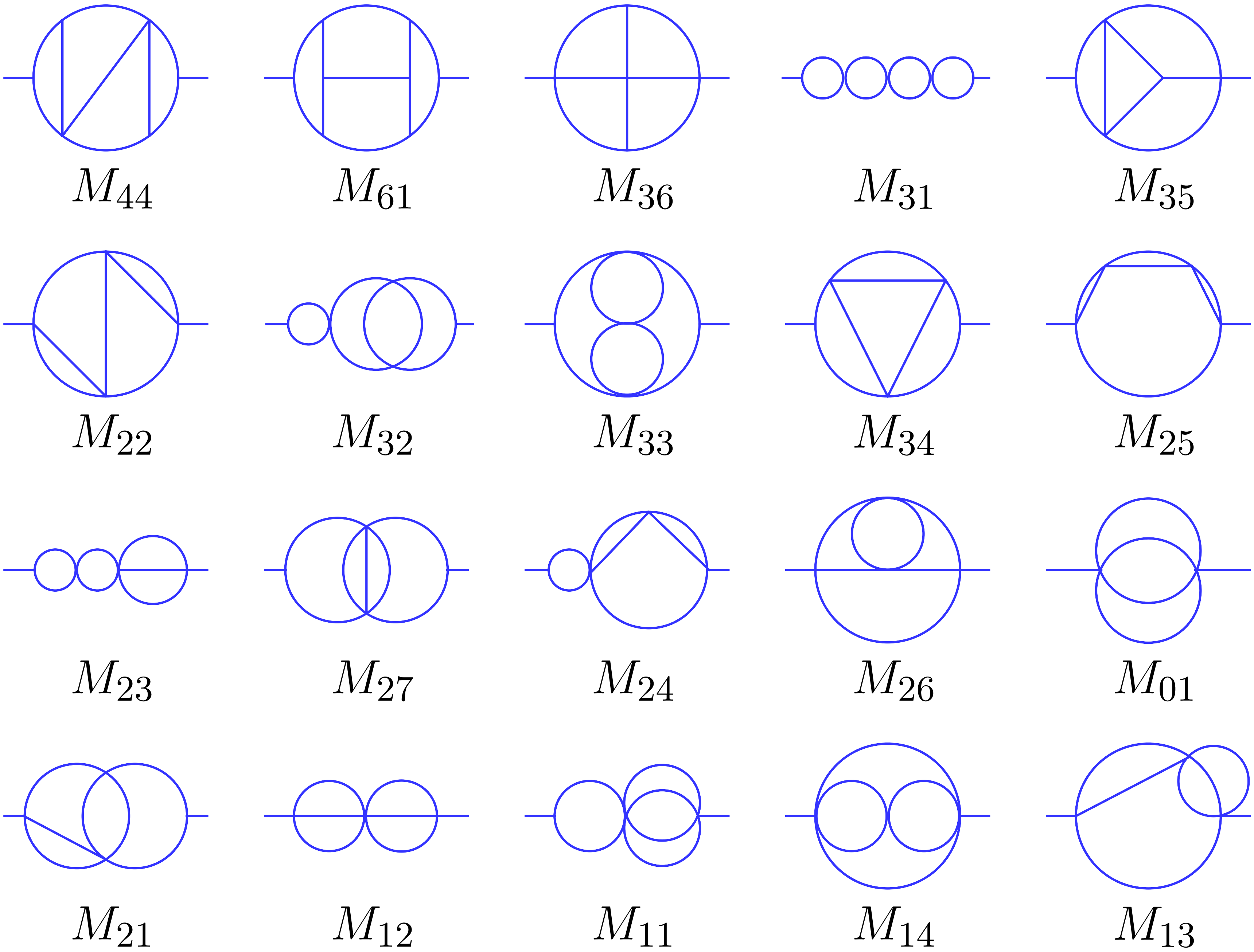
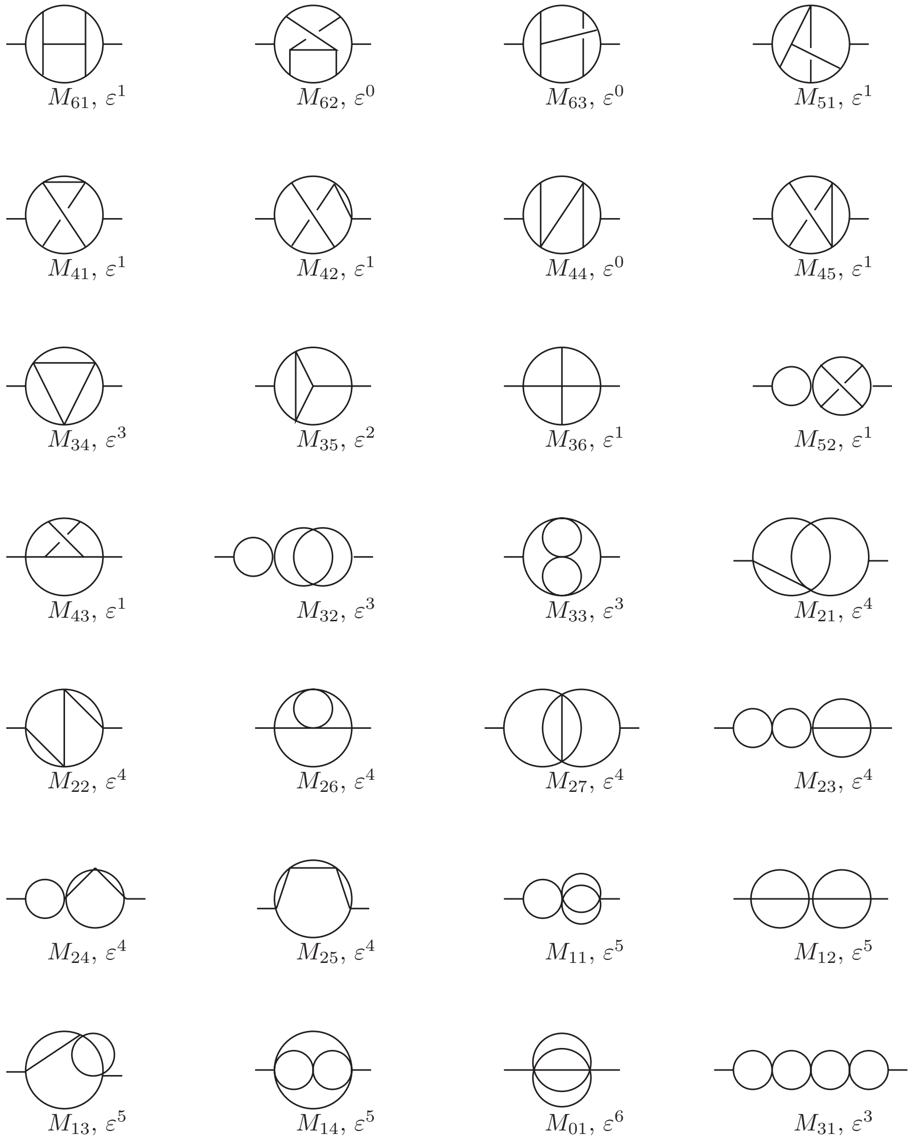
Only I_1 and I_2, are associated with non-planar graphs.

20 master integrals $M_{44}, \ldots, M_{13}$ correspond to planar graphs and can be represented, via duality, as four-loop propagator master (momentum) integrals.


M_{44}
M_{61}
M_{36}
M_{31}
M_{35}
M_{22}
M_{32}
M_{33}
M_{34}
M_{25}
M_{23}
M_{27}
M_{24}
M_{26}
M_{01}
M_{21}
M_{12}
M_{11}
M_{14}
M_{13}


M_{61}, ε^1
M_{62}, ε^0
M_{63}, ε^0
M_{51}, ε^1
M_{41}, ε^1
M_{42}, ε^1
M_{44}, ε^0
M_{45}, ε^1
M_{34}, ε^3
M_{35}, ε^2
M_{36}, ε^1
M_{52}, ε^1
M_{43}, ε^1
M_{32}, ε^3
M_{33}, ε^3
M_{21}, ε^4
M_{22}, ε^4
M_{26}, ε^4
M_{27}, ε^4
M_{23}, ε^4
M_{24}, ε^4
M_{25}, ε^4
M_{11}, ε^5
M_{12}, ε^5
M_{13}, ε^5
M_{14}, ε^5
M_{01}, ε^6
M_{31}, ε^3

Results in an ϵ expansion up to transcendentality weight seven [P.A. Baikov & K.G. Chetyrkin'10]
and up to weight twelve [R.N. Lee, A.V. Smirnov & V.A. Smirnov]

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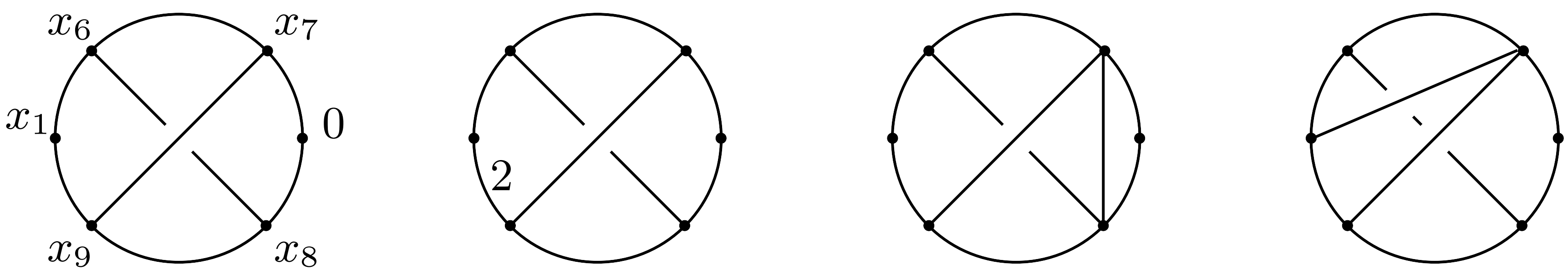
The two non-planar master integrals I_1 and I_2



I_1 I_2 $I_3(0)$ $I_4(0)$

We did not use the method by R. Lee [R. Lee'09]
based on dimensional recurrence relations.

Its applications [R. Lee, A. and V. Smirnovs'10,11]

$$I_1 = \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_1}{\epsilon} + b_1 + c_1\epsilon + O(\epsilon^2)\,,$$

$$I_2 = \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2\epsilon + O(\epsilon^2)$$

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$$I_2 = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2\epsilon + O(\epsilon^2)$$

$$\left(\frac{3a_1}{80} + \frac{9a_2}{160} + \frac{15\zeta_5}{16}\right)\epsilon^{-2}$$
$$+\left(-\frac{21a_1}{80} - \frac{9a_2}{80} + \frac{3b_1}{80} + \frac{9b_2}{160} + \frac{15{\zeta_3}^2}{16} + \frac{5\pi^6}{2016}\right)\epsilon^{-1}$$
$$+\left(\frac{741a_1}{640} + \frac{807a_2}{320} - \frac{21b_1}{80} - \frac{9b_2}{80} + \frac{3c_1}{80} + \frac{9c_2}{160} - \frac{225\zeta_7}{64} - \frac{5\pi^2\zeta_5}{16}\right.$$
$$\left.+\frac{7035\zeta_5}{128} + \frac{81{\zeta_3}^2}{16} + \frac{\pi^4\zeta_3}{32} - \frac{27\zeta_3}{4} - \frac{237}{16}\right) + O(\epsilon)$$

The absence of poles $\to$ two relations between coefficients $a_1, a_2, b_1, b_2, c_1, c_2$

To evaluate a_1, a_2 take a Fourier transform:

$$\mathcal{F}\left[I_1\right] = \mathcal{F}\left[\frac{a_1}{\epsilon}(x_1^2)^{-4\epsilon} + O(\epsilon^0)\right] = \left(64\, a_1 + O(\epsilon)\right)\left(p^2\right)^{-2+5\epsilon}.$$

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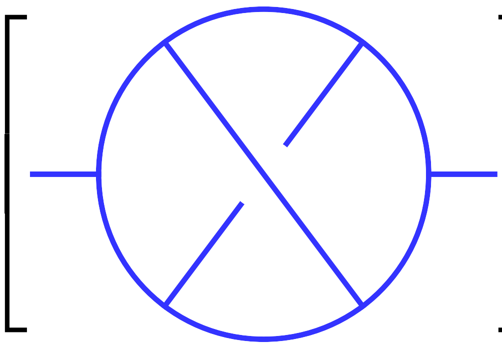
$$\mathcal{F}\left[I_1\right] = 16 \Big[$$



$$\Big] = 16\big(20\zeta_5 + O(\epsilon)\big)(p^2)^{-2+5\epsilon}\,.$$

so that $a_1 = 5\zeta_5$.

$$\mathcal{F}[I_2] = F\left[\frac{a_2}{\epsilon}(x_1^2)^{-1-4\epsilon} + O(\epsilon^0)\right] = 4\left(\frac{a_2}{\epsilon} + O(\epsilon)\right)(p^2)^{-1+5\epsilon}\,.$$

$$\mathcal{F}\left[\frac{1}{(x_{19}^2)^2}\right] = 2^{-2\epsilon}\Gamma(-\epsilon)(p^2)^{\epsilon} = -\frac{1}{\epsilon} + O(\epsilon^0)\,.$$

so that taking the residue at the pole reduces to shrinking the corresponding line to a point.

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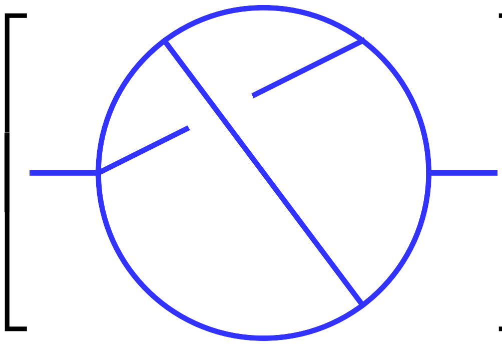
$$\mathcal{F}[I_2] = -\frac{4}{\epsilon}\Bigg[$$



$$\Bigg] = -\frac{4}{\epsilon}(20\zeta_5 + O(\epsilon))(p^2)^{-1+5\epsilon}\,.$$

We obtain $a_2 = -20\zeta_5\,.$

Introduce the following auxiliary integrals

$$I_3(\kappa) \;=\; \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}}\,,$$

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with

$$I_i(\kappa) = b_i + \epsilon(c_i + \kappa\, d_i) + O(\epsilon^2)\,, \qquad i = 3, 4$$

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with

$$I_i(\kappa) = b_i + \epsilon(c_i + \kappa\, d_i) + O(\epsilon^2)\,, \qquad i = 3,4$$

Evaluate $I_3(0)$ and $I_4(0)$. They are not master integrals.
We use FIRE to reduce them to master integrals,

in particular, I_1 and I_2.

$$b_3 = -\frac{2}{3}\,b_1 - \frac{7}{3}\,b_2 - 70\,\zeta_5 + \frac{26}{3}{\zeta_3}^2 - \frac{65}{567}\pi^6\,,$$

$$b_4 = -b_1 - 2b_2 - 45\zeta_5 + 7{\zeta_3}^2 - \frac{5}{54}\pi^6\,,$$

$$c_3 = \frac{14}{3}\,b_1 + \frac{14}{3}\,b_2 - \frac{2}{3}\,c_1 - \frac{7}{3}\,c_2 - \frac{4667}{6}\zeta_7 + \frac{130}{9}\pi^2\zeta_5 - \frac{100}{3}\zeta_5 + \frac{13}{45}\pi^4\zeta_3\,,$$

$$c_4 = 2b_1 - 6b_2 - c_1 - 2c_2 - \frac{4193}{4}\zeta_7 + \frac{35}{3}\pi^2\zeta_5 - 275\zeta_5 + 35{\zeta_3}^2$$
$$+\frac{7}{30}\pi^4\zeta_3 - \frac{25}{54}\pi^6\,.$$

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Evaluate I_3 and I_4 at $\kappa = 1/2$ and $\kappa = 1$ and obtain I_1 and I_2, i.e. b_1, b_2 and c_1, c_2.

$I_i(\kappa)$, $i = 3, 4$ is a linear function of κ at $O(\epsilon) \to$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

$I_i(\kappa)$, $i=3,4$ is a linear function of κ at $O(\epsilon)$ $\rightarrow$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

Let $\kappa = 1$, i.e. with propagators $1/(x^2)^{1-\epsilon}$ $\rightarrow$
$\mathcal{F}[I_3(1)]$ and $\mathcal{F}[I_4(1)]$ are given by conventional four-loop momentum Feynman integrals with propagators $1/p^2$.
$\mathcal{F}[I_3(1)] \rightarrow M_{45}$ of Baikov and Chetyrkin.

$$I_3(1) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + 288{\zeta_3}^2 - 378\,\zeta_7\Big) + O(\epsilon^2)$$

$I_i(\kappa)$, $i=3,4$ is a linear function of κ at $O(\epsilon) \to$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

Let $\kappa = 1$, i.e. with propagators $1/(x^2)^{1-\epsilon} \to$
$\mathcal{F}[I_3(1)]$ and $\mathcal{F}[I_4(1)]$ are given by conventional four-loop momentum Feynman integrals with propagators $1/p^2$.
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The second integral $\mathcal{F}[I_4(1)]$ is not a master integral.
We applied `FIRE` to reduce it to master integrals

$M_{01}, M_{11}, M_{35}, M_{13}, M_{36}, M_{12}, M_{21}$.

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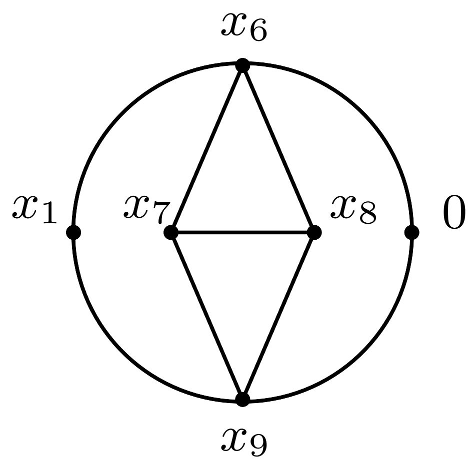
Using inversion $x_i^\mu \to x_i^\mu / x_i^2$ we obtain

$$I_3(1/2) \quad = \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_6^2 x_9^2 x_{89}^2)^{1-\epsilon/2}} =$$



The two-loop subintegral over x_7 and x_8 equals

$$\frac{e^{2\gamma\epsilon}}{\pi^D} \int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)}{(x_{69}^2)^{1-\epsilon/2}}$$

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Method of gluing

[K.G. Chetyrkin & F.V. Tkachov'81, P.A. Baikov & K.G. Chetyrkin,10]

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Gluing by vertex and gluing by line.

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Gluing by vertex and gluing by line.

Let $F_\Gamma(q;d)$ be an l-loop dimensionally regularized scalar propagator massless Feynman integral corresponding to a graph Γ,

$$F_\Gamma(q;d) = C_\Gamma(\epsilon)(q^2)^{\omega/2-l\epsilon} \; ,$$

where $\omega = 4l - 2L$ is the degree of divergence and $C_\Gamma(\epsilon)$ is a meromorphic function which is finite at $\epsilon = 0$ if the integral is convergent.

Gluing

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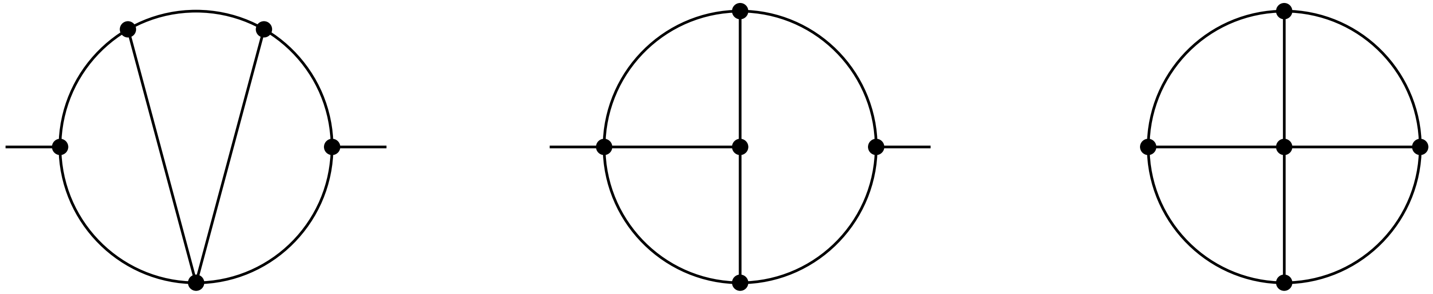

$$C_{\Gamma_1}(0) = C_{\Gamma_2}(0) = 20\zeta_5$$

Let us prove (without calculation) that $I_3(0) = I_4(0)$.

$$I_i(0) = \frac{c_i(\epsilon)}{(x_1^2)^{1+4\epsilon}}, \qquad i = 3, 4\,.$$

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Add to each of these diagrams a new line with the usual propagator $1/x_1^2$, i.e. multiply $I_i(0)$ by $1/x_1^2$ (gluing by a line).

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$$\mathcal{F}\left[\frac{I_i(0)}{x_1^2}\right] = \mathcal{F}\left[\frac{c_i(\epsilon)}{(x_1^2)^{2+4\epsilon}}\right] = c_i(\epsilon)\frac{2^{-10\epsilon}\Gamma(-5\epsilon)}{\Gamma(2+4\epsilon)}(p^2)^{5\epsilon}$$

The pole part in ϵ is independent of p,
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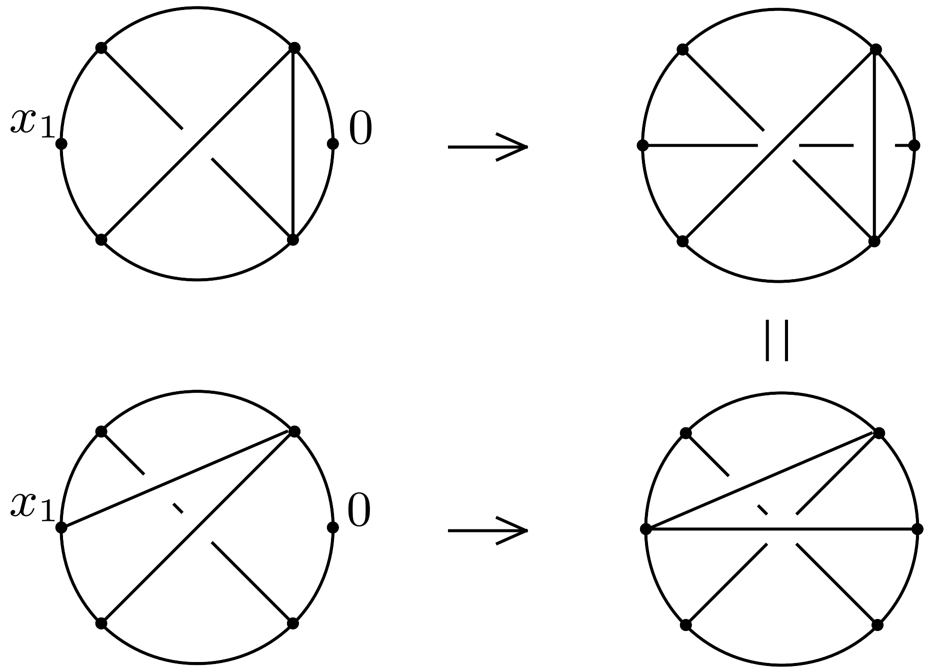

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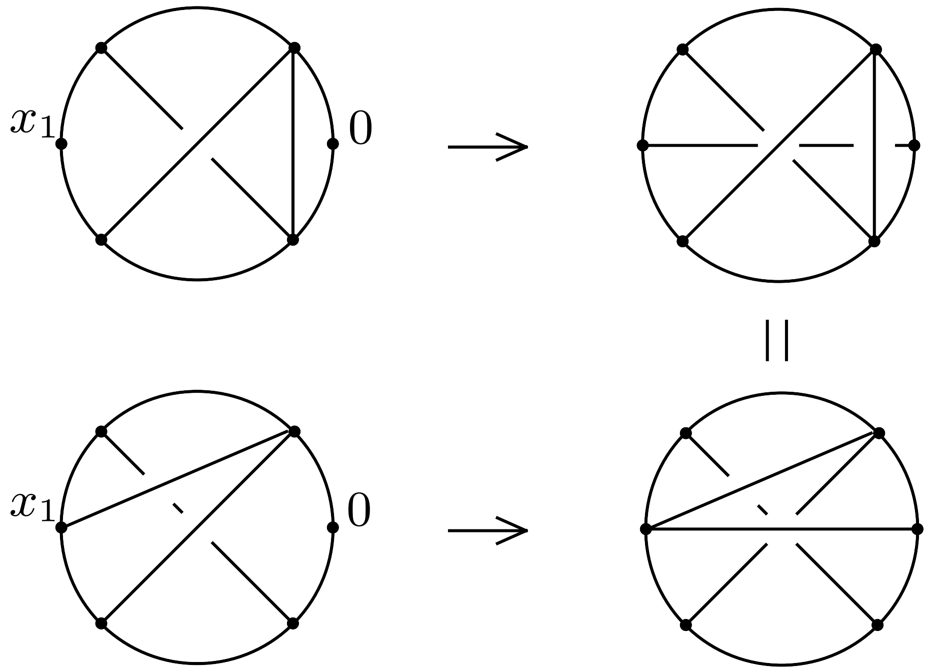

So, $c_3(0) = c_4(0)$ and, therefore, $I_4(0) = I_3(0)$ at $\epsilon = 0$.

Consider I_3 and I_4 with all the indices equal to $1 - \epsilon/2 - \lambda/10$.
Formally, these are $I_3(\kappa)$ and $I_4(\kappa)$ at $\kappa = 1/2 - \lambda/(10\epsilon)$.

$$I_i(1/2 + \lambda/(10\epsilon)) = \frac{c_i(\epsilon, \lambda)}{(x_1^2)^{1-\epsilon/2-9\lambda/10}}, \qquad i = 3, 4\,.$$

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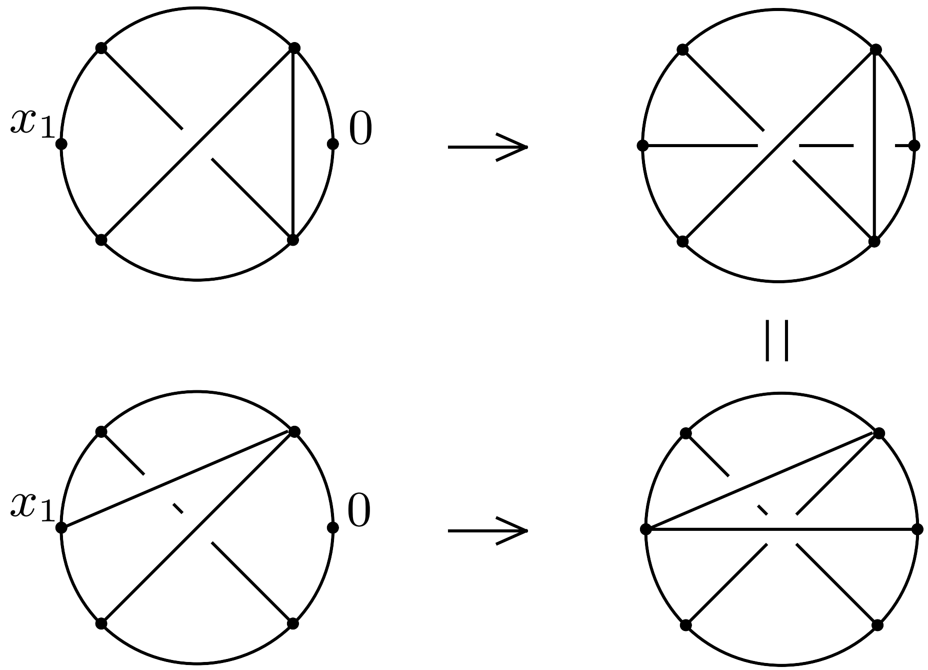

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We obtain

$$\begin{aligned} b_3 &= b_4 = 36{\zeta_3}^2\,, \\ c_3 &= 108\,\zeta_3\zeta_4 + 378\,\zeta_7\,, \\ c_4 &= 108\,\zeta_3\zeta_4 + 180{\zeta_3}^2 - \frac{189}{2}\,\zeta_7\,. \end{aligned}$$

This gives a system of linear relations for b_1, b_2 and c_1, c_2, with the solution

$$a_1 = 5\,\zeta_5\,, \quad b_1 = \frac{5}{378}\pi^6 - 13\,\zeta_3{}^2 + 35\,\zeta_5\,,$$

$$a_2 = -20\,\zeta_5\,, \quad b_2 = -\frac{10}{189}\pi^6 - 8\,\zeta_3{}^2 - 40\,\zeta_5\,,$$

$$c_1 = -\frac{13}{30}\pi^4\zeta_3 - 91\zeta_3{}^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6\,,$$

$$c_2 = -\frac{4}{15}\pi^4\zeta_3 - 16\zeta_3{}^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6$$

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$$I_1 = \frac{5\,\zeta_5}{\epsilon} + \frac{5}{378}\pi^6 - 13\,\zeta_3{}^2 + 35\,\zeta_5$$

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[A.V. Smirnov & M.N. Tentyukov'08]

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Modern sector decompositions

[T. Binoth & G. Heinrich'00; C. Bogner & S. Weinzierl'07; A.V. Smirnov & M.N. Tentyukov'08;

A.V. Smirnov, V.A. Smirnov, & M.N. Tentyukov'10; J. Carter & G. Heinrich'10]

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Why have we succeeded?

$$\gamma_{\mathcal{K}}^{(5)} = \frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7$$

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An analytic six-loop calculation?