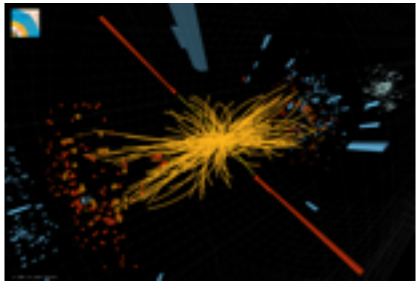
Two-loop form factors in N=4 SYM and QCD



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Brandhuber, Yang, GT 1201.4170 [hep-th]

Brandhuber, Gurdogan, Mooney, Yang, GT 1107.5067 [hep-th]
Brandhuber, Spence, Yang, GT 1011.1899 [hep-th]

Amplitudes 2012, DESY Hamburg 8th March 2012

Plan

- Why form factors ?

- Form factors in N=4 SYM

- Three-point form factor of 1/2 BPS operators in N=4 SYM at two loops:
  - 1. from unitarity 2. from symbols

- Compare N=4 form factor to Higgs + multi-gluon amplitudes in QCD and to N=4 amplitude remainders

Form Factors

- Partially **off-shell** quantities

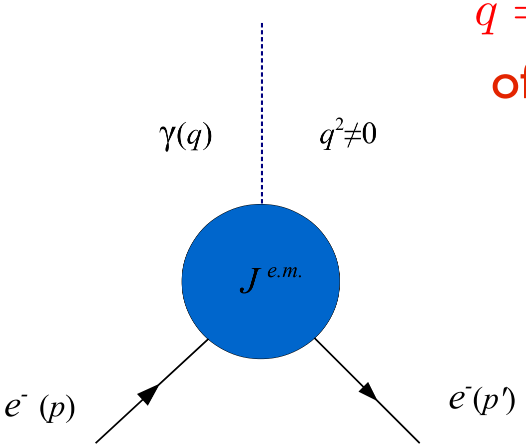
$$F \;=\; \int d^4x\, e^{-iqx}\,\langle state|\mathcal{O}(x)|0\rangle \;=\; \delta^{(4)}(q-p_{state})\,\langle state|\mathcal{O}(0)|0\rangle$$

- Electromagnetic form factor

$q = p - p'$

off shell

$$\langle e^-(p')|\,J_\mu^{e.m.}(0)\,|e^-(p)\rangle \;=$$

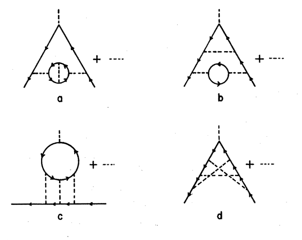


$$J_\mu^{e.m.} \;=\bar\psi\gamma_\mu\psi$$

on shell

on shell

- Three-loop correction to electron g–2

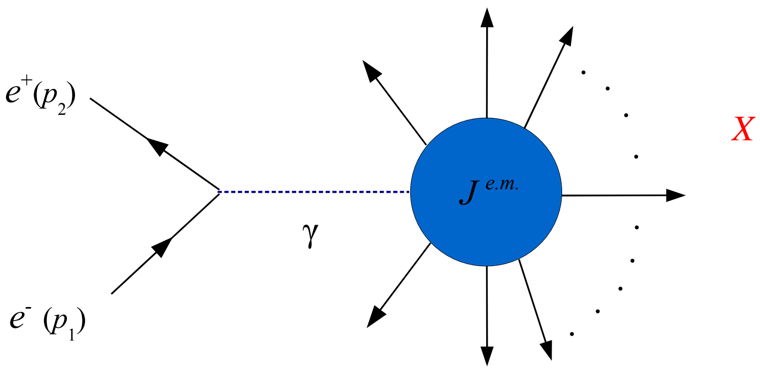
72 diagrams like  $= \ (1.181241456\ldots)\,(\alpha_{\rm e.m.}/\pi)^3$

(Cvitanovic & Kinoshita '74)

(Laporta & Remiddi '96)

  - wild oscillations between the values of each integral
  - final result is $O(1)$
  - another example of unexplained simplicity...

- Form factors appear in several interesting contexts:
  - deep inelastic scattering ($e^- + p \to e^- +$ hadrons)
  - $e^+ e^- \to$ hadrons :



$$= e\,\bar{v}(p_2)\gamma_\mu u(p_1)\,\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}\,(-e)\langle X|\,J_\nu^{e.m.}(0)\,|0\rangle$$

$e^+ e^- \to$ hadrons (X)

all orders in α_{strong}, first order in $\alpha_{\text{e.m.}}$

hadronic electromagnetic current

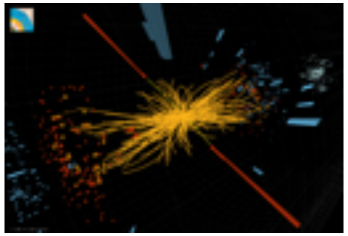
- Total cross section: $$\sigma = \frac{e^4}{2(q^2)^3} L^{\mu\nu} W_{\mu\nu}$$

  - $L^{\mu\nu} = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \frac{q^2}{2}\eta^{\mu\nu}$ from LHS $(q = p_1 + p_2)$

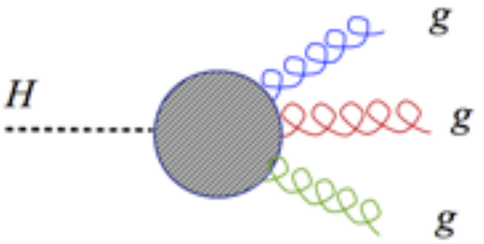
  - $W_{\mu\nu} = \frac{1}{\pi}\mathrm{Im} \int d^4x\, e^{-iqx} \langle 0| T\left[J_\mu^{e.m.}(x)\, J_\nu^{e.m.}(0)\right]|0\rangle$ from RHS

    - encodes our ignorance of QCD dynamics
    - usually evaluated using OPE / models

- Correlation functions appear in the picture



- Higgs + multi-gluon amplitudes
  - at low M_H , dominant Higgs production at the LHC through gluon fusion



  - coupling to gluons through a fermion loop
    - proportional to the mass of the quark $\Rightarrow$ top quark dominates
  - for $M_H < 2\, m_t$ integrate out the top quark

- Effective Lagrangian description $\mathcal{L}_{\rm eff} \;\sim\; H\, \mathrm{Tr}\, F^2$
  - coupling is independent of m_t
  - efficient MHV rules (Dixon, Glover & Khoze; Badger, Glover & Risager; Boels & Schwinn)

- In our language:
  - form factor of $\mathrm{Tr}\,(F_{\mathrm{SD}})^2$ (= amplitude of a different theory!)

$$F_{\mathrm{Tr}F^2_{\mathrm{SD}}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state|\,\mathrm{Tr}\,F^2_{\mathrm{SD}}(x)\,|0\rangle$$

  - in N=4 SYM, this is related to the form factor of $\mathrm{Tr}\,(\phi_{12})^2$

$$F_{\mathrm{Tr}\phi^2_{12}}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\;\; \langle state'|\,\mathrm{Tr}\,\phi^2_{12}(x)\,|0\rangle$$

    - $\mathrm{Tr}\ \phi^2{}_{12}$ and $\mathrm{Tr}\ F_{\mathrm{SD}}{}^2$ part of the same 1/2 BPS supermultiplet
    - supersymmetric form factor of the chiral part of the stress tensor multiplet (Brandhuber, Gurdogan, Mooney, Yang, GT)

- Number of Feynman diagrams for the 3-point form factor in QCD (Gehrmann, Glover, Jaquier, Koukoutsakis)

    - 1 loop: 60

    - 2 loops: 1306

Form Factors in N=4 super YM

i. Tree level

(Brandhuber, Spence, GT, Yang; + Gurdogan & Mooney)

- Simplest form factors: scalar 1/2 BPS operators
  - e.g. $O(x) = \mathrm{Tr}\,(\phi_{12}\,\phi_{12})(x)$ where $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\,\bar{\phi}^{CD}$
  - Sudakov form factor: $< \phi_{12}\,(p_1)\;\;\phi_{12}\,(p_2)\,|\,O\,(0)\,|\,0 >$
    Important note: O is a colour singlet
  - equal to 1 at tree level

- "MHV" family: add positive-helicity gluons

$$\int d^4x\; e^{-iqx}\;\langle g^+(p_1)\cdots\phi_{12}(p_i)\cdots\phi_{12}(p_j)\cdots g^+(p_n)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(x)\;\;|0\rangle$$

tree

$$= \frac{\langle i\,j\rangle^2}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}\;\;\delta^{(4)}(q-\sum_i p_i)$$

- $$F_{\mathrm{MHV}}(1,\ldots,i\ldots,j,\ldots,n) \;=\; \frac{\langle i\, j\rangle^2}{\langle 1\, 2\rangle\cdots\langle n\, 1\rangle}$$

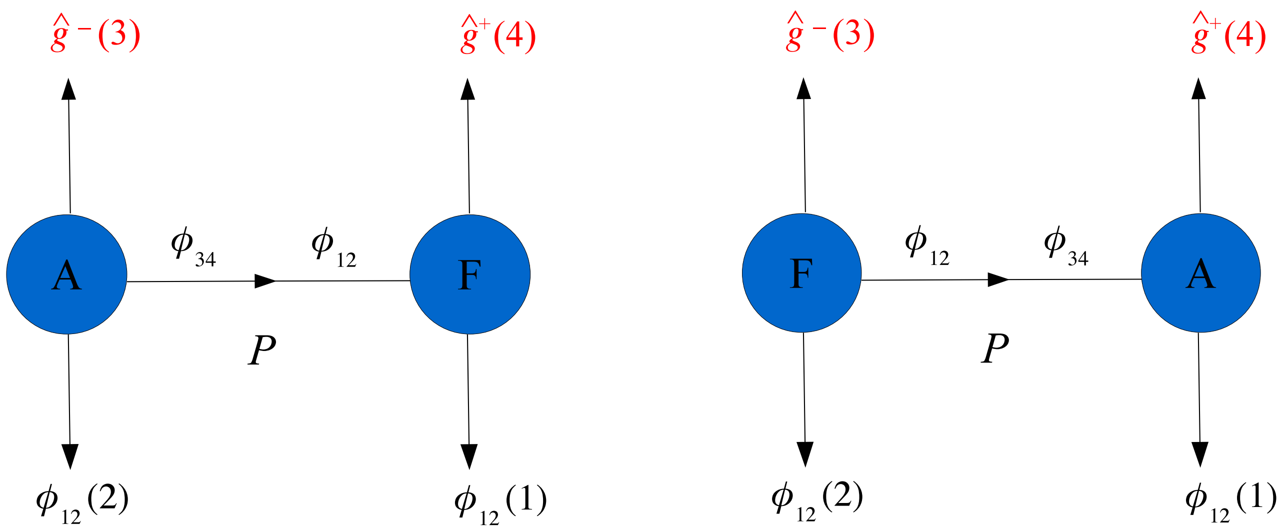
- structure very similar to that of MHV amplitudes in N=4
  - holomorphic function of spinor variables
  - localises on a line in Penrose's twistor space, as MHV amplitudes
  - numerator can be derived from a supersymmetric δ-function

- Non-MHV form factors: add g^-'s in the external state

$$F_{\mathrm{NMHV}}(1,\ldots,4) \;=\; \langle \phi_{12}(p_1)\,\phi_{12}(p_2)\,g^-(p_3)\,g^+(p_4)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\,\rangle$$

- Use on shell techniques, e.g. BCFW recursion relation:

2 recursive diagrams



$$F_{\mathrm{NMHV}}(1,\ldots,4) \;=\; \frac{[24]^2}{[23][34]}\,\frac{1}{s_{234}}\,\frac{\langle 1\,|q|\,4]}{\langle 1\,|q|\,2]} \;+\; \frac{\langle 13\rangle^2}{\langle 34\rangle\langle 41\rangle}\,\frac{1}{s_{341}}\,\frac{\langle 3\,|q|\,2]}{\langle 1\,|q|\,2]}$$

ii. Supersymmetric form factors

(Brandhuber, GT, Yang; + Gurdogan & Mooney; Bork, Kazakov, Vartanov)

- In N=4 SYM: re-package component amplitudes into superamplitudes

- From form factors to super form factors:
  - supersymmetrise the state (Nair)
  - we can also supersymmetrise the operator
  - supersymmetry relates $\mathrm{Tr}\ \phi^2{}_{12}$ and $\mathrm{Tr}\ F_{\mathrm{SD}}{}^2$ form factors

Chiral part of the stress-tensor multiplet

(Eden, Heslop, Korchemsky, Sokatchev)

$$\mathcal{T}(x,\theta^+) \;:=\; \mathcal{T}(x,\theta^+,\bar\theta_- = 0; u)$$
$$= \; \mathrm{Tr}(\phi^{++}\phi^{++}) + \cdots + \frac{1}{3}(\theta^+)^4\,\mathcal{L}$$

- here $\theta_\alpha^{+a} := \theta_\alpha^A u_A^{+a}$, $\quad a = 1{,}2 \quad \alpha = 1{,}2$ (harmonic projection)
- natural choice in order to match to Nair chiral superspace
- lowest component is $\mathrm{Tr}\,(\phi_{12}\;\phi_{12})(x)$; top component is the (on-shell) Lagrangian

$$\mathcal{L} \;=\; \mathrm{Tr}\Big[-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{2}g\lambda^{\alpha A}[\phi_{AB},\lambda_\alpha^B] - \frac{1}{8}g^2[\phi^{AB},\phi^{CD}][\phi_{AB},\phi_{CD}]\Big]$$

- obtained from the complete stress tensor multiplet by setting $\bar\theta_- = 0$

- Our super form factor is then:

$$\mathcal{F} \;=\; \int d^4x\, d^4\theta^+ \; e^{-iqx \,-\, i\gamma^{\alpha}_{+a}\theta^{+a}_{\alpha}} \; \langle 1 \cdots n | \mathcal{T}(x, \theta^+) | 0 \rangle$$

  - depends on q and γ^{α}_{+a} (conjugate to x and θ^{+a}_{α})
  - $<1 \,....\, n \,|\; = \;< 0 \,|\, \Phi(p_1,\eta_1) \;......\; \Phi(p_n,\eta_n)$ Nair superstate

- From supersymmetric Ward identities:

$$\mathcal{F} \;=\; \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\; \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+;i})\; \delta^{(4)}(\sum_i \lambda_i \eta_{-;i})\; R$$

  - Grassmann variables associated to particles:

$$\eta_{\pm a,i} \;:=\; \bar{u}^{A}_{\pm a}\eta_{A,i}$$

- Our super form factor is then:

$$\mathcal{F} \;=\; \int d^4x\, d^4\theta^+ \; e^{-iqx\,-\,i\gamma^{\alpha}_{+a}\theta^{+a}_{\alpha}} \;\langle 1\cdots n|\mathcal{T}(x,\theta^+)|0\rangle$$

  - depends on q and γ^{α}_{+a} (conjugate to x and θ^{+a}_{α})
  - <1 .... n | = < 0 | $\Phi(p_1,\eta_1)$ ...... $\Phi(p_n,\eta_n)$ Nair superstate

$$\Phi(p,\eta) \;=\; g^+(p) + \eta_A\psi^A(p) + \frac{\eta_A\eta_B}{2!}\phi^{AB}(p) + \epsilon^{ABCD}\frac{\eta_A\eta_B\eta_C}{3!}\tilde{\psi}_D(p) + \eta_1\eta_2\eta_3\eta_4 g^-(p)$$

- From supersymmetric Ward identities:

$$\mathcal{F} \;=\; \delta^{(4)}\big(q - \sum_i \lambda_i\tilde{\lambda}_i\big)\; \delta^{(4)}\big(\gamma_+ - \sum_i \lambda_i\eta_{+;i}\big)\; \delta^{(4)}\big(\sum_i \lambda_i\eta_{-;i}\big)\; R$$

  - Grassmann variables associated to particles:

$$\eta_{\pm a,i} \;:=\; \bar{u}^{A}_{\pm a}\eta_{A,i}$$

iii. Some explicit examples

- $\mathrm{Tr}\,\phi^2{}_{12}$ and $\mathcal{L}$ ($\ni \mathrm{Tr}\,F_{\mathrm{SD}}{}^2$) form factors:

$$\mathcal{F}_{\mathrm{Tr}(\phi^{++})^2} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\ \delta^{(4)}(\sum_i \lambda_i \eta_{-;i})\ R$$

$$\mathcal{F}_{\mathcal{L}} = \delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\ \delta^{(8)}(\sum_i \lambda_i \eta_i)\ R$$

valid to all loops

- Super MHV, tree: $R^{\mathrm{MHV}} = \frac{1}{\langle 12 \rangle \cdots \langle n1 \rangle}$

$$\mathcal{F}_{\mathcal{T}}^{\mathrm{MHV}} = \frac{\delta^{(4)}(q - \sum_i \lambda_i \tilde{\lambda}_i)\ \delta^{(4)}(\gamma_+ - \sum_i \lambda_i \eta_{+;i})\ \delta^{(4)}(\sum_i \lambda_i \eta_{-;i})}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

MHV form factor of on-shell Lagrangian

$$\langle 1_{g^+} \cdots i_{g^-} \cdots j_{g^-} \cdots n_{g^+} | \, \mathrm{Tr}\, F_{\mathrm{SD}}^2(0) \, |0\rangle \;=\; \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}$$

- same as Higgs + multi-gluon amplitude (Dixon, Glover, Khoze)
- same as gluon MHV amplitude for $q=0$: $\mathcal{F}_{\mathcal{L}}|_{q=0} \sim \frac{\partial}{\partial(1/g^2)} \mathcal{A}$
  (Lagrangian insertion trick) (Intriligator; Eden, Howe, Schubert, Sokatchev, West)

scalar MHV form factor

$$\langle 1_{g^+} \cdots i_{\phi_{12}} \cdots j_{\phi_{12}} \cdots n_{g^+} | \, \mathrm{Tr}\,(\phi_{12}\phi_{12})(0) \, |0\rangle \;=\; \frac{\langle i\, j\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n\,1\rangle}$$

maximally non-MHV (from Grassmann Fourier transform)

$$\langle 1_{g^-} \cdots n_{g^-} | \, \mathrm{Tr}\, F_{\mathrm{SD}}^2(0) \, |0\rangle \;=\; \frac{q^4}{[1\,2]\,[2\,3] \cdots [n\,1]}$$

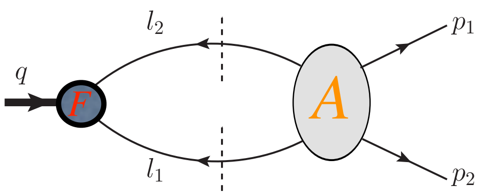
- same as "minus only" $A_n(H,\, g_1^-, \cdots, g_n^-)$ (Dixon, Glover, Khoze)

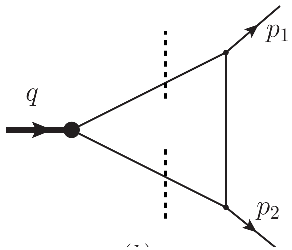
iv. One loop

(Brandhuber, Spence, GT, Yang; + Gurdogan & Mooney)

- Form factors from unitarity
  - simplest application: Sudakov form factor

$$F(q^2) \;:=\; \langle \phi_{12}(p_1)\phi_{12}(p_2)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\rangle \qquad\qquad q \;:=\; p_1+p_2$$





$$\left[F(q^2)\right]^{1\,\mathrm{loop}} \;=\; 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big] \qquad D=4-2\epsilon \;\; ☜$$

regulates infrared divergences

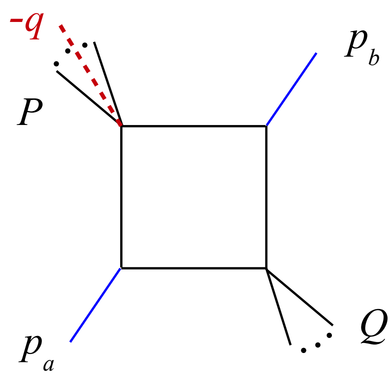
- agrees with a pioneering calculation of van Neerven (van Neerven '86)

- MHV: $F_{\text{MHV}} = \langle g^+(p_1)\cdots\phi_{12}(p_i)\cdots\phi_{12}(p_j)\cdots g^+(p_n)|\,\text{Tr}(\phi_{12}\phi_{12})(0)\,|0\rangle$

  - one loop:

$$F^{(1)} = F^{(0)}\Big[-\sum_{i=1}^{n}\frac{(-s_{ii+1})^{-\epsilon}}{\epsilon^2} + \sum_{a,b}\text{Fin}^{2\text{me}}(p_a,p_b,P,Q)\Big]$$

  - one-loop result proportional to tree $F^{(0)}$
  - sum of finite two-mass easy box functions
  - result very similar to the MHV amplitude...
  - ...except that q can be inserted in all possible ways ("nonplanarity" of momentum flow)



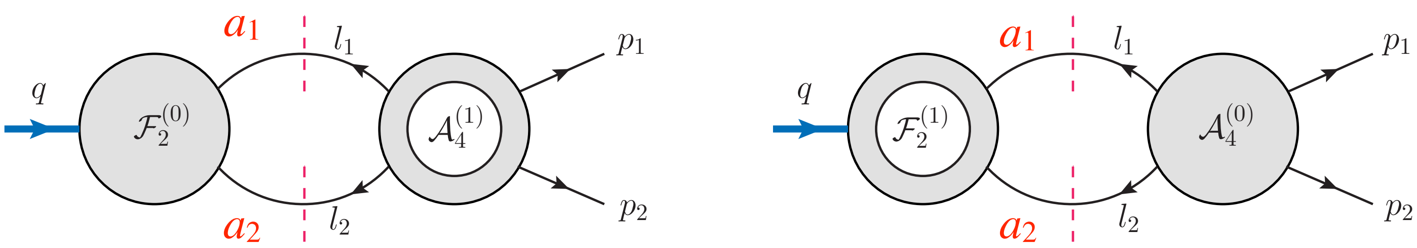
a "two-mass easy" box function: two opposite legs, p_a and $p_{b,}$ are massless

v. Two loops

- Sudakov:

$$F(q^2) \;:=\; \langle \phi_{12}(p_1)\phi_{12}(p_2)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\rangle$$

  - simple illustration of the technique



  - F proportional to $\delta^{a_1 a_2}$

  - non-planar one-loop amplitude are also relevant in the cuts!

- One-loop complete amplitude (planar + non-planar)

Complete: $\mathcal{A}^{(1)} \;=\; A^{(1)}_{\rm P} \;+\; A^{(1)}_{\rm NP}$ where

P: $$A_{\rm P}^{(1)} \;=\; N \sum_{\sigma\in S_n/\mathbb{Z}_n} {\rm Tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}})\, A^{[1]}_{n;1}(\sigma_1,\ldots,\sigma_n)$$

NP: $$A_{\rm NP}^{(1)} \;=\; \sum_{c=2}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_{n;c}} {\rm Tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_{c-1}}})\,{\rm Tr}(T^{a_{\sigma_c}}\cdots T^{a_{\sigma_n}})\, A^{[1]}_{n;c}(\sigma_1,\ldots,\sigma_n)$$

- $A^{[1]}{}_{n;c}$ linear combinations of colour-ordered amplitudes $A^{[1]}{}_{n;1}$
  (Bern, Dixon, Dunbar, Kosower)

- contracting with tree form factor ~ $\delta^{a_1 a_2}$ we get:

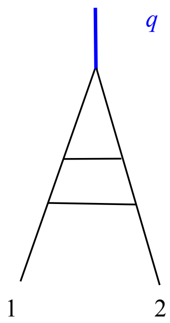
P: $$N\,\delta^{a_1a_2}{\rm Tr}(T^{a_1}T^{a_2}T^XT^Y) \;=\; N^2\,{\rm Tr}(T^XT^Y) \;=\; N^2\,\delta^{XY}$$

NP $$\delta^{a_1a_2}\,{\rm Tr}(T^{a_1}T^{a_2})\,{\rm Tr}(T^XT^Y) \;=\; N^2\,{\rm Tr}(T^XT^Y) \;=\; N^2\,\delta^{XY}$$

both leading in colour!

Final result obtained very easily:

$$F^{(2)}(q^2) = \quad 4$$



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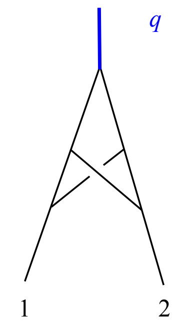


- agrees with van Neerven
- two-loop result exponentiates as expected:

$$\Big[F(q^2)\Big]^{1\,\mathrm{loop}} \;=\; 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big]$$

$$\Big[\mathrm{Log}\;F(q^2)\Big]^{2\,\mathrm{loop}} \;=\; (-q^2)^{-2\epsilon}\Big[\frac{\zeta_2}{\epsilon^2}+\frac{\zeta_3}{\epsilon}+\mathcal{O}(\epsilon)\Big]$$

- result is transcendental (non-planar integral topology)
- recent three-loop calculation (Henn, Huber, Gehrmann)

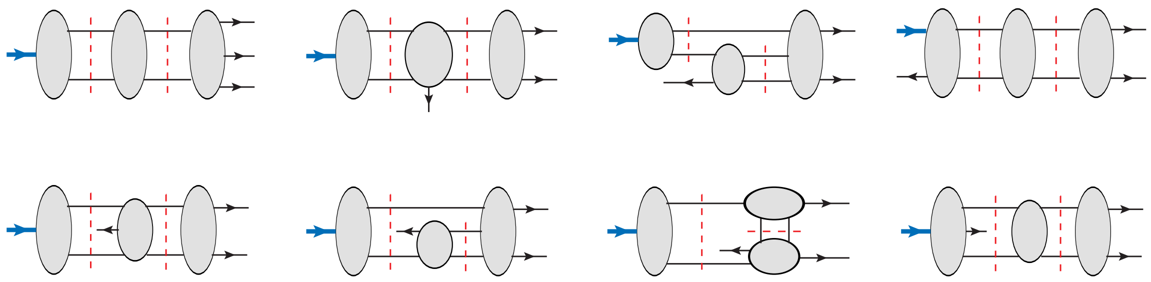
3-point form factor at 2 loops

(Brandhuber, GT, Yang)

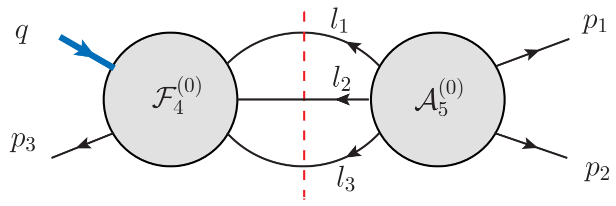
- MHV $F_3(1,2,3) = \langle \phi_{12}(p_1)\, \phi_{12}(p_2) g^+(p_3)|\, \mathrm{Tr}(\phi_{12}\phi_{12})(0)\ |0\rangle$
  - Tree: $F_3^{\mathrm{tree}} = \dfrac{\langle 1\,2\rangle}{\langle 2\,3\rangle\langle 3\,1\rangle}$
  - Loops: $F_3^{(L)} = F_3^{\mathrm{tree}}\, \mathcal{G}_3^{(L)}(1,2,3)$
    - $\mathcal{G}_3^{(L)}$ helicity-blind function
    - totally symmetric under legs exchange
    - one-loop: IR divergences + sum of finite 2me box

- First 2-loop calculation: with generalised unitarity

  1. detect all possible integrals and coefficients with iterated two-particle cuts



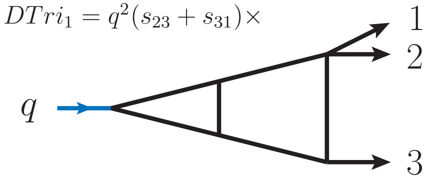
  2. next, fix all remaining ambiguities using three-particle cuts, such as



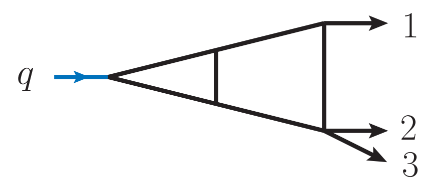
- **Final result:**

$$\boxed{\frac{F_3^{(2)}}{F_3^{\rm tree}} = \sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}}$$

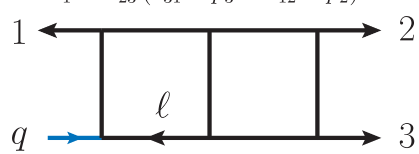
$DTri_1 = q^2(s_{23}+s_{31})\times$



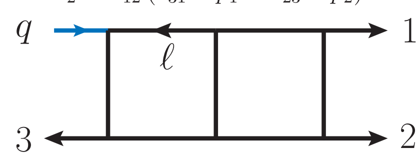
$DTri_2 = q^2(s_{12}+s_{31})\times$



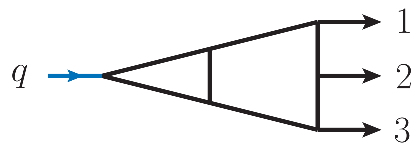
$DBox_1 = s_{23}\left(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2\right)\times$



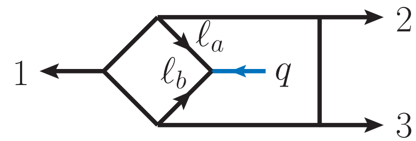
$DBox_2 = s_{12}\left(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2\right)\times$



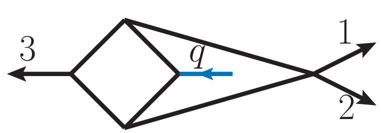
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



- result expressed in terms of two-loop planar and non-planar integrals

- Evaluate integrals with sophisticated technologies:
    - AMBRE (Gluza, Kajda, Riemann, Yundin) (only for planar or non-planar with 1 scale)
    - MB.m (Czakon):
    - MBresolve.m (Smirnov & Smirnov)
- Several analytic results (Gehrmann & Remiddi)
    - variables: $u := \frac{s_{12}}{q^2}\,,\ v := \frac{s_{23}}{q^2}\,,\ w := \frac{s_{31}}{q^2}$, with $q = p_1 + p_2 + p_3$
    - all known integrals appearing in our answer are transcendental
    - expressed in terms of Goncharov polylogarithms...
    - ...which disappear in our final expression for the remainder

- Numerical results for the two-loop form factor
  - for various values of $(-s_{12}, -s_{23}, -s_{31})$:

$$\begin{aligned}
(1,1,1):\quad & \frac{4.5}{\epsilon^4}+\frac{0.}{\epsilon^3}+\frac{6.12223}{\epsilon^2}-\frac{16.7052}{\epsilon}-18.2484\pm 0.02+\mathcal{O}(\epsilon)\,,\\
(1,1,2):\quad & \frac{4.5}{\epsilon^4}-\frac{2.07944}{\epsilon^3}+\frac{7.98765}{\epsilon^2}-\frac{18.9491}{\epsilon}-7.3182\pm 0.02+\mathcal{O}(\epsilon)\,,\\
(1,2,2):\quad & \frac{4.5}{\epsilon^4}-\frac{4.15888}{\epsilon^3}+\frac{9.2099}{\epsilon^2}-\frac{23.0025}{\epsilon}+1.8686\pm 0.02+\mathcal{O}(\epsilon)\,,\\
(1,2,3):\quad & \frac{4.5}{\epsilon^4}-\frac{5.37528}{\epsilon^3}+\frac{11.6703}{\epsilon^2}-\frac{25.9714}{\epsilon}+10.6624\pm 0.03+\mathcal{O}(\epsilon)\,.
\end{aligned}$$

  - sanity check: exponentiation of infrared divergences
  - next: construct finite remainder

The form factor remainder

- Construct ABDK/BDS remainder, $\mathcal{R}$

$$\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)} - \frac{1}{2}\left(\mathcal{G}_n^{(1)}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\,\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)$$

  - Ingredients:
    - two-loop form factor $\mathcal{G}_n^{(2)}$
    - one-loop form factor $\mathcal{G}_n^{(1)}$ to higher orders in ϵ
    - $f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2$ contains cusp and collinear anomalous dimensions (integrability!), $C^{(2)}(\epsilon) = 4\,\zeta_4$
  - Properties:
    - finite
    - trivial collinear limits $\mathcal{R}_n^{(2)} \to \mathcal{R}_{n-1}^{(2)}$
    - in particular: $\mathcal{R}_3^{(2)} \to 0$ (there is no Sudakov remainder $\mathcal{R}_2^{(2)}$!)

- Numerical checks:

  - recall: $u = s_{12} / q^2$, $v = s_{23} / q^2$, $w = s_{31} / q^2$ where $q^2 = (p_1 + p_2 + p_3)^2$

(u, v, w)	numerical $\mathcal{R}_3^{(2)}$	est. error
(1/3, 1/3, 1/3)	-0.1519	0.02
(1/4, 1/4, 1/2)	-0.1203	0.02
(1/5, 2/5, 2/5)	-0.1301	0.02
(1/2, 1/3, 1/6)	-0.1080	0.03

  - ϵ^{-4}, ϵ^{-3} infrared poles cancel with negligible errors
  - ϵ^{-2}, ϵ^{-1} poles cancel with precision 10^{-13}, 10^{-7} respectively
  - remainder is very small!

The remainder from symbols

- Strategy:
  - define an appropriate remainder function:
    - finite
    - trivial/understood collinear limits
  - determine its symbol (Goncharov, Spradlin, Vergu, Volovich)
    - remainder is a transcendentality-four function (two loops)
    - impose symmetries and physical constraints
  - fix beyond the symbol terms
  - lift symbols to functions
    - doable at least in certain controlled cases

Examples of this strategy so far:

- Six-point MHV remainder (Goncharov, Spradlin, Volovich, Vergu)
- MHV remainder in $(1+1)$-dim kinematics (Heslop & Khoze)
  - 2 loops, all n
  - 3 loops, all n (7 undetermined constants)
- MHV remainder, any n (Caron-Huot)
- Six-point, MHV remainder at 3 loops
  (Dixon, Drummond, Henn; Caron-Huot, He)
- Six-point NMHV remainder at 2 loops (Dixon, Drummond, Henn)
- Our example: three-point (1 leg off shell, 3 on shell) form factor remainder at 2 loops

Crash review of symbols

- The symbol of a transcendentality-k function is an element of the k-fold tensor product of rationals
(Goncharov, Spradlin, Vergu, Volovich)
    - $$f^{(k)} \longrightarrow \mathcal{S}[f^{(k)}] \; = \; R_1 \otimes \cdots \otimes R_k$$

- Recursive definition:
    - $$df^{(k)} \; = \; \sum_a f_a^{(k-1)} \, d \, \log R_a \qquad \Rightarrow \qquad \mathcal{S}[f^{(k)}] \; = \; \sum_a \mathcal{S}[f_a^{(k-1)}] \otimes R_a$$

- Two key properties:
    - $$\cdots \otimes R_a \, R_b \otimes \cdots \; = \; \cdots \otimes R_a \otimes \cdots + \cdots \otimes R_b \otimes \cdots$$
    - $$\cdots \otimes c \, R_a \otimes \cdots \; = \; \cdots \otimes R_a \otimes \cdots$$ where c = constant

- Examples:
  - $\mathcal{S}\,[\log x] = x\,,\quad \mathcal{S}\,[\mathrm{Li}_2\,(x)] = -\,((1-x)\otimes x\,),\quad \mathcal{S}\,[\mathrm{Li}_3\,(x)] = -\,((1-x)\otimes x\otimes x)$
  - $\mathcal{S}\,[\log x\,\log y] = x\otimes y + y\otimes x$ (<u>note</u>: $x\otimes y$ is not the symbol of a function)

- The symbol transforms complicated polylogarithmic identities into algebraic ones, e.g.
  - $\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(z)\log(1-z) - \dfrac{\pi^2}{6} = 0$ translates into

$$-((1-z)\otimes z) - (z\otimes(1-z)) + (1-z)\otimes z + z\otimes(1-z) = 0$$

  - loss of information on π's (beyond-the-symbol terms) and branch cuts where the function has to be evaluated

Constructing the symbol of $\mathcal{R}$

- Entries: $(u, v, w, 1\text{-}u, 1\text{-}v, 1\text{-}w)$ $u = s_{12} / q^2$, $v = s_{23} / q^2$, $w = s_{31} / q^2$
  - from inspecting the relevant integrals in Gehrmann & Remiddi
- First entry: (u, v, w) for correct branch cuts
  (Gaiotto, Maldacena, Sever, Vieira)
  - $\mathcal{S}[\mathcal{R}^{(2)}] = \sum_{i,j} P_{i,j}^2 \otimes \mathcal{S}[\mathrm{disc}_{i,j}\mathcal{R}^{(2)}]$ with $P_{ij} := p_i + \ \ldots + p_j$
  - also satisfied at the GR integral function level
- Further constraints on entries
  (Gaiotto, Maldacena, Sever, Vieira; Caron-Huot; Dixon, Drummond, Henn)
  - second & last entries

- Trivial collinear limits
  - $\mathcal{R}_3^{(2)}(u,v,w) \rightarrow 0$ as u or v or w $\rightarrow 0$
  - reminder: $u := \frac{s_{12}}{q^2}\,,\ v := \frac{s_{23}}{q^2}\,,\ w := \frac{s_{31}}{q^2}$
- Symmetry $\mathcal{R}_3^{(2)}(u,v,w) = \mathcal{R}_3^{(2)}(v,u,w) = \mathcal{R}_3^{(2)}(w,v,u)$
- Integrability (Goncharov; Goncharov, Spradlin, Vergu, Volovich)
  - the symbol must correspond to a function!
  - for any two adjacent entries i and $i+1$:

$$\sum d\log R_i \wedge d\log R_{i+1} \left[R_1 \otimes \cdots \otimes R_{i-1} \otimes R_{i+2} \otimes \cdots \otimes R_k\right] = 0$$

- The unique symbol satisfying these requirements:

$$
\begin{aligned}
\mathcal{S}^{(2)} \;=\; & -2u\otimes(1-u)\otimes(1-u)\otimes\frac{1-u}{u}+u\otimes(1-u)\otimes u\otimes\frac{1-u}{u}\\
& -u\otimes(1-u)\otimes v\otimes\frac{1-v}{v}-u\otimes(1-u)\otimes w\otimes\frac{1-w}{w}\\
& -u\otimes v\otimes(1-u)\otimes\frac{1-v}{v}-u\otimes v\otimes(1-v)\otimes\frac{1-u}{u}\\
& +u\otimes v\otimes w\otimes\frac{1-u}{u}+u\otimes v\otimes w\otimes\frac{1-v}{v}\\
& +u\otimes v\otimes w\otimes\frac{1-w}{w}-u\otimes w\otimes(1-u)\otimes\frac{1-w}{w}\\
& +u\otimes w\otimes v\otimes\frac{1-u}{u}+u\otimes w\otimes v\otimes\frac{1-v}{v}\\
& +u\otimes w\otimes v\otimes\frac{1-w}{w}-u\otimes w\otimes(1-w)\otimes\frac{1-u}{u}\\
& +\text{ cyclic permutations}\,.
\end{aligned}
$$

  - overall coefficient fixed from numerics for $n = 3$
    (from collinear limits for $n > 3$)
  - can we determine uniquely the function with this symbol?

- Yes!
  - $\mathcal{S}^{(2)}$ satisfies a particular relation of Goncharov, Spradlin, Vergu & Volovich:

$$\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$$

  - $\Rightarrow$ can re-express as a linear combination of classical polylogarithms only

$$\log x_1 \log x_2 \log x_3 \log x_4\;,\; \mathrm{Li}_2(x_1)\log x_2 \log x_3\;,\; \mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\;,\; \mathrm{Li}_3(x_1)\log x_2 \text{ and } \mathrm{Li}_4(x_i)$$

  - we find the following arguments:

$$\left(u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right)$$

- Final answer fits on one line (for appropriately chosen fonts):

- Final answer:

$$\mathcal{R}_3^{(2)} = -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i^{-1})\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4$$

  - $u_1 = u\,,\ \ u_2 = v\ \ ,\ u_3 = w$
  - $\mathrm{J}_4(z) := \mathrm{Li}_4(z)-\log(-z)\mathrm{Li}_3(z)+\frac{\log^2(-z)}{2!}\mathrm{Li}_2(z)-\frac{\log^3(-z)}{3!}\mathrm{Li}_1(z)-\frac{\log^4(-z)}{48}\ .$
  - beyond the symbol terms: fixed using collinear limits
  - no Goncharov polylogarithms!

- Next: QCD

Form factors in QCD

- Higgs + 3 partons (Koukoutsakis 2003; Gehrmann, Glover, Jaquier & Koukoutsakis 2011)
  - $H\, g^+\, g^-\, g^-$ MHV
  - $H\, g^+\, g^+\, g^+$ maximally non-MHV
  - $H\, q\ \ \bar{q}\ \ g$ fundamental quarks

$$F^{\rm tree}(H, g_1^-, g_2^-, g_3^+) \;=\; \frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

$$F^{\rm tree}(H, g_1^+, g_2^+, g_3^+) \;=\; \frac{q^4}{[1\,2]\,[2\,3]\,[3\,1]}$$

$$q^2 \;=\; M_H^2$$

- In N=4 SYM:
  - $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$ both derived from super form factor
  - from supersymmetric Ward identities:

$$\frac{F^{(L)}(g_1^-, g_2^-, g_3^+)}{F^{\rm tree}(g_1^-, g_2^-, g_3^+)} \;=\; \frac{F^{(L)}(g_1^+, g_2^+, g_3^+)}{F^{\rm tree}(g_1^+, g_2^+, g_3^+)} \;=\; \mathcal{G}^{(L)}(u,v,w) \quad \leftarrow \text{what we computed}$$

- QCD answer from Gehrmann, Glover, Jaquier & Koukoutsakis :
    - expressed in terms of (a few pages of) Goncharov polylogarithms
    - entirely expected because of expansion as $\sum$ (coefficient x integral) !
        - e.g. scalar non-planar double box does not satisfy the Goncharov et al criterion
- Next, relate N=4 SYM and QCD form factors:
    - take maximally transcendental piece of $(H\, g^{+}\, g^{-}\, g^{-})$ and $(H\, g^{+}\, g^{+}\, g^{+})$
    - convert the Catani remainder into our ABDK/BDS-type remainder

in practice: $$\mathcal{R}^{(2)} = F^{(2)}_{\mathrm{GGJK}} - \frac{1}{2}\,(F^{(1)}_{\mathrm{GGJK}})^2$$

- We find a surprising relation...

$$\boxed{\mathcal{R}^{(2)}_{H\,g^-g^-g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{H\,g^+g^+g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{\mathcal{N}=4\,\text{SYM}}}$$

!

  - from symbol and numerics
  - all Goncharov polylogarithms in QCD results can be eliminated in favour of classical polylogarithms

see also Claude Duhr's talk ☜

- Nothing similar seems to hold for the $(H, q, \bar{q}, g)$ form factor

  - maximally transcendental part does not satisfy Goncharov et al criterion
  - interesting simplifications may still occur...

• Final surprise: amplitude vs form factor remainders

- the six-point MHV amplitude remainder is built out of six variables $(u, v, w; y_u, y_v, y_w)$:
  - cross ratios: $u := \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,, \quad v := \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}\,, \quad w := \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$
  - y variables: $y_u := \frac{u - z_+}{u - z_-}\,, \quad y_v := \frac{v - z_+}{v - z_-}\,, \quad y_w := \frac{w - z_+}{w - z_-}$

$$z_\pm := \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]\,, \quad \Delta := (1 - u - v - w)^2 - 4\,uvw$$

- amplitude remainder is dual conformal invariant
  (Drummond, Henn, Korchemsky & Sokatchev)

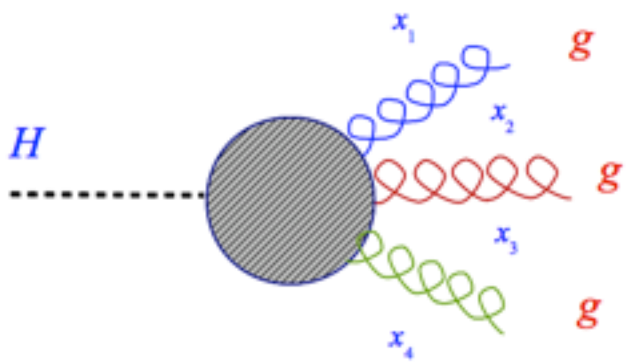
- form factor remainder has no dual conformal invariance

- three-point form factor variables:

$$u := \frac{x_{13}^2}{x_{14}^2}\,, \quad v := \frac{x_{24}^2}{x_{14}^2}\,, \quad w := \frac{x_{34}^2}{x_{14}^2}$$

$$u + v + w = 1$$

written in a slightly provocative way...



- Symbol of 6-pt MHV amplitude remainder has two parts:

$$\mathcal{S}^{(2)}_{6,\,\mathrm{ampl}} = \hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w) + \tilde{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w;y_u,y_v,y_w)$$

  - both $\hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w)$ and $\tilde{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w;y_u,y_v,y_w)$ have trivial collinear limits (independently)

- We find: $\boxed{\mathcal{S}^{(2)}_{3,\,\mathrm{form\ factor}}(u,v,w) = \hat{\mathcal{S}}^{(2)}_{6,\,\mathrm{ampl}}(u,v,w)}$

  - identify the (independent) cross ratios (u,v,w) with the (dependent) form factor ratios (u,v,w)
  - In general, form factor remainder depends on $3n$ - 7 ratios, amplitude remainder depends on $3n$ - 15 cross ratios.

- Reminiscent of a strong coupling observation...
  - 4-pt form factor in (1+1)-dimensional kinematics expressed in terms of the octagon remainder function (Maldacena, Zhiboedov)
- More investigations are under way
  - (1+1)-dimensional kinematics
  - 3 loops

Summary

- Hidden structures in (amplitudes &) form factors

- Form factors in N=4 super Yang-Mills
  - tree, one and two loops

- Three-point form factor in N=4 super Yang-Mills & QCD
  - remainder function from symbols and explicit calculations
  - relation to Higgs + multi-gluon QCD remainder...
  - ...and to the N=4 six-point MHV remainder

Open questions

- Further relations between amplitude and form factor remainders? (there are no coincidences in N=4 SYM...)
- More loops, more legs
- Further applications of symbol to QCD?
- Connection to correlation functions?
- Representation in terms of Wilson lines?
- Recursion relations for form factors integrands?
- ....