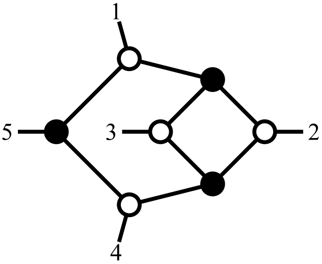
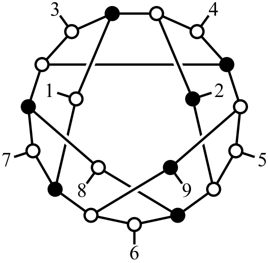
Non-planar leading singularities in $\mathcal{N}=4$ SYM





Jaroslav Trnka

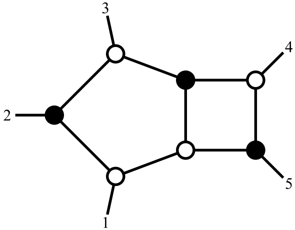
Princeton University

With Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo and Alexander Postnikov

Leading singularities

Leading singularities in planar $\mathcal{N}=4$ SYM:

- contain all information about on-shell scattering amplitudes.
- finite number of them for given n and k at all loops, as a consequence of their Grassmanian origin.
- They are represented by a special class of on-shell diagrams.

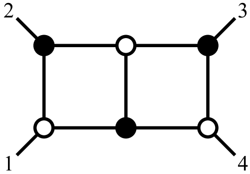


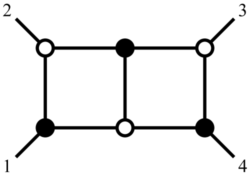
- In this case leading singularity $=$ maximal cut $=$ on-shell diagram.

Composite leading singularities

Let us take a trivial example of 4pt 2-loop amplitude. It is given by one double box + cyclic.

- Cut 7 propagators you see and get two graphs that contribute:





- Each of them is represented by a function like

$$\int dz\, F(z) = \int \frac{dz}{z(z-1)\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

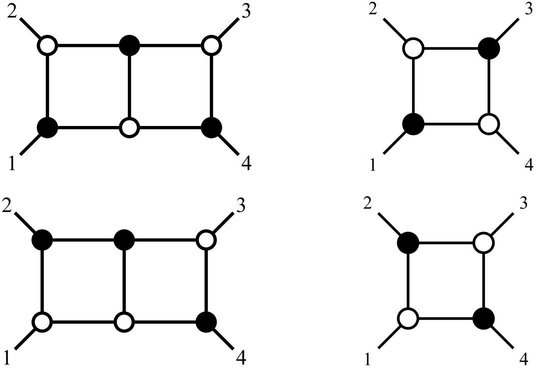
Different contours can give different residues $R_1, R_2, \ldots$.

Composite leading singularities

Two different ways how to store information about the cut:

- Take the form $F(z) \to$ maximal cuts.
- List of residues $R_i \to$ leading singularities.

This leading singularity diagram is an on-shell diagram which is not reduced.

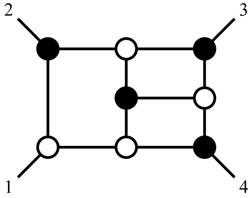


$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Composite leading singularities

Composite leading singularity is a list of rational functions = different ways how to reduce the initial reducible on-shell graph. Let us take

3-loop 4pt example.



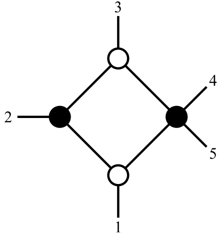
- This represents a function with two integrations left

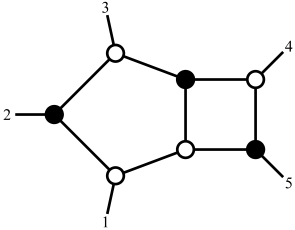
$$\int dz_1\, dz_2\, F(z_1, z_2)$$

- Chain of two reductions to get reduced graph (here again 4pt box).
- The LS is a list of residues of all possible z_1 and z_2 contours.

Colored graphs and on-shell diagrams

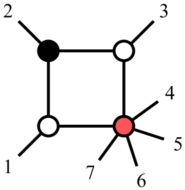
- For the leading singularity it is often convenient to use *colored graphs* when we glue together also higher n and k amplitudes but they are equivalent to on-shell diagrams.
- In the on-shell diagram these vertices are expanded into $k=1$ and $k=2$ 3pt amplitudes.





Colored graphs and on-shell diagrams

- In general case the colored graph can have vertices with $k \geq 3$, then one colored graph corresponds to sum of on-shell diagrams.
- Example: 7pt NMHV



where the 6pt NMHV tree-level can be written using black and white vertices via BCFW.

Colored graphs are equivalent to sums of on-shell diagrams.

Leading singularities in planar case

Calculate leading singularities as algebraic functions.

- Field theory: we glue tree-level amplitudes together evaluated at the cut momenta.

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New much more powerful ways using the Grassmannian (Nima's talk):

- Local: on-shell gluing and constructing the Grassmannian.
- Global: using permutations - also tells us that there is finite number of them.

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Presence of poles that are not present in the amplitude,
e.g. 6pt NMHV leading singularity

$$\frac{1}{[12][23]\langle 45\rangle\langle 56\rangle s_{123}\langle 6|1+2|3]\langle 4|2+3|1]}$$

Leading singularities in planar case

MHV case is special

- The only leading singularity is Parke-Taylor factor

$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\dots\langle n1\rangle}$$

  The leading singularity corresponds to the unique top form - Freddy's talk.

- The integrand at any loop order can be written then as

$$M_n^{\ell-loop} = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\dots\langle n1\rangle}\cdot I_n^{\ell-loop}$$

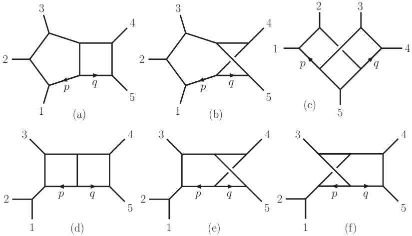
  where $I_n^{\ell-loop}$ is dual conformal invariant and evaluates to 1 (or zero) on any T^{4L} contour.

This is no longer true for the non-planar case.

Example of a leading singularity of non-planar amplitude

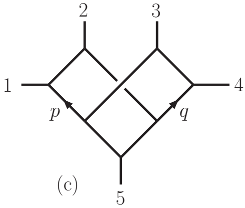
Let us extract the LS directly from the 5pt 2-loop non-planar amplitude:

Carrasco, Johannson, 2011



$\mathcal{I}^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a),(b)	$\frac{1}{4}\Big(\gamma_{12}(2s_{45}-s_{12}+\tau_{2p}-\tau_{1p})+\gamma_{23}(s_{45}+2s_{12}-\tau_{2p}+\tau_{3p})$ $+2\gamma_{45}(\tau_{5p}-\tau_{4p})+\gamma_{13}(s_{12}+s_{45}-\tau_{1p}+\tau_{3p})\Big)$
(c)	$\frac{1}{4}\Big(\gamma_{15}(\tau_{5p}-\tau_{1p})+\gamma_{25}(s_{12}-\tau_{2p}+\tau_{5p})+\gamma_{12}(s_{34}+\tau_{2p}-\tau_{1p}+2s_{15}+2\tau_{1q}-2\tau_{2q})$ $+\gamma_{45}(\tau_{4q}-\tau_{5q})-\gamma_{35}(s_{34}-\tau_{3q}+\tau_{5q})+\gamma_{34}(s_{12}+\tau_{3q}-\tau_{4q}+2s_{45}+2\tau_{4p}-2\tau_{3p})\Big)$
(d)-(f)	$\gamma_{12}s_{45}-\frac{1}{4}\Big(2\gamma_{12}+\gamma_{13}-\gamma_{23}\Big)s_{12}$

Example of a leading singularity of non-planar amplitude



(c)

(c)	$\frac{1}{4}\Big(\gamma_{15}(\tau_{5p}-\tau_{1p})+\gamma_{25}(s_{12}-\tau_{2p}+\tau_{5p})+\gamma_{12}(s_{34}+\tau_{2p}-\tau_{1p}+2s_{15}+2\tau_{1q}-2\tau_{2q})$ $+\gamma_{45}(\tau_{4q}-\tau_{5q})-\gamma_{35}(s_{34}-\tau_{3q}+\tau_{5q})+\gamma_{34}(s_{12}+\tau_{3q}-\tau_{4q}+2s_{45}+2\tau_{4p}-2\tau_{3p})\Big)$

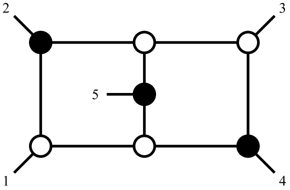
where $\tau_{ip}=2(k_i\cdot p)$ and $\gamma_{12}=\beta_{12345}-\beta_{21345}$ with

$$\beta_{12345}=\frac{[12][23][34][45][51]}{\langle 12\rangle[23]\langle 35\rangle[51]-[12]\langle 23\rangle[35]\langle 51\rangle}$$

No reason to expect that the leading singularities would be simple.

Example of a leading singularity of non-planar amplitude

Let us calculate one leading singularity explicitly for integral (c):



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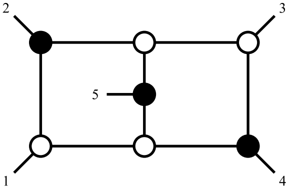

$$= \frac{\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\langle 15\rangle\langle 53\rangle}$$

$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \frac{1}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle}$$

- And the same for all other colorings. Very unexpected!

All non-planar MHV leading singularities

What can we say from the obvious symmeties?

- Superconformal symmetry fixes the LS to be holomorphic.
- They can contain spurious poles.
- There may be infinite number of them for fixed n (new LS for higher loops) of any kind.

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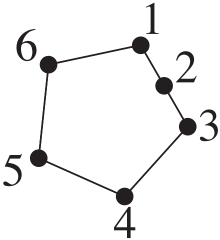
Main result

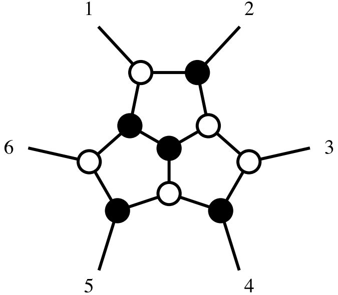
All non-planar MHV leading singularities can be written as linear combinations of Parke-Taylor factors with different orderings. It is easy to calculate all of them explicitly.

Proof of main result

Step 0: Configuration of points and poles

From Nima's talk: configuration of n points in P^{k-1} are equivalent to the point in the Grassmannian $G(k,n)$ and it is equivalent to the on-shell diagram.





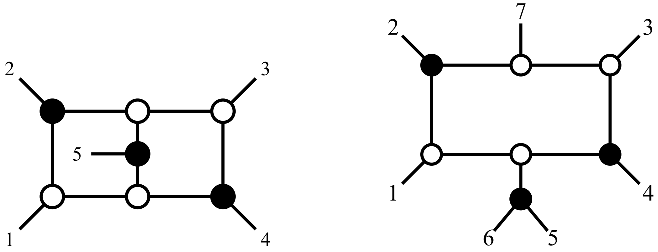
The poles correspond to imposing one more constraint on the geometric configuration.

Proof of main result

In planar case all the constraints are between consecutive points. What to do in the non-planar case?

Any non-planar diagram can be planarized by cutting internal lines.

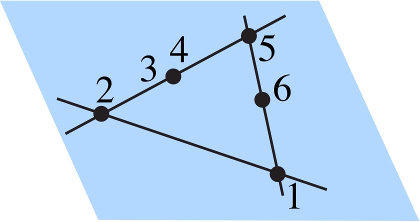
Any non-planar on-shell diagram can be obtained from the planar diagram by identifications of pairs of external lines.



Gluing 7 and 6 we get a non-planar graph.

Proof of main result

What is the gluing of external legs from the point of view of configuration of points in P^{k-1}?



- We draw the line that connects the points we want to get rid of.
- Choose an arbitrary point on this line and project all other points through this point on an arbitrary P^{k-2} plane.

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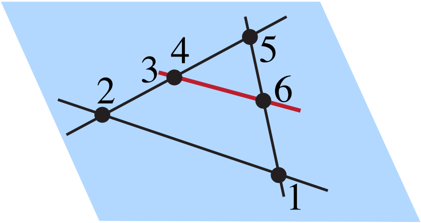

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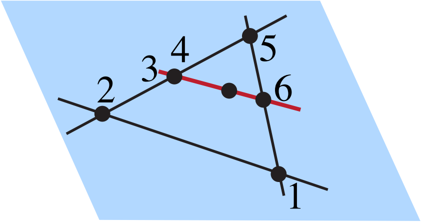

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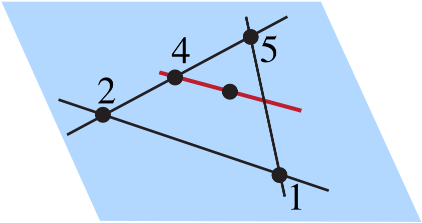

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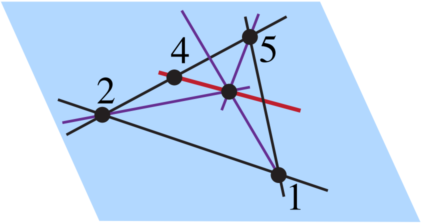

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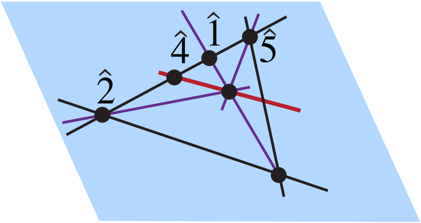

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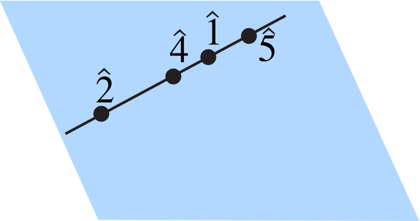

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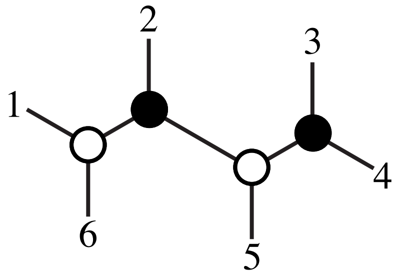

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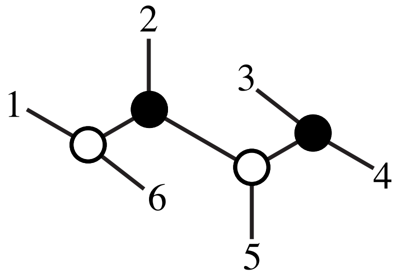

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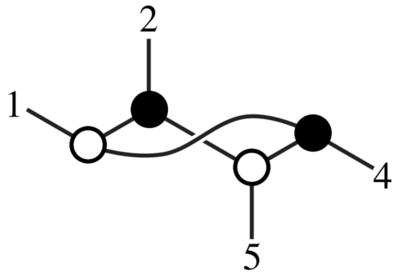

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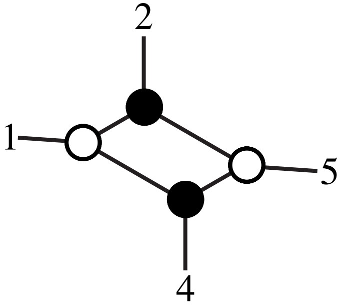


Proof of main result

We start with a $d + r$ dimensional configuration of points in $G(n + 2r, k + r)$

- After r gluing we land on a d dimensional configuration in $G(n, k)$.
- It is not in the positive part of Grassmannian anymore, so the linear dependencies are not just between consecutive points.

This is true for any on-shell diagram/configuration of points. For a special case of MHV leading singularities, $k = 2$ and $d = 2n - 4$.

The leading singularity is a top cell of $G(2, n)$ so the configuration is always n generic points in P^1 (with no ordering).

Question: can we learn about the pole structure of a given leading singularity?

Proof of main result

For MHV the minors of Grassmannian $(i\,j) = \langle i\,j\rangle$. To probe a pole we need to set $(i\,j) = 0$ which means that points i and j sit on top of each other.

In the planar case, we know the boundary operator - forces two points to sit on top of each other.

For the non-planar case, we do not know the boundary operator. There can be also other boundaries where e.g. $(12)(34)(56) = (14)(25)(36)$ which would produce a pole $(\langle 12\rangle\langle 34\rangle\langle 56\rangle - \langle 14\rangle\langle 25\rangle\langle 36\rangle)$ in the denominator.

Proof of main result

Step 1: Absence of spurious poles

We want to prove that the only poles in the algebraic expressions for the MHV leading singularities are $\langle i\, j\rangle$. It means that the boundary operator produces only doubled points. The boundary operator is defined implicitly:

- Cut the lines of the on-shell diagram and planarize it.
- Calculate the boundary (erase the edge in the diagram or use the boundary operator in the configuration of points).
- Glue again all the lines you cut before.

If the resulting configuration has (at least) one double point, then the only poles are $\langle i\, j\rangle$.

Proof of main result

But we will prove even stronger statement.

Theorem
For a cell in $G(k,n)$ without zeros of dimension d, the number of distinct points is less than or equal to $d-n+2k$.

We use the equivalence between cells in $G(k,n)$ and on-shell diagrams. For the configuration without any repeated points following relations are true:

- $F-E+V=1$ Euler's formula
- $d=F-1$ dimension formula
- $V=B+W$ vertices are black or white
- $B-W=k-(n-k), \quad 3V+n=2E, \quad E\leq 3B+W$

Here F is number of faces, E edges, V vertices, B black vertices, W white vertices. This gives us $n\leq d-n+2k$.

Proof of main result

If we remove the repeated point, then d decreases by 1, n decreases by 1 and k is unchanged. Therefore, we get

$$P \leq d - n + 2k$$

where P is number of distinct points which completes a proof.

The number of double points for our case ($k = 2$, $d = 2n - 5$) is then

$$\Delta = n - P \geq 2n - 2k - d = 1$$

The only boundaries are with at least one double point $\rightarrow$ only poles $\langle ij \rangle$.

Proof of main result

Step 2: No double poles

The dlog form of the Grassmannian measure (nothing to do with planarity)

$$\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}\dots\frac{dx_n}{x_n}\,\delta(\cdots)$$

- obvious that the leading singularities do not have any double poles.
- also we do not get any double poles if we take residues on $x_j \to 0$ which corresponds to setting $\langle i\, j\rangle \to 0$. A very non-trivial property! E.g. it is true for Parke-Taylor factor

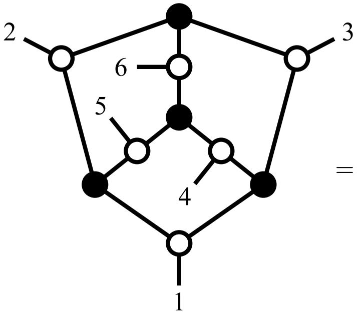
$$\lim_{\langle 56\rangle\to 0}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} = \frac{1}{\langle 56\rangle}\cdot\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

but for another simple function

$$\lim_{\langle 56\rangle\to 0}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 45\rangle\langle 56\rangle\langle 64\rangle} = \frac{1}{\langle 56\rangle}\cdot\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle^2\langle 64\rangle}$$

Proof of main result

Let us take a more complicated example



$$= \frac{(\langle 15\rangle\langle 26\rangle\langle 34\rangle - \langle 14\rangle\langle 25\rangle\langle 36\rangle)^2}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}$$

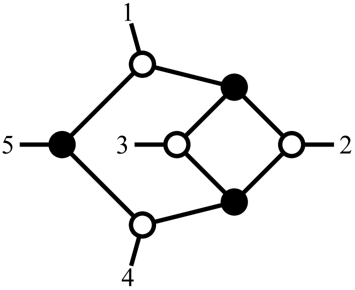
Calculate the limit $\langle 56\rangle \to 0$ and then $\langle 23\rangle \to 0$,

$$\to \frac{1}{\langle 56\rangle}\cdot\frac{\langle 13\rangle}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 35\rangle} \to \frac{1}{\langle 56\rangle}\cdot\frac{1}{\langle 23\rangle}\cdot\frac{1}{\langle 14\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle}$$

Proof of main result

Step 3: Inverse soft factor terms

We can easily prove that inverse soft factor acting on Parke-Taylor factor produces a particular sum of Parke-Taylor factors with different orderings.



$$= \frac{\langle 41\rangle}{\langle 45\rangle\langle 51\rangle}\cdot\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$= PT(12453) + PT(12435)$$

The general formula is simple and straightforward:

$$\frac{\langle i\,j\rangle}{\langle i\,n\rangle\langle n\,j\rangle}\cdot PT(123\ldots n-1) = \sum_{k=i}^{j-1} PT(123\ldots k\,n\,k+1\ldots n-1)$$

Proof of main result

Step 4: Recursive construction

Knowing that the leading singularities are holomorphic functions with $\langle ab\rangle$ only and no double poles (even for any chain of residues), we can prove our conjecture. Suppose that we have the result for leading

singularity A,

- Pick two indices, say 1 and j and do the shift $\widehat{\lambda_1} = \lambda_1 + z\lambda_j$.
- Use the Cauchy theorem to write

$$A = \sum (\text{Res}A)_{\langle\widehat{1}\,r\rangle=0} = \sum \frac{\langle 1\,j\rangle}{\langle 1\,r\rangle\langle r\,j\rangle} \cdot \begin{pmatrix} \text{new function} \\ \text{with no } \lambda_r \end{pmatrix}$$

- We can continue until the function that is left is Parke-Taylor factor

$$A = \sum S \cdot S \dots S \cdot PT(\dots).$$

If we want, we can go down to the 3pt amplitude.

Proof of main result

We already know that

$$S \cdot PT = \sum_i PT_i$$

Therefore, for any leading singularity

$$A = \sum_i PT_i$$

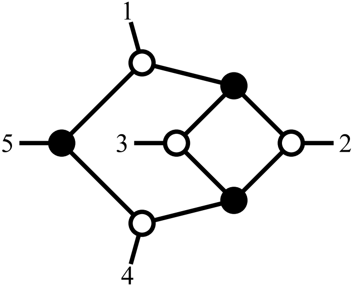
And our conjecture is proven.

Q.E.D.

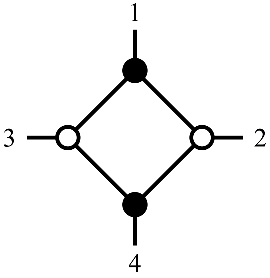
Calculating all leading singularities

If the leading singularity an is inverse soft-factor applied to lower point one, we can strip off the factor $S_k^{(i,j)}$ where we add particle k between i and j.

Reminder of an example, we have already worked out

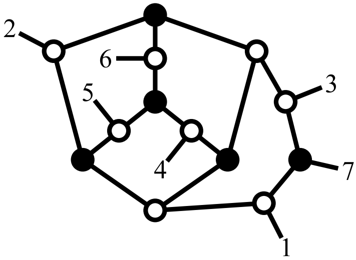


$= S_5^{(1,4)} \times$



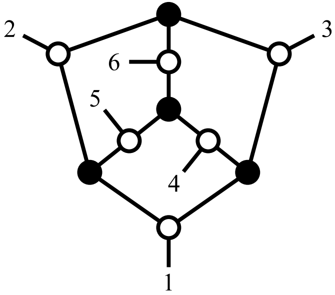
$$= \frac{\langle 41\rangle}{\langle 45\rangle\langle 51\rangle} \cdot \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = PT(12453) + PT(12435)$$

Calculating all leading singularities



The 6pt leading singularity is not an inverse soft factor anymore.

$$= S_7^{(1,3)} \cdot$$

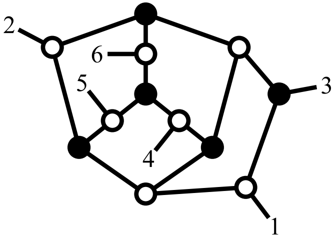


Calculating all leading singularities

Step 4: BCFW bridge

Invese soft factor terms are special cases, in general we have to use the BCFW construction.

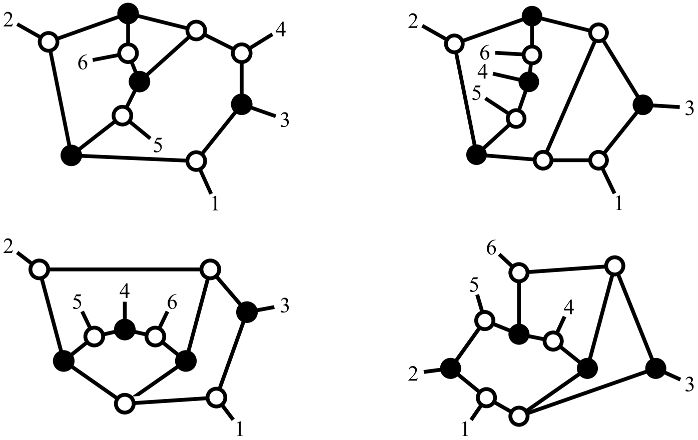
- Attach the bridge to the graph:



- Remove one edge at a time. Removable edges are those when the resulting configuration is non-singular, these represent the terms in Cauchy's formula.

Calculating all leading singularities

There are four diagrams that do contribute



They are guaranteed to be $S \cdot (\text{lower point diagram})$ as we showed before.

Calculating all leading singularities

If we iterate this procedure, we find that the result can be written as a sum of six terms,

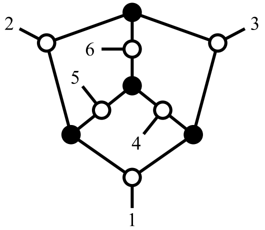
$$S_1^{(2,3)} \cdot S_6^{(2,3)} \cdot PT(2345) + S_1^{(2,3)} \cdot S_6^{(2,4)} \cdot PT(2534) + S_1^{(2,5)} \cdot S_6^{(2,5)} \cdot PT(2345)$$
$$+ S_1^{(2,5)} \cdot S_6^{(2,4)} \cdot PT(2354) + S_6^{(3,5)} \cdot S_1^{(2,4)} \cdot PT(2354) + S_1^{(2,4)} \cdot S_6^{(3,4)} \cdot PT(2345)$$

which can be written as a sum of six Parke-Taylor factors

$$PT(126435) + PT(123564) + PT(123456) + PT(125463) + PT(126453) + PT(125364)$$

which surprisingly shrinks to

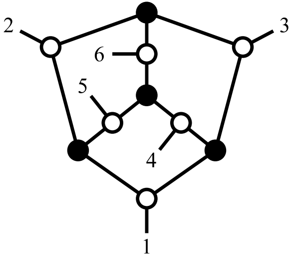
$$\frac{(\langle 15\rangle\langle 26\rangle\langle 34\rangle - \langle 14\rangle\langle 25\rangle\langle 36\rangle)^2}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}$$



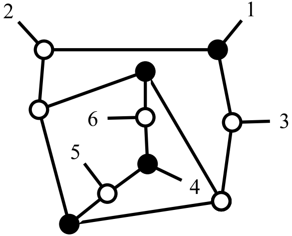
Calculating all leading singularities

Faster BCFW bridge:

- From the diagram we can determine what are all possible poles in the denominator.
- In our case these are $\langle 12\rangle$, $\langle 13\rangle$, $\langle 14\rangle$, $\langle 15\rangle$.
- Erase an edge that leads to the given pole (line 1 and j would be connected to white vertex) and then attach BCFW bridge to 1 and any other fixed line.
- This provides us directly the terms of the form $S \cdot (n\text{–}1\text{pt diagram})$.



for pole $\langle 13\rangle$



Reduction

The BCFW construction can be used for all reduced diagrams.

In general, we do not know how to reduce a graph for general k. Mathematicians also do not know how to do it in the non-planar case.

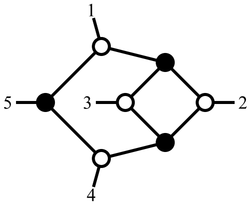
However, we have the physical insight: the only poles of MHV leading singularities are $\langle ab\rangle$ and there are no double poles.

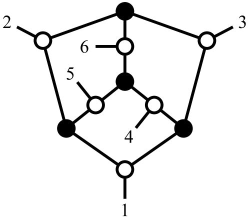
If the diagram is reduced,

- Removing an edge corresponds to $x_j \to 0$ which means $\langle ab\rangle \to 0$.
- The resulting diagram must have two external lines attached to a white vertex. This corresponds to $\lambda_i \sim \lambda_j$ which is equivalent to $\langle i\,j\rangle = 0$.
- Not all edges can be removed from the graph. For non-removable edges we get even more external lines attached to the white vertex (or more pairs to more vertices).
- We have to check all removable edges.

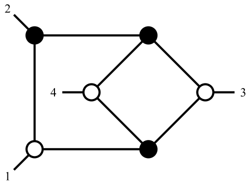
Reduction

We can show that these graphs are reduced:

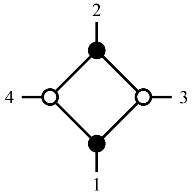




But e.g. other even simpler graph:



remove edge (12)

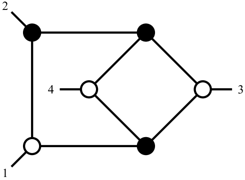


is not reduced. If the graph is not reduced, we have to reduce it.

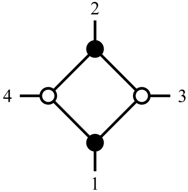
Reduction

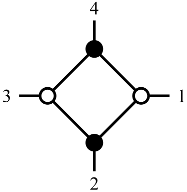
How does the reduction work?

Remove the edge, and list all the diagrams where none of white vertices has two external legs attach to it.



of we get a list





These diagrams are already reduced, and as a result we get: $\{PT(1234), PT(1423)\}$. These are two different composite LS of this diagram.

General method

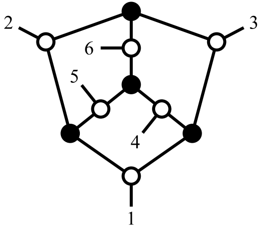
We can evaluate arbitrary MHV on-shell diagram (leading singularity):

- We check the reducibility. If the diagram is not reduced, reduce it.
- Once we have the reduced diagrams only (one or a list), we use BCFW to calculate them.
- The result is a sum of Parke-Taylor factors.

Evidence for new structure

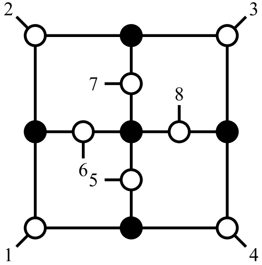
Computational evidence that the leading singularities are even simpler.

We do not get arbitrary linear combinations of Parke-Taylor factors, but the special ones that form compact expressions:



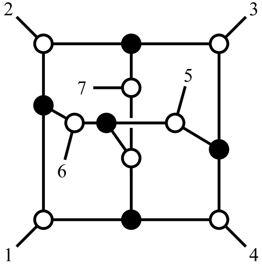
$$\frac{(\langle 15\rangle\langle 26\rangle\langle 34\rangle - \langle 14\rangle\langle 25\rangle\langle 36\rangle)^2}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}$$

Evidence for new structure



$$\frac{(\langle 16\rangle\langle 27\rangle\langle 38\rangle\langle 45\rangle - \langle 15\rangle\langle 26\rangle\langle 37\rangle\langle 48\rangle)^2}{\langle 12\rangle\langle 14\rangle\langle 15\rangle\langle 16\rangle\langle 23\rangle\langle 26\rangle\langle 27\rangle\langle 34\rangle\langle 37\rangle\langle 38\rangle\langle 45\rangle\langle 48\rangle\langle 56\rangle\langle 58\rangle\langle 67\rangle\langle 78\rangle}$$

Evidence for new structure



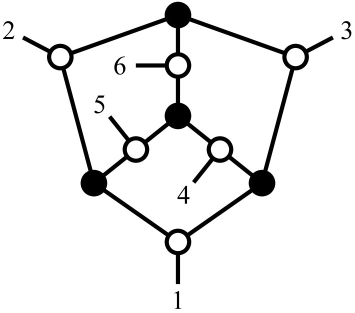
$$\frac{(\langle 16\rangle\langle 23\rangle\langle 47\rangle\langle 57\rangle - \langle 14\rangle\langle 25\rangle\langle 37\rangle\langle 67\rangle)^2}{\langle 12\rangle\langle 14\rangle\langle 16\rangle\langle 17\rangle\langle 23\rangle\langle 25\rangle\langle 27\rangle\langle 34\rangle\langle 35\rangle\langle 37\rangle\langle 46\rangle\langle 47\rangle\langle 56\rangle\langle 57\rangle\langle 67\rangle}$$

Evidence for new structure

The compact expressions seem to be fundamental.

The expansion in terms of Parke-Taylor factors looks like some "triangulation".

An idea: the result can be read from the diagram directly without any calculation. We do not know if it is true but there is an interesting evidence that it is not completely crazy:



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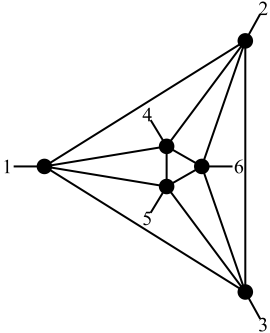

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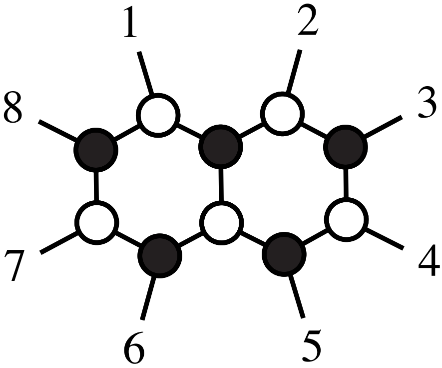
$$= \frac{1}{2}\sum_{\text{all closed paths}} PT(i_1,i_2,i_3,i_4,i_5,i_6)$$

Beyond MHV

There are definitely new objects we get even for NMHV.

Example of gluing legs from $8pt$ N^{2}MHV to $6pt$ NMHV,



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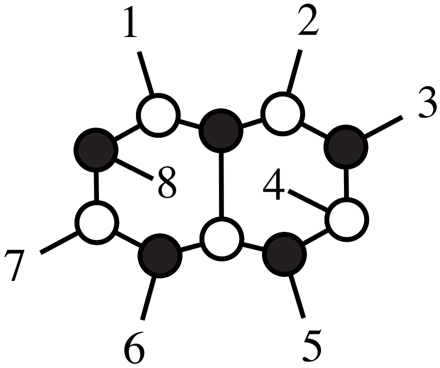

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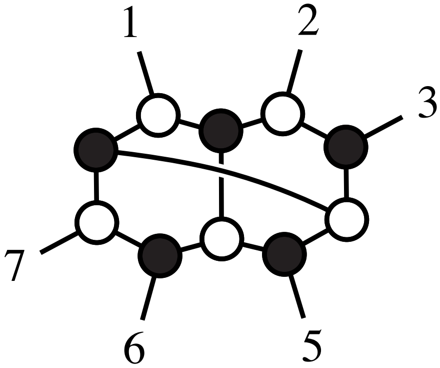
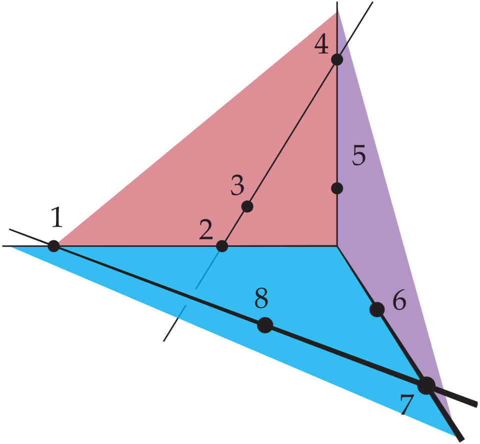


Beyond MHV

For the projection of points from P^3 to P^2,



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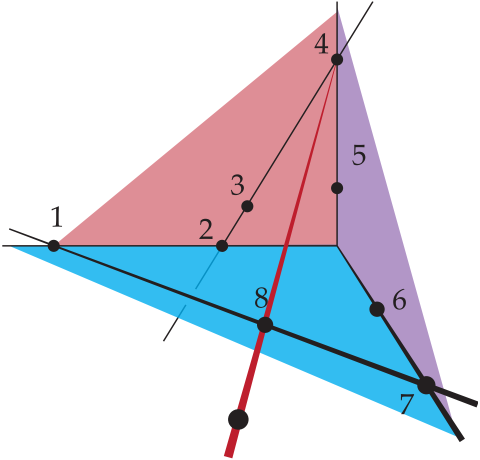

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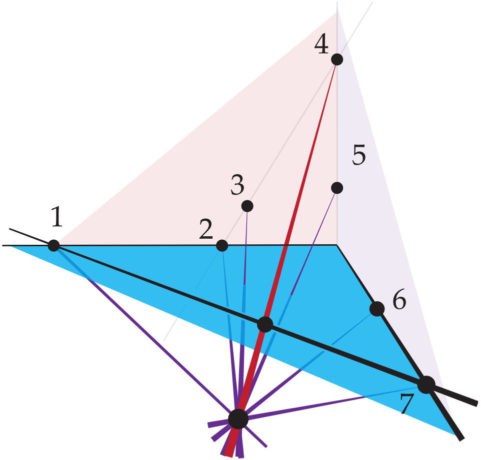

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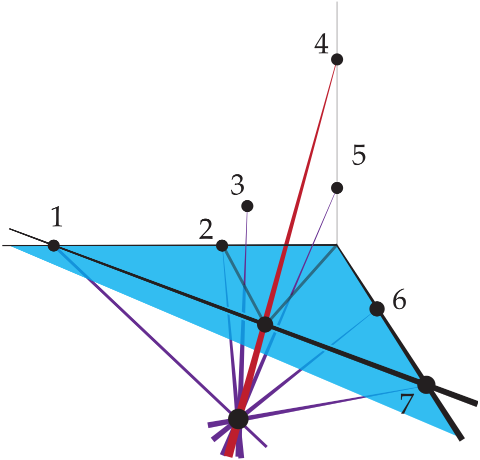

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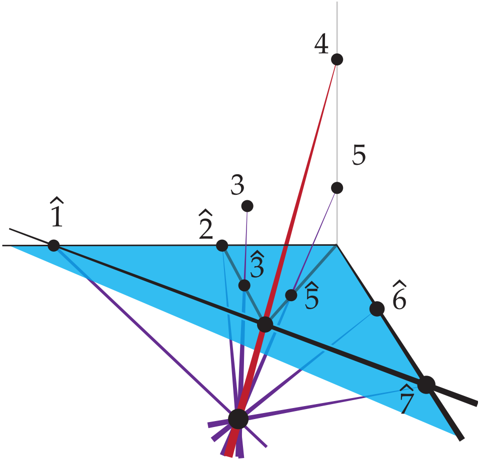

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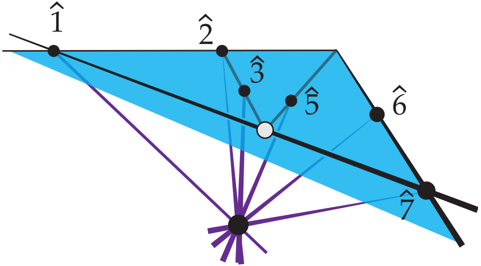

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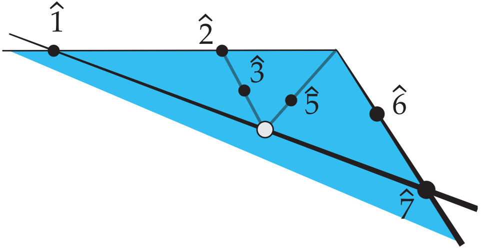

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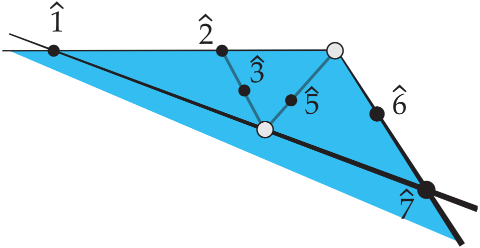

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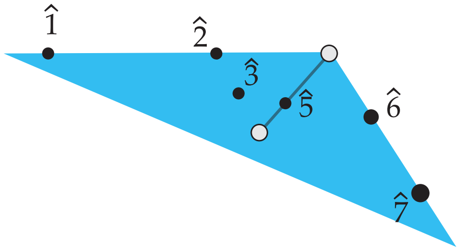

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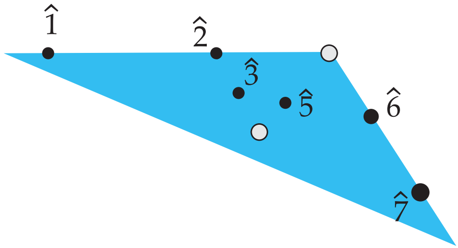


From leading singularities to amplitudes

How can we use the knowledge of leading singularities to construct the MHV non-planar amplitudes?

Natural idea: recursion relations (as in planar case).

Obvious problems with uniqueness of the integrand, etc. $\rightarrow$ we do not have recursion relations now.

The other natural proposal is to write the amplitude in a form

$$M = \sum_i (LS)_i \cdot I_i$$

This work is in progress now ... but there is some interesting aspect even for $4pt$ 2-loop case.

From leading singularities to amplitudes

Work by *Bern,Rozowsky,Yan (1997)*,*Bern,Dixon,Dunbar,Perelstein,Rozowsky (1998)*:

The complete 2-loop 4pt amplitude is

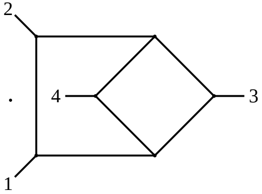
$$A_4^{(2)} \sim \mathcal{K} \sum_{S_4} \left[C_{1234}^{(P)} I_{1234}^{(P)} + C_{1234}^{(NP)} I_{1234}^{(NP)} \right]$$

where C_{1234} is a color factor and

$$\mathcal{K} = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle} \delta^{(8)}(\lambda \cdot \eta)$$

The non-planar diagram is

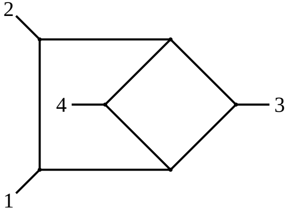
$$I_{1234}^{(NP)} = s_{12} \cdot$$



From leading singularities to amplitudes

This integral was calculated by Tausk in 1999 and the result does not have uniform transcendentality despite the final result does!

Let us look at the leading singularities of this integral.

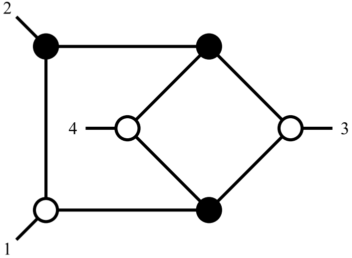


Let us calculate the quadrupole cut of the box part. It gives the Jacobian $J = (p - k_1 - k_3)^2(p + k_2 + k_3)^2$,

$$\frac{1}{(p-k_1)^2 p^2 (p+k_2)^2 \cdot (p-k_1-k_3)^2 (p+k_2+k_3)^2}$$

From leading singularities to amplitudes

The remaining p-integral has four quadrupole cuts (one is vanishing). Two of them just correspond to the leading singularity



but the other two (when we cut both $(p-k_1-k_3)^2$, $(p+k_2+k_3)^2$) not.

Why? After we cut $(p-k_1-k_2)^2$, $(p+k_2+k_3)^2$ and p^2 we generate a double-pole and the numerator does not kill it because it is just scalar function $N=(k_1+k_2)^2$.

From leading singularities to amplitudes

Cutting both factors in Jacobian corresponds to setting both $s = t = 0$ in 4pt SYM amplitude, but SYM does not have this singularity, it is $1/s + 1/t$.

We should find the numerator that prevents us from having this cut,

$$N = \left[(p - k_1 - k_2)^2 + (p + k_2 + k_3)^2\right]$$

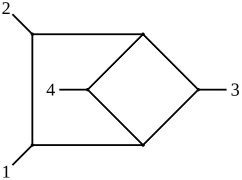
How is it related to the original numerator $N = (k_1 + k_2)^2$?

$$\left[(p - k_1 - k_2)^2 + (p + k_2 + k_3)^2\right] = (k_1 + k_2)^2 + (p - k_1)^2 + (p + k_2)^2$$

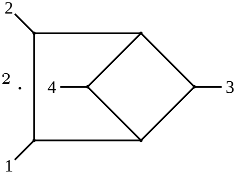
From leading singularities to amplitudes

The integrals are then related via:

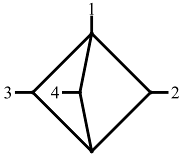
$$\left[(p-k_1-k_2)^2+(p+k_2+k_3)^2\right]\cdot$$



$$=\quad (k_1+k_2)^2\cdot$$



$+$



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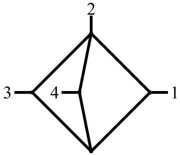


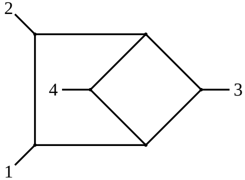
Fortunately, the 6-propagator integral was also calculated by Tausk, and it has just transcendentality 3.

Note: it does not have any leading singularities.

From leading singularities to amplitudes

Combining the pieces calculated by Tausk we find that our integral

$$\left[(p-k_1-k_2)^2+(p+k_2+k_3)^2\right]\cdot$$



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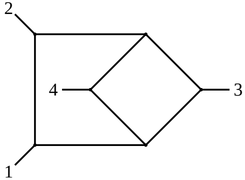

has uniform transcendentality!

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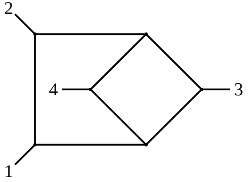


has uniform transcendentality!

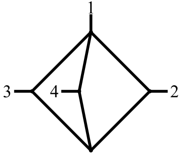
It supports the idea that the leading singularities are related to uniform transcendentality.

But the former integral gave a correct answer for the amplitude (after the color sum is performed). Do we still get the right answer with this integral?

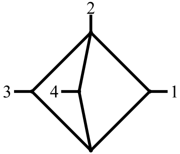
From leading singularities to amplitudes

We need to evaluate a sum:

$$\sum_{S_4} C^{(NP)}_{1234} \times$$



$+$

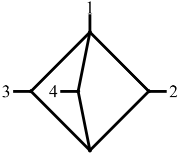


Note that each integral is a function $F(s,t,u)$ which is completely symmetric in s,t,u. It is obvious if you write $s=(k_1+k_2)^2$, $t=(k_1+k_3)^2$, $u=(k_1+k_4)^2$. Then we can factor out this function and get

From leading singularities to amplitudes

We need to evaluate a sum:

$$\sum_{S_4} C^{(NP)}_{1234} \times$$



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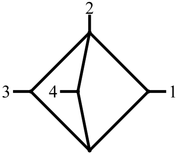

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$$2F(s,t,u)\sum_{S_4} C^{(NP)}_{1234} = 0$$

and the answers are identical. Same for 4pt 2-loop gravity amplitude where we use momentum conservation instead of color identity.

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THANK YOU!