Eidgenössische Technische Hochschule Zürich
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Hopf algebras and Higgs amplitudes

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The life-cycle of a loop computation

- Multi-loop computations are generically very difficult.
- For this reason, physicists have split the problem of computing loop amplitudes into various building blocks:
  - ➡ Group loop integrals into topologies.
  - ➡ Reduce every topology to master integrals using, e.g., IBP identities.
  - ➡ The remaining (scalar) integrals are computed by whatever means necessary:
    - ★ Direct integration.
    - ★ Mellin-Barnes.
    - ★ Differential equations.
    - ★ Dimensional recurrence.
    - ★ ...

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    - ★ Elliptic functions.
- In this talk: will concentrate exclusively on polylogarithms.

The life-cycle of a loop computation

- Recursive definition of multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t) \quad \Big| \quad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{n-1}(t)$$

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  - ➡ Cyclotomic harmonic polylogarithms: roots of unity.

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  - ➡ separate the problem into conceptually different subproblems (reduction vs. analytical evaluation),
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- At the same time, a lot of harm can be done to the amplitude, because
  - ➡ symmetries might be lost along the way,
  - ➡ while easier to compute, the master integrals might have a more complicated analytical structure.
- In other words, even if an amplitude is simple, it might be that our approach to the problem leads to a difficult answer.

The ‘classical’ example

- The ‘classical’ example of this is the six-point remainder function in N=4 SYM.
- By evaluating the individual diagrams one arrives at a very complicated combination of multiple polylogarithms (17 pages),

$$
\begin{aligned}
&R_{6,WL}^{(2)}(u_1,u_2,u_3) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
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&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
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\end{aligned}
$$

[Del Duca, CD, Smirnov]

The 'classical' example

- ... but the result can be rewritten in a much more compact form

$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$

$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{J^4}{24}+\chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right)$$

$$L_4(x^+,x^-)=\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))+\frac{1}{8!!}\log(x^+x^-)^4$$

$$\ell_n(x)=\frac{1}{2}\left(\mathrm{Li}_n(x)-(-1)^n\,\mathrm{Li}_n(1/x)\right)$$

$$x_i^\pm=u_i x^\pm,\qquad\qquad x^\pm=\frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},$$

$$\Delta=(u_1+u_2+u_3-1)^2-4u_1u_2u_3.$$

[Goncharov, Spradlin, Vergu, Volovich]

Maybe amplitudes are simple...?

- Could Feynman integrals be simpler than we thought...?
- Long term goal: get to the simple answer (the function) without the 'divide and conquer' strategy.
- In the mean time: gather data, and try to find a way to get the simple answer out of the 'divide and conquer' approach.

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- Could Feynman integrals be simpler than we thought...?
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- In the mean time: gather data, and try to find a way to get the simple answer out of the 'divide and conquer' approach.
- Outline:
  - ➡ Symbols: advantages and disadvantages.
  - ➡ Symbols vs. coproducts: recovering the lost pieces.
  - ➡ An application: Two-loop Higgs boson amplitudes.

Symbols

Advantages and disadvantages

Symbols

- Symbols were the main tool used to simplify the six-point remainder function.
- Main idea: Combinatorics of functional equations among multiple polylogarithms is mapped to the combinatorics of a certain tensor algebra.
- We can then simplify the symbols (easy) rather than the functions (difficult).
- Finally, we must find a simpler function that has the same symbol (most difficult step).

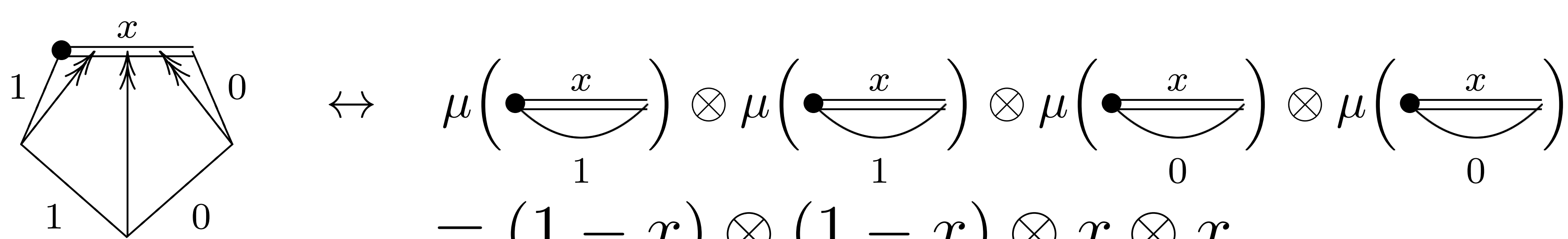
Symbols

- Two definitions were introduced in physics:
  - ➡ via differential equations: [Goncharov, Spradlin, Vergu, Volovich]

    If $dF_w = \sum_i F_{i,w-1}\, d\ln R_i$ then

    $$\mathcal{S}(F_w) = \sum_i \mathcal{S}(F_{i,w-1}) \otimes R_i$$

  - ➡ via a correspondence to certain decorated polygons



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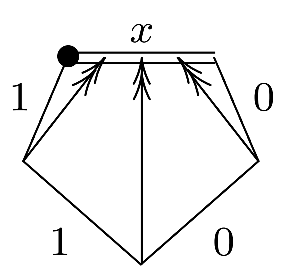
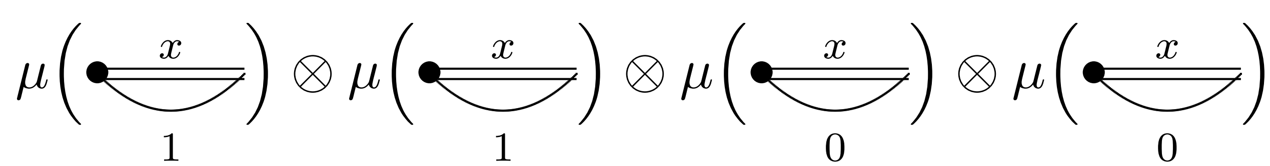


$\leftrightarrow$



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- The two definitions can be shown to be equivalent, except for constants, e.g., while

$d\mathrm{Li}_2\left(\frac{1}{2}\right) = 0$, the polygons give $\mathcal{S}\left[\mathrm{Li}_2\left(\frac{1}{2}\right)\right] = -(2 \otimes 2)$

Symbols

- Examples:

$$\mathcal{S}(\ln x\ \ln y) = x \otimes y + y \otimes x$$

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- The kernel of the symbol map is non trivial, *e.g.*,

$$\mathcal{S}(i\pi) = 0 \quad \text{and} \quad \mathcal{S}(\zeta_n) = 0$$

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- The aim of this talk: the symbol is only 'the tip of the iceberg' of a more general structure:
  - ➡ that contains the symbol in a certain limit.
  - ➡ incorporates almost all the terms that the symbol misses.
  - ➡ the terms that are missed are transcendental constants.

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  - ➡ the terms that are missed are transcendental constants.
- This more general structure is the Hopf algebra of multiple polylogarithms.

[Goncharov]

Symbols vs. coproducts

Recovering the lost pieces

Algebras and coalgebras

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  - ➡ with a multiplication
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$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

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$$\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$$

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If we iterate,

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the order in which we do this is immaterial, because

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

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- Next, we iterate this procedure to split the word into three.

- Two choices, e.g,

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- As long as we sum over all possibilities, it does not matter which way we iterate, and always arrive at the same result.

Hopf algebras

- A Hopf algebra is
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  - ➡ that is at the same time a coalgebra
  - ➡ such that the product and coproduct are compatible

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  - ➡ and with an additional structure, the antipode (which we will not use in the following).
- Goncharov showed that multiple polylogarithms form a Hopf algebra with coproduct

$$\Delta(I(a_0; a_1, \ldots, a_n; a_{n+1}))$$

$$= \sum_{0=i_1<i_2<\ldots<i_k<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \left[\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})\right]$$

Polylogarithm Hopf algebra

- Examples:

$$\Delta(\ln z) \;= 1 \otimes \ln z + \ln z \otimes 1$$

$$\Delta(\mathrm{Li}_n(z)) \;= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

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- We can compute the coproduct of any combination of logarithms and polylogarithms, e.g.,

$$\begin{aligned}\Delta(\ln x \ln y) &= \Delta(\ln x)\, \Delta(\ln y) = [1 \otimes \ln x + \ln x \otimes 1]\, [1 \otimes \ln y + \ln y \otimes 1] \\ &= 1 \otimes (\ln x \, \ln y) + \ln x \otimes \ln y + \ln y \otimes \ln x + (\ln x \, \ln y) \otimes 1\,.\end{aligned}$$

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$$\mathcal{S}(\ln x\,\ln y) = x\otimes y + y\otimes x$$

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- We can compute the coproduct of any combination of logarithms and polylogarithms, e.g.,

$$\begin{aligned}\Delta(\ln x \ln y) \ &= \Delta(\ln x)\, \Delta(\ln y) = [1 \otimes \ln x + \ln x \otimes 1]\, [1 \otimes \ln y + \ln y \otimes 1] \\ &= 1 \otimes (\ln x \ \ln y) + \ln x \otimes \ln y + \ln y \otimes \ln x + (\ln x \ \ln y) \otimes 1 \,.\end{aligned}$$

$$\mathcal{S}(\ln x \ \ln y) = x \otimes y + y \otimes x$$

Polylogarithm Hopf algebra

- Examples:

$$\Delta(\ln z) \;= 1 \otimes \ln z + \ln z \otimes 1$$

$$\Delta(\mathrm{Li}_n(z)) \;= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

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$$\mathcal{S}(\ln x\, \ln y) = x \otimes y + y \otimes x$$

Is this a coincidence..?

Polylogarithm Hopf algebra

- Let's look at the classical polylogarithm

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

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$$\Delta(\mathrm{Li}_2(z)) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \ln(1-z) \otimes \ln z$$

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- More generally, we have

$$\Delta_{n-1,1}(\mathrm{Li}_n(z)) = \mathrm{Li}_{n-1}(z) \otimes \ln z$$

and so if we iterate

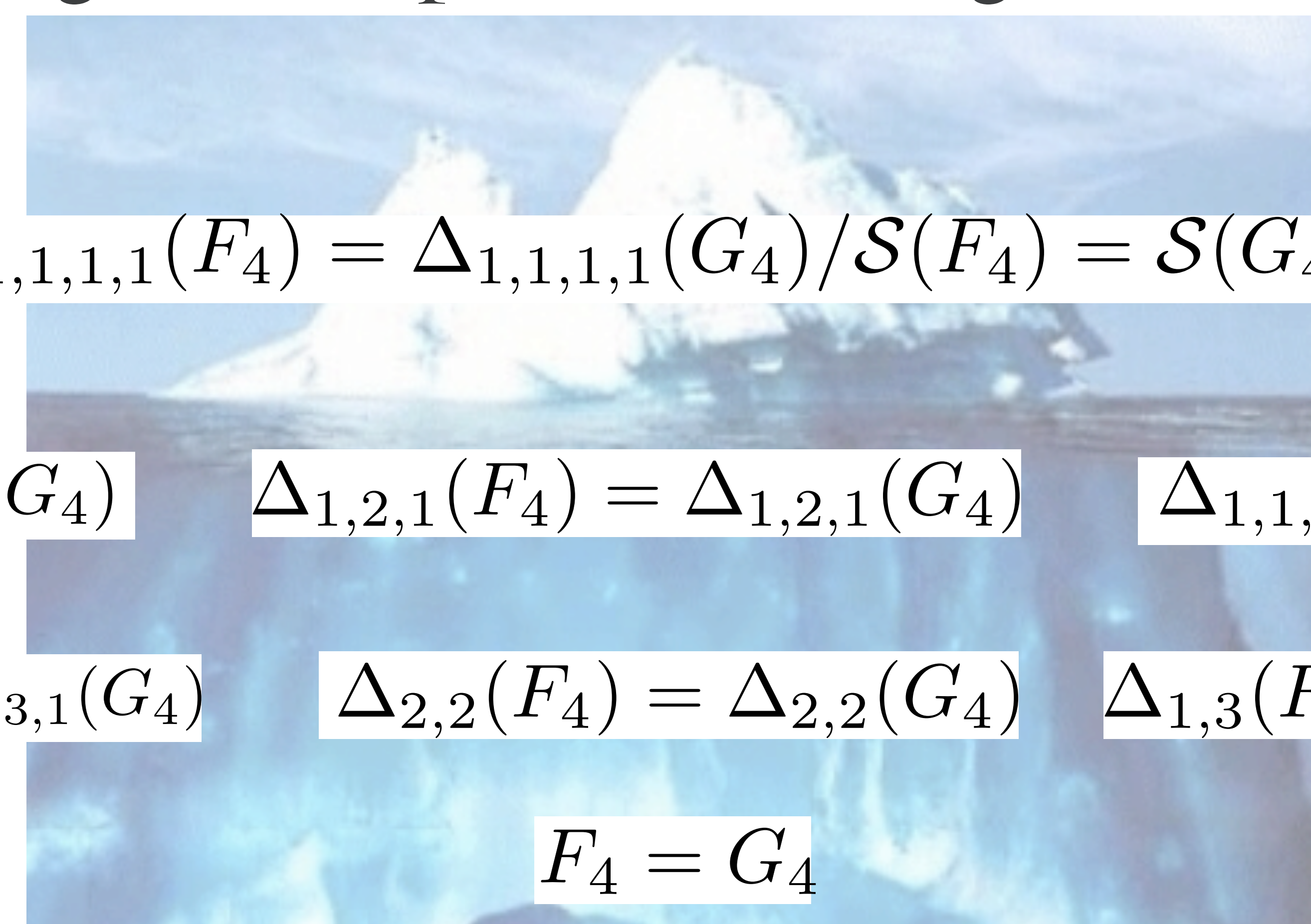
$$\Delta_{1,\ldots,1}(\mathrm{Li}_n(z)) = -\ln(1-z) \otimes \underbrace{\ln z \otimes \ldots \otimes \ln z}_{n-1}$$

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1}$$

Symbols vs. coproducts

- This is a general feature: the symbol agrees with the maximal iteration of the coproduct where we have decomposed a polylogarithm into logarithms (up to a technical detail).

- This shows why the symbol loses so much information: we are only looking at the tip of the iceberg



$$\Delta_{1,1,1,1}(F_4) = \Delta_{1,1,1,1}(G_4) / \mathcal{S}(F_4) = \mathcal{S}(G_4)$$

$$\Delta_{2,1,1}(F_4) = \Delta_{2,1,1}(G_4) \qquad \Delta_{1,2,1}(F_4) = \Delta_{1,2,1}(G_4) \qquad \Delta_{1,1,2}(F_4) = \Delta_{1,1,2}(G_4)$$

$$\Delta_{3,1}(F_4) = \Delta_{3,1}(G_4) \qquad \Delta_{2,2}(F_4) = \Delta_{2,2}(G_4) \qquad \Delta_{1,3}(F_4) = \Delta_{1,3}(G_4)$$

$$F_4 = G_4$$

Symbols vs. coproducts

- Advantage of symbols: Functional equations are reduced to functional equations for ordinary logarithms.
- Disadvantage of symbols: loses information, e.g.,

$$\mathcal{S}(\zeta_3^2) = 0$$

while the coproduct is non zero,

$$\Delta(\zeta_3^2) = 1 \otimes \zeta_3^2 + \zeta_3^2 \otimes 1 + 2\zeta_3 \otimes \zeta_3$$

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- ... so we should look also at these terms, and then...
- ... there is a problem...

Coproduct on zeta values

- Putting z=1 in

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

we arrive at

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- On the other hand, from $\zeta_4 = \frac{1}{15}\zeta_2^2$ we get

$$\Delta(\zeta_4) = \frac{1}{15}\Delta(\zeta_2)^2 = \frac{1}{15}[1 \otimes \zeta_2 + \zeta_2 \otimes 1]^2 = \frac{1}{15}[1 \otimes \zeta_2^2 + \zeta_2^2 \otimes 1 + 2\zeta_2 \otimes \zeta_2]$$

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- So there is a contradiction, unless $\Delta(\zeta_{2n}) = 0$.

- But then, we have not gained much...

Coproduct on zeta values

- In a recent paper on multiple zeta values, Francis Brown argues that one can also define

$$\Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1$$

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- We obtain a consistent way to include all the zeta values.
- I even argue that we can do better and define

$$\Delta(\pi) = \pi \otimes 1$$

- This will allow to include also $i\pi$.

Using coproducts in computations

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- More precisely, I argue that if we have a function F of weight n, and if we can find a (simpler) function G such

$$\Delta_{i,j,\ldots}(F) = \Delta_{i,j,\ldots}(G)$$

then

$$F = G + \sum_i c_i \, P_{w,i}$$

where the sum is over the primitive elements of weight n.

Using coproducts in computations

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then
$$F = G + \sum_i c_i\, P_{w,i}$$
where the sum is over the primitive elements of weight n.
- The function is not completely fixed, but we are in a much better shape than with the symbol:
  - ➡ only constants are missed.
  - ➡ there are only very few for a given weight, in practice most of the time just ζ_n.

Example: inversion relations

- The symbol does not entirely fix the inversion relation
  - ➡ it misses terms proportional to $i\pi$ and zeta values.

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- Weight 1: trivial

$$\mathrm{Li}_1\left(\frac{1}{x}\right) = -\ln\left(1-\frac{1}{x}\right) = -\ln(1-x) + \ln(-x) = -\ln(1-x) + \ln x - i\pi$$

Example: inversion relations

- Weight 2:

$$
\begin{aligned}
\Delta_{1,1}\left[\mathrm{Li}_2\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right) \\
&= \ln(1-x)\otimes\ln x - \ln x\otimes\ln x + i\pi\otimes\ln x \\
&= \Delta_{1,1}\Big[-\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x\Big]\,.
\end{aligned}
$$

Example: inversion relations

- Weight 2:

$$\begin{aligned}\Delta_{1,1}\left[\mathrm{Li}_2\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\ &= \ln(1-x)\otimes\ln x - \ln x\otimes\ln x + i\pi\otimes\ln x\\ &= \Delta_{1,1}\left[-\mathrm{Li}_2(x)-\frac{1}{2}\ln^2 x + i\pi\ln x\right].\end{aligned}$$

Thus

$$\mathrm{Li}_2\left(\frac{1}{x}\right) = -\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x + c\,\pi^2$$

and c = 1/3 from x=1.

➡ We have gained the imaginary part over the pure symbol approach!

➡ Note that it is crucial to assume $\Delta(\pi) = \pi\otimes 1$!

Example: inversion relations

- Weight 3:

$$
\begin{aligned}
\Delta_{1,1,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\
&= -\ln(1-x)\otimes\ln x\otimes\ln x+\ln x\otimes\ln x\otimes\ln x-i\pi\otimes\ln x\otimes\ln x\\
&= \Delta_{1,1,1}\left[\mathrm{Li}_3(x)+\frac{1}{6}\ln^3 x-\frac{i\pi}{2}\ln^2 x\right].
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$$= -\ln(1-x)\otimes\ln x\otimes\ln x + \ln x\otimes\ln x\otimes\ln x - i\pi\otimes\ln x\otimes\ln x$$
$$= \Delta_{1,1,1}\left[\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right].$$

➡ Again, we get the imaginary part, but we can get even more!

$$\Delta_{1,2}\left[\mathrm{Li}_3\left(\frac{1}{x}\right) - \left(\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right)\right] = 0\,.$$

$$\Delta_{2,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right) - \left(\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right)\right] = -\frac{1}{3}\pi^2\otimes\ln x = \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln x\right)$$

Thus $\mathrm{Li}_3\left(\frac{1}{x}\right) = \mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x - \frac{\pi^2}{3}\ln x + \alpha\zeta_3 + \beta\, i\pi^3$

and $\alpha = \beta = 0$ from x=1.

An application

Two-loop Higgs boson amplitudes

Higgs + 3 gluons

- Gehrmann, Jaquier, Glover and Koukoutsakis have recently computed the two-loop helicity amplitudes for a Higgs boson + 3 gluons
  - ➡ in the decay region

$$H \to g^+ g^+ g^+ \qquad H \to g^+ g^+ g^-$$

  - ➡ and the scattering region

$$g^+ H \to g^+ g^+ \qquad g^+ H \to g^+ g^- \qquad g^- H \to g^+ g^+$$

- Kinematics (in the decay region):

$$x_1 = \frac{s_{12}}{m_H^2}, \qquad x_2 = \frac{s_{23}}{m_H^2}, \qquad x_3 = \frac{s_{31}}{m_H^2}$$

$$0 < x_i < 1 \quad \text{and} \quad x_1 + x_2 + x_3 = 1$$

Higgs + 3 gluons

- The result was expressed in terms of complicated combinations of 2d harmonic polylogarithms.
  - ➡ Symmetries completely lost (e.g. Bose symmetry).
  - ➡ Very long and complicated.
  - ➡ Numerical evaluation of complicated special functions.
  - ➡ Analytic continuation from decay to scattering region very complicated.

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  - ➡ Symmetries completely lost (e.g. Bose symmetry).
  - ➡ Very long and complicated.
  - ➡ Numerical evaluation of complicated special functions.
  - ➡ Analytic continuation from decay to scattering region very complicated.
- Brandhuber, Gang and Travaglini observed that the symbol of the leading color weight 4 part (after substracting the one-loop squared) is equal to the symbol of the form factor remainder in N=4 SYM.
  - ➡ A simpler representation of the Higgs amplitudes in terms of classical polylogarithms only should exist.

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[Brandhuber, Gang, Travaglini]

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$$\mathcal{S}\left(\overline{A}^{(2)}_{\alpha,\text{ weight }4}\right) = \mathcal{S}\left(\mathcal{R}_3^{(2)}\right)$$

[Brandhuber, Gang, Travaglini]

$$\Delta_{2,1,1}\left[\overline{A}^{(2)}_{\alpha,\text{ weight }4} - \mathcal{R}_3^{(2)}\right] = -\frac{1}{6}\,\pi^2 \otimes \Delta_{1,1}\left[A_\alpha^{(1)}\right] = \Delta_{2,1,1}\left[-\frac{\pi^2}{6}\,A_\alpha^{(1)}\right]$$

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$$\overline{A}^{(2)}_{\alpha,\ \text{weight } 4} - \mathcal{R}_3^{(2)} + \frac{\pi^2}{6}\,A_\alpha^{(1)} + \frac{1}{4}\,\zeta_3\,B_\alpha^{(1)} = -0.03382260105347\ldots = -\frac{\pi^4}{2880}$$

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- We can of course do the same for all other color structures.

Higgs + 3 gluons

$$
\begin{aligned}
\overline{A}_{\alpha}^{(2)} &= \mathcal{R}_3^{(2)} - \frac{\pi^2}{6} A_{\alpha}^{(1)} - \frac{1}{4} \zeta_3 B_{\alpha}^{(1)} - \frac{\pi^4}{2880} \\
&\frac{11}{6} \Bigg\{ \Lambda_3\left(-\frac{x_1 x_3}{x_2}\right) + \Lambda_3\left(-\frac{x_2 x_3}{x_1}\right) + \Lambda_3\left(-\frac{x_1 x_2}{x_3}\right) - \sum_{i=1}^{3} \mathrm{Li}_3\left(1 - \frac{1}{x_i}\right) \\
&- \Lambda_3\left(-\frac{x_1}{x_2}\right) - \Lambda_3\left(-\frac{x_2}{x_1}\right) - \Lambda_3\left(-\frac{x_1}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_1}\right) - \Lambda_3\left(-\frac{x_2}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_2}\right) \\
&+ \frac{1}{2} \ln(x_1\, x_2\, x_3)\, A_{\alpha}^{(1)} + \frac{7}{2} \sum_{i=1}^{3} \left[\mathrm{Li}_2\left(1 - x_i\right)\, \ln x_i\right] + \frac{3}{4} \ln x_1\, \ln x_2\, \ln x_3 + \frac{1}{6} \ln^3\left(x_1 x_2 x_3\right) \\
&- \frac{5}{16} \pi^2 \ln(x_1 x_2 x_3) - \frac{3}{8} \zeta_3 + i\pi\, A_{\alpha}^{(1)} + \frac{i\pi^3}{16} - \frac{1}{3} \sum_{i=1}^{3} \ln^3 x_i \Bigg\} \\
&+ \frac{1}{36} \sum_{i=1}^{3} \left[\frac{P_1(x_i, x_{i-1}, x_{i+1})}{x_{i-1}^2 x_{i+1}^2} \mathrm{Li}_2(1 - x_i) + \frac{P_2(x_i, x_{i-1}, x_{i+1})}{x_i^2} \ln x_{i-1} \ln x_{i+1} + \frac{121}{4} \ln^2 x_i \right] \\
&+ \frac{P_3(x_1, x_2, x_3)}{144 x_1^2 x_2^2 x_3^2} \pi^2 - \frac{121}{72} i\pi \ln(x_1 x_2 x_2) + \frac{11}{36} i\pi \left(x_1 x_2 + x_2 x_3 + x_3 x_1\right) + \frac{185}{24} i\pi \\
&+ \frac{1}{72} \sum_{i=1}^{3} \frac{P_4(x_i, x_{i-1}, x_{i+1})}{x_{i-1} x_{i+1}} \ln x_i - \frac{1}{72} \left(x_1 x_2 + x_3 x_2 + x_1 x_3\right)^2 + \frac{247}{108} \left(x_1 x_2 + x_3 x_2 + x_1 x_3\right) \\
&+ \frac{1321}{216},
\end{aligned}
$$

➡ Kummer' function

$$
\Lambda_n(z) = \int_0^z \mathrm{d}t\, \frac{\ln^{n-1}|t|}{1+t} = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{k!} \ln^k |z|\, \mathrm{Li}_{n-k}(z)
$$

Higgs + 3 gluons

$$
\begin{aligned}
\overline{D}_\alpha^{(2)} &= -\zeta_3 + \frac{i\pi}{4} - \frac{1}{6}\left(x_1x_2 + x_3x_2 + x_1x_3\right) + \frac{67}{48} + \frac{P_5(x_1,x_2,x_3)}{72x_1^2x_2^2x_3^2}\pi^2 \\
&\quad + \frac{1}{12}\sum_{i=1}^{3}\left[\frac{P_6(x_i,x_{i-1},x_{i+1})}{x_{i-1}^2x_{i+1}^2}\,\mathrm{Li}_2(1-x_i) + \frac{P_7(x_i,x_{i-1},x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1}\right. \\
&\quad \left. + \frac{P_8(x_i,x_{i-1},x_{i+1})}{2x_{i-1}x_{i+1}}\ln x_i\right]
\end{aligned} \tag{7.19}
$$

$$
\begin{aligned}
\overline{E}_\alpha^{(2)} &= -\frac{i\pi^3}{48} - \frac{i\pi}{3}A_\alpha^{(1)} - \frac{1}{12}\ln\left(x_1x_2x_3\right)\left(\ln x_1\,\ln x_2 + \ln x_1\,\ln x_3 + \ln x_2\,\ln x_3\right) \\
&\quad + \frac{P_{13}(x_1,x_2,x_3)}{432} + \frac{7}{12}\ln x_1\,\ln x_2\,\ln x_3 - \frac{5}{48}\pi^2\ln\left(x_1x_2x_3\right) - \frac{29}{24}\zeta_3 \\
&\quad + \frac{11}{18}i\pi\,\ln(x_1x_2x_3) + \frac{P_{11}(x_1,x_2,x_3)}{288x_1^2x_2^2x_3^2}\pi^2 + \sum_{i=1}^{3}\left[\mathrm{Li}_3(x_i) - \frac{1}{3}\mathrm{Li}_3(1-x_i)\right. \\
&\quad + \frac{1}{6}\mathrm{Li}_2(1-x_i)\,\ln x_i + \frac{1}{2}\ln(1-x_i)\,\ln^2 x_i + \frac{1}{6}\ln(x_1x_2x_3)\,\mathrm{Li}_2(1-x_i) \\
&\quad + \frac{P_9(x_i,x_{i-1},x_{i+1})}{36x_{i-1}^2x_{i+1}^2}\mathrm{Li}_2(1-x_i) + \frac{P_{10}(x_i,x_{i-1},x_{i+1})}{36x_i^2}\ln x_{i-1}\,\ln x_{i+1} \\
&\quad \left. + \frac{11}{36}\ln^2 x_i + \frac{P_{12}(x_i,x_{i-1},x_{i+1})}{216x_{i-1}x_{i+1}}\ln x_i\right] - \frac{13}{36}i\pi\left(x_1x_2 + x_3x_2 + x_1x_3\right) - \frac{71}{18}i\pi\,,
\end{aligned} \tag{7.20}
$$

Higgs + 3 gluons

$$\overline{F}_{\alpha}^{(2)} = -\frac{i\pi}{18}\ln(x_1x_2x_3) - \frac{11}{144}\pi^2 + \frac{1}{36}\sum_{i=1}^{3}\ln^2 x_i - \frac{5}{54}\ln(x_1x_2x_3) + \frac{5i\pi}{18}$$
$$+ \frac{i\pi}{18}(x_1x_2 + x_2x_3 + x_3x_1) + \frac{5}{54}(x_1x_2 + x_3x_2 + x_1x_3)$$
$$- \frac{1}{72}(x_1x_2 + x_3x_2 + x_1x_3)^2 - \frac{x_1x_2x_3}{18}\sum_{i=1}^{3}\frac{\ln x_i}{x_i},$$

- Originally, the expressions filled up more than 6 pages!
- Bose symmetry is now completely manifest.
- Only simple functions (classical polylogarithms) with simple arguments.
  - ➡ easy numerical evaluation.
- Similar results can be obtained for $H \to g^+g^+g^-$.

Higgs + 3 gluons

- We can even do more!
- The analytic continuation to the scattering region is completely trivial now.
- In the scattering region, we have for example

$$s_{23} \to |s_{23}|\, e^{i\pi} \quad \text{and} \quad s_{13} \to |s_{13}|\, e^{i\pi}$$

- The analytic continuation formulas for the polylogarithms can again be worked using the coproduct, *e.g.*, for z > 0,

$$\mathrm{Li}_3\left(1 - z\, e^{i\delta\pi}\right) = \mathrm{Li}_3\left(\frac{1}{1+z}\right) - \frac{1}{6}\ln^3(1+z) - \frac{i}{2}\delta\pi\ln^2(1+z) + \frac{\pi^2}{3}\ln(1+z)$$

$$\mathrm{Li}_4\left(1 - z\, e^{i\delta\pi}\right) = -\mathrm{Li}_4\left(\frac{1}{1+z}\right) - \frac{1}{24}\ln^4(1+z) - \frac{i\delta\pi}{6}\ln^3(1+z) + \frac{\pi^2}{6}\ln^2(1+z) + \frac{\pi^4}{45}$$

Conclusion & Outlook

- The symbol is only the tip of the iceberg of a much deeper structure!
- This new way of looking at the problem is not only useful to simplify complicated expressions, but it might open new directions.
- We we have a way to determine the symbol of an amplitude, we can get additional information from the coproduct, by e.g., integrating in the first to two components

  ➡ Information about the zeta valued terms.

- Is there a way to determine directly the coproduct of an amplitude (instead of just its symbol)...?

Conclusion & Outlook

- Is there a way to determine directly the coproduct of an amplitude (instead of just its symbol)...?
- There is indeed a coproduct on Feynman graphs (the 'core Hopf algebra')

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma = \cup \gamma_i} \gamma \otimes \Gamma/\gamma.$$

- Could one make these two match...?
    - ➡ 'Break' the Feynman graph into subgraph, for which the analytic result is known.
    - ➡ Obtain in this way the coproduct (in terms of polylogarithms) of the original graph.
    - ➡ It is not obvious if/how this could work...

Conclusion & Outlook

- Is there a way to determine directly the coproduct of an amplitude (instead of just its symbol)...?
- There is indeed a coproduct on Feynman graphs (the 'core Hopf algebra') [Kreimer]

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