Towards the all-loop S-matrix of planar $\mathcal{N} = 4$ Super Yang-Mills

Song He

Max-Planck-Institut für Gravitationsphysik (AEI), Potsdam

Simon Caron-Huot & SH arXiv: 1112.1060.

Amplitudes 2012, DESY, Hamburg
March 8th, 2012

Plan of the talk

- Review of the S-matrix in planar $\mathcal{N} = 4$ SYM
- The S-matrix from symmetries
  - New differential equations
  - Solving the equations
- Jumpstarting amplitudes
  - Two-loop MHV
  - Two-loop NMHV
  - Three-loop MHV
  - Two-dimensional kinematics
- Outline of a derivation
- Summary and outlook

Review of the S-matrix in planar $\mathcal{N}=4$ SYM

- All the on-shell states in $\mathcal{N}=4$ SYM can be combined into an on-shell superfield,

$$\Phi := G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\varepsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\bar{\Gamma}^{D} + \frac{1}{4!}\varepsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-},$$

which depends on the Grassmann variable η^{A}, and a null momenta $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}}$.

- All color-ordered amplitudes are then packaged into a superamplitude $\mathcal{A}(\{\lambda_i, \bar{\lambda}_i, \eta_i\})$, which has an expansion in terms of Grassmann degrees $4k+8$,

$$\mathcal{A}_n = \mathcal{A}_{n,\mathrm{MHV}} + \mathcal{A}_{n,\mathrm{NMHV}} + \cdots + \mathcal{A}_{n,\overline{\mathrm{MHV}}} = \frac{\delta^4(\sum_i \lambda_i\bar{\lambda}_i)\delta^{0|8}(\sum_i \lambda_i\eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{k=0}^{n-3} A_{n,k},$$

where $A_{n,k}$ denotes the N^{k}MHV amplitude, with MHV tree, $\mathcal{A}^{\mathrm{tree}}_{n,\mathrm{MHV}}$, stripped off.

- $\mathcal{N}=4$ SYM is a *superconformal* field theory, which should be reflected in the structure of scattering amplitudes. The tree-level S-matrix is invariant under this $\mathfrak{psu}(2,2|4)$ symmetry: $\{\mathfrak{q}^{\alpha}_{A}, \bar{\mathfrak{q}}^{A}_{\dot{\alpha}}, \mathfrak{p}_{\alpha\dot{\alpha}}, \mathfrak{m}_{\alpha\beta}, \bar{\mathfrak{m}}_{\dot{\alpha}\dot{\beta}}, \mathfrak{s}^{A}_{\alpha}, \bar{\mathfrak{s}}^{\dot{\alpha}}_{A}, \mathfrak{k}_{\alpha\dot{\alpha}}, \mathfrak{d}, \mathfrak{r}^{A}_{B}\}$. At loop level, the superconformal symmetry of the S-matrix is broken by infrared divergences.

Review of the S-matrix in planar $\mathcal{N}=4$ SYM: dual symmetries

- A *dual* conformal symmetry has been observed at both weak [Drummond Henn Smirnov Sokatchev 2006] and strong couplings [Alday Maldacena 2007]. The symmetry has been generalized to a dual super-conformal symmetry [Drummond Henn Korchemsky Sokatchev 2008] of the dual chiral superspace,

$$x_i^{\alpha\dot\alpha} - x_{i-1}^{\alpha\dot\alpha} = \lambda_i^\alpha \bar\lambda_i^{\dot\alpha}, \qquad \theta_i^{\alpha A} - \theta_{i-1}^{\alpha A} = \lambda_i^\alpha \eta_i^A .$$

  The tree-level S-matrix is invariant under the dual $\mathfrak{psu}(2,2|4)$ symmetry.

- An all-loop, exponentiated ansatz for MHV amplitude in $4-2\epsilon$ dimensions has been proposed, which encodes infrared and collinear behavior [Anastasiou Bern Dixon Kosower 2003] [Bern Dixon Smirnov 2005],

$$A_n^{\rm BDS} = 1 + \sum_{\ell=1}^{\infty} g^{2\ell} A_n^{(\ell)}(\epsilon) := \exp\left[\sum_{\ell=1}^{\infty} g^{2\ell}\left(\Gamma_{\rm cusp}^{(\ell)}(\epsilon) A_{n,0}^{(1)}(\ell\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right)\right].$$

- MHV loop amplitudes satisfy an anomalous Ward identity for the dual conformal symmetry [Drummond Henn Korchemsky Sokatchev 2007]. For $n=4,5$, the only solution is given by the BDS ansatz, since there is no cross-ratios. A finite remainder function of $3(n-5)$ cross-ratios is allowed for n-point MHV amplitude, e.g. $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$ etc. for $n=6$.

Review of the S-matrix in planar $\mathcal{N}=4$ SYM: Wilson loops

- There is strong evidence for a duality between MHV amplitude and a null polygonal Wilson loop in dual spacetime [Brandhuber Heslop Travaglini 2007] [Drummond Henn Korchemsky Sokatchev 2007/08] [Bern Dixon Kosower Roiban Spradlin Vergu Volovich 2008]. On the string side, (fermionic) T-duality maps the original superconformal symmetry of the amplitude to the dual symmetry of the Wilson loop [Berkovits Maldacena 2008] [Beisert Ricci Tseytlin Wolf 2008], and their closure is the Yangian symmetry, $y[\mathfrak{psu}(2,2|4)]$ [Drummond Henn Plefka 2009].

- A generalized duality between the superamplitude and a supersymmetric Wilson loop has been derived at the integrand level [Mason Skinner 2010][Caron-Huot 2010], although a rigorous UV regularization for the super-loop has not been carried out [Belitzky Korchemsky Sokatchev 2011],

$$A_n(\lambda_i,\bar{\lambda}_i,\eta_i)=W_n(x_i,\theta_i)(1+\mathcal{O}(\epsilon)),\quad W_n=\frac{1}{N_c}\langle \mathrm{Tr}\mathcal{P}e^{-\oint \mathbf{A}(x_i,\theta_i)}\rangle.$$

- The chiral formalism obscures one chiral half of superconformal symmetries. As a natural generalization, Wilson loops in non-chiral $\mathcal{N}=4$ superspace generally manifest the full symmetry [Caron-Huot 2011] [Beisert Vergu 2012] [Beisert SH Schwab Vergu 2012], see Niklas's talk.

- One can obtain amplitudes by setting $\bar{\theta}=0$, but there is no obvious way to define non-chiral amplitudes dual to non-chiral Wilson loops. They contain additional terms, which can play a role for compensating symmetry anomalies of amplitudes.

Review of the S-matrix in planar $\mathcal{N}=4$ SYM: momentum twistors

- It is convenient to introduce unconstrained *momentum-twistor* variables [Hodges 2009],

$$\mathcal{Z}_i = (Z_i^a, \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha\dot\alpha}\lambda_{i\alpha}, \theta_i^{\alpha A}\lambda_{i\alpha}),$$

which are twistors of the dual (super)space. Then one can construct invariants,

$$\text{four-bracket}: \quad \langle ijkl\rangle := \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d, \quad \text{e.g.} \quad u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}.$$

$$\text{R-invariant}: \quad [i\,j\,k\,l\,m] := \frac{\delta^{0|4}(\chi_i^A\langle jklm\rangle + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}.$$

Yangian invariant tree amplitudes and leading singularities are built from (generally shifted) R-invariants, e.g. NMHV tree: $A_{n,1}^{\text{tree}} (= R_{n,1}^{\text{tree}}) = \sum_{1<i<j<n} [1\,i\,i{+}1\,j\,j{+}1]$.

- The dual superconformal generators all become first-order differential operators,

$$Q_A^a = (\mathfrak{Q}_A^\alpha, \bar{\mathfrak{S}}_A^{\dot\alpha}) := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial \chi_i^A}, \qquad \bar{Q}_a^A = (\mathfrak{S}_\alpha^A, \bar{\mathfrak{Q}}_{\dot\alpha}^A = \bar{s}_{\dot\alpha}^A) := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a},$$

$$K_b^a = (\mathfrak{P}_{\alpha\dot\alpha}, \mathfrak{K}_{\alpha\dot\alpha}, \mathfrak{M}_{\alpha\beta}, \bar{\mathfrak{M}}_{\dot\alpha\dot\beta}, \mathfrak{D}) := \sum_{i=1}^n Z_i^a \frac{\partial}{\partial Z_i^b}, \qquad R_B^A = \mathfrak{R}_B^A := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial \chi_i^B}.$$

The S-matrix from symmetries: new differential equations

- We define *BDS-subtracted* S-matrix: $A_{n,k} = A_n^{\mathrm{BDS}} \times R_{n,k}$, which is finite, depends on conformal cross-ratios and R-invariants, and has simple collinear limits: the k-preserving limit, $R_{n,k} \to R_{n-1,k}$, and the k-decreasing one, $\frac{\int d^4\chi_n R_{n,k}}{\int d^4\chi_n [n{-}2\, n{-}1\, n\, 1\, 2]} \to R_{n-1,k-1}$. By construction, $R_{4,0} = R_{5,0} = R_{5,1}/R_{5,1}^{\mathrm{tree}} = 1$.

- The BDS-subtracted S-matrix is invariant under Q_A^a, R_B^A, K_b^a, but not for (naive) $\bar{Q}_a^A$. We propose an all-loop equation in terms of collinear integral (see also [Bullimore Skinner 2011]),

$$\bar{Q}_a^A R_{n,k} = \Gamma_{\mathrm{cusp}}\, \mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3}\mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\mathrm{tree}}\right] + \text{cyclic},$$

  where the cusp anomalous dimension is known $\Gamma_{\mathrm{cusp}} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots$.

- For $\mathcal{Z}_{n+1}$, we integrate over $0 \le \tau < \infty$, and extract the coefficient of $d\epsilon/\epsilon$ as $\epsilon \to 0$,

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + \epsilon\tau \frac{\langle n{-}1 n 2 3\rangle}{\langle n 1 2 3\rangle} \mathcal{Z}_1 + \epsilon^2 \frac{\langle n{-}2\, n{-}1\, n 1\rangle}{\langle n{-}2\, n{-}1\, 2 1\rangle} \mathcal{Z}_2,$$

$$\mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3}\mathcal{Z}_{n+1})_a^A = \frac{\langle n{-}1 n 2 3\rangle}{\langle n 1 2 3\rangle} (n{-}1\, n\, 1)_a \oint_{\epsilon=0} \epsilon\, d\epsilon \int_0^\infty d\tau (d^{0|3}\chi_{n+1})^A.$$

The S-matrix from symmetries: new differential equations

- Using the discrete parity symmetry, we derive an equivalent equation for level-one generator, $Q^{(1)a}_A = (s^\alpha_A, \ldots) := \frac{1}{2}\sum_{i,j} \mathrm{sgn}(j-i)\left(Z^a_i \frac{\partial}{\partial Z^b_i} Z^b_j \frac{\partial}{\partial \chi^A_j} - Z^a_i \frac{\partial}{\partial \chi^B_i} \chi^B_j \frac{\partial}{\partial \chi^A_j}\right)$,

$$Q^{(1)a}_A R_{n,k} = \Gamma_{\text{cusp}} Z^a_n \lim_{\epsilon \to 0} \int_0^\infty \frac{d\tau}{\tau} (d\eta_{n+1})_A \left(R_{n+1,k} - \sum_{i,j} C_{i,j} \frac{\partial R_{n,k}}{\partial \chi_j} \right) + \text{cyclic.}$$

- The differential equations are finite, regulator independent, and manifest the transcendentality of loop amplitudes. On the RHS, the measures of integrating out a particle carry correct quantum numbers, and $1d$ integrals reflect that naive generators are violated since they cause asymptotic states to radiate collinearly.

- Given RHS of both equations as linear operators acting on S-matrix, they can be interpreted as *quantum corrections* to the naive generators [Bargheer Beisert Galleas Loebbert McLoughlin 2009] [Sever Vieira 2009], in which sense the BDS-subtracted S-matrix is Yangian invariant!

- We claim the equations to be valid for any value of the coupling (the explicit dependence is only through Γ_{cusp}), and they determine the all-loop S-matrix.

The S-matrix from symmetries: solving the equations

- The RHS of $\bar{Q}$ equation can be evaluated at the τ-integrand level, $X = Z_n \wedge Z_{n+1}$,

$$\int d^{2|3}\mathcal{Z}_{n+1}[i\,j\,k\,n\,n{+}1]f(\tau,\epsilon) = \bar{Q}\log\frac{\langle \bar{n}j\rangle}{\langle \bar{n}i\rangle}\int_0^\infty d\log\frac{\langle Xij\rangle}{\langle Xjk\rangle}f(\tau,\epsilon\to 0) + (j \leftrightarrow k).$$

  This immediately gives $\bar{Q}$ of all one-loop N^{k}MHV amplitudes (see [Beisert Henn McLoughlin Plefka 2010]).

- Generally for computing $\bar{Q}$ of N^{k}MHV amplitudes, we write $R_{n+1,k+1} - R_{n,k}R^{\text{tree}}_{n+1,1}$ in terms of N^{k+1}MHV leading singularities $\times$ pure functions f, and the integral gives a finite set of prefactors as N^{k}MHV leading singularities $\times$ $\bar{Q}\log(\ldots)$, which can be determined to all loops, and some pure functions $F = \int d\log(\ldots)f(\epsilon\to 0)$.

- For MHV amplitude, since $R_{n,0}$ is independent of Grassmann variables, $\bar{Q}$ equation gives all derivatives, $\frac{\partial}{\partial\chi_i^1}\bar{Q}_a^1 = \frac{\partial}{\partial Z_i^a}$ for $a = 1,\ldots 4$ and $i = 1,\ldots,n$, and uniquely determine MHV amplitudes up to a constant, to be fixed by a collinear limit.

- The total derivative of MHV remainder is $dR_{n,0} = \sum_{i,j} F_{i,j}\, d\log\langle\bar{i}j\rangle$, with $F_{i,j}$ from $1d$ integrals of pure functions of NMHV, which proves the conjecture of [Caron-Huot 2011].

The S-matrix from symmetries: solving the equations

- Any Q-invariant NMHV expression can be expanded in a basis of R-invariants,

$$C = \sum_{2 \le j < k < l < m \le n} [1\, j\, k\, l\, m] C_{j,k,l,m}(Z).$$

If it is also invariant under naive $\bar{Q}$, we extract the component $\chi_i^1 \chi_j^1 \chi_k^2 \chi_l^3 \chi_m^4$ of $Z_j^a \bar{Q}_a^1 C$, which can only arise from $[1\, j\, k\, l\, m]$, thus $Z_j^a \frac{\partial}{\partial Z_i^a} C_{j,k,l,m} = 0$.

- Repeating for k, l, m and all i's we deduce $C_{j,k,l,m}$ depends only on $1, j, k, l, m$. No non-trivial conformal invariant functions of five twistors means $C_{j,k,l,m} = \mathrm{const}$.

- Thus NMHV is uniquely determined by $\bar{Q}$ equation up to a linear combination of R-invariants, which is again fixed by collinear limits. In practice, the crucial step is to complete the arguments of $\bar{Q} \log$ in prefactors into conformal cross-ratios.

- Beyond NMHV level, we also need to use $Q^{(1)}$ equation. It is known [Korchemsky Sokatchev 2010][Drummond Ferro 2010] that all invariants under naive generators Q, $\bar{Q}$ and $Q^{(1)}$ are given by Grassmannian residues [Arkani-Hamed Cachazo Cheung Kaplan 2009]. Up to linear combinations of such invariants, all-loop N^{k}MHV amplitudes are determined by both equations!

Jumpstarting amplitudes: two-loop MHV

- The $\bar{Q}$ of two-loop MHV hexagon is given by the collinear integral of $R_{7,1}^{\text{1-loop}}$,

$$\bar{Q} R_{6,0}^{\text{2-loop}} = (I_{1,1} + I_{1,2}) \bar{Q} \log \frac{\langle 5613 \rangle}{\langle 5612 \rangle} + (I_{2,1} + I_{2,2}) \bar{Q} \log \frac{\langle 5614 \rangle}{\langle 5612 \rangle} + \text{cyclic},$$

where it is of paramount importance to us that upon τ-integral $I_{1,2}$ and $I_{2,2}$ vanish,

$$I_{1,2} = \log \epsilon^2 \times \int_0^\infty d \left(\log \frac{u_3(\tau+1)}{\tau+u_3} \log(\frac{\tau}{\tau+u_3}) + \log(\tau+1) \log \frac{\tau+u_3}{\tau+1} \right) = 0,$$
$$I_{2,2} = \log \epsilon^2 \times \int_0^\infty d \left(\log \frac{\tau+u_3}{\tau} \log \frac{u_3}{\tau+u_3} \right) = 0.$$

It is straightforward to obtain the finite integrals, in terms of 6D hexagon integral,

$$I_{1,1} = \left(\frac{1}{3} \log^2 u_3 + \log u_1 \log u_2 + \sum_{i=1}^{3} \mathrm{Li}_2(1-u_i) \right) \log u_3 - 2\mathrm{Li}_3(1-\frac{1}{u_3}),$$

$$I_{2,1} = -\frac{1}{2} I_6^{6D} + \sum_{i=1}^{3} (-)^{\delta_{3i}} \mathrm{Li}_3(1-\frac{1}{u_i}) + \frac{1}{2} \log \frac{u_2 u_3}{u_1} \sum_{i=1}^{3} \mathrm{Li}_2(1-\frac{1}{u_i}) + \frac{1}{12} \log^3 \frac{u_2 u_3}{u_1}.$$

Jumpstarting amplitudes: two-loop MHV

- Therefore, we obtain the total differential of $R_{6,0}^{\text{2-loop}}$ in a very compact form,

$$dR_{6,0}^{\text{2-loop}} = I_6^{6D}\, d\log\frac{x^+}{x^-} + \left(I_{1,1}\, d\log\frac{1-u_3}{u_3} + \text{two cyclic images}\right),$$

which can be integrated and agrees precisely with [Del Duca Duhr Smirnov 2010][Goncharov Spradlin Vergu Volovich 2010],

$$R_{6,0}^{\text{2-loop}} = 4\sum_{i=1}^{3}\left(L_4^+(u_i) - \frac{1}{2}\mathrm{Li}_4(1-\frac{1}{u_i})\right) - \frac{1}{2}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-\frac{1}{u_i})\right)^2 + \frac{1}{6}J^4 + \frac{\pi^2}{3}J^2 + \frac{\pi^4}{18}.$$

- There is no qualitative difference between $n > 6$ cases and the hexagon. The $\log\epsilon^2$ terms integrate to zero, leaving finite, conformal integrals, which can be easily evaluated at the level of symbol. The result agrees with [Caron-Huot 2011] up to $n = 10$.

- Furthermore, we can choose an integral path connecting a collinear $(n-1)$-gon to the original n-gon, and obtain an integral representation for two-loop n-point MHV. We hope to compare [Caron-Huot SH unpublished 2011] with numerical results in [Anastasiou Brandhuber Heslop Khoze Spence Travaglini 2009].

Jumpstarting amplitudes: two-loop NMHV

- NMHV hexagon is given by collinear integral of N^{2}MHV heptagon. From its leading singularities, we get 7×6 prefactors, and conformal symmetry removes one,

$$dR_{6,1} = \sum_{i=1}^{41} (\text{R-invariants})_i \times F_i \times d\log(\text{cross-ratios})_i,$$

which holds to all loops! We compute F_i at two-loop and write (see James's talk),

$$R_{6,1}^{\text{2-loop}} = [(1){+}(4)]V_3{+}[(2){+}(5)]V_1{+}[(3){+}(6)]V_2{+}[(1){-}(4)]\tilde{V}_3{+}[(5){-}(2)]\tilde{V}_1{+}[(3){-}(6)]\tilde{V}_2$$

where V's and $\tilde{V}$'s are degree-4 functions, with differentials as follows,

$$dV_3 = -\frac{1}{2} I_6^{6D} d\log\frac{y_2}{y_3} + (dV_3)_1 d\log\frac{u_1}{(1-u_2)(1-u_3)} + (dV_3)_2 d\log\frac{1-u_1}{u_2 u_3}$$
$$+ \left((dV_3)_3 d\log\frac{u_2}{1-u_2} + (u_2 \leftrightarrow u_3) \right),$$

$$d\tilde{V}_3 = \frac{1}{2} I_6^{6D} d\log\frac{u_2(1-u_3)}{(1-u_2)u_3} + (d\tilde{V}_3)_1 d\log y_1 + (d\tilde{V}_3)_2 d\log y_2 y_3 + (d\tilde{V}_3)_3 d\log\frac{y_2}{y_3}.$$

We find the *function* agrees with the results in [Kosower Vergu Roiban 2011] and [Dixon Drummond Henn 2011].

Jumpstarting amplitudes: three-loop MHV

- We can get all-loop predictions and the symbol at two loops for higher-point NMHV amplitudes. For heptagon, there are 288 independent prefactors, and its symbol has been obtained, but the complexity increases rapidly with n [Caron-Huot SH unpublished 2011].

- Based on physical considerations and assumptions on the symbol, an ansatz for the symbol of three-loop MHV hexagon was proposed [Dixon Drummond Henn 2011] (James's talk). From NMHV hexagon we confirm the assumptions, fix the two free parameters,

$$S[R_{6,0}^{\text{3-loop}}] = \left(S[X] - \frac{3}{8}S[f_1] + \frac{7}{32}S[f_2]\right)(u_1, u_2, u_3),$$

- It is possible that by fixing two-loop N^{2}MHV (e.g. N^{2}MHV heptagon is given by the parity conjugate of NMHV, and the octagon is doable.), one could obtain the symbol of three-loop NMHV and even four-loop MHV using $\bar{Q}$ equations.

- Furthermore, one can make all-loop predictions, e.g. determine the final entry of NMHV symbol, which in turn gives the next-to-final entry of MHV symbol. Together with $Q^{(1)}$ equation, we can certainly go on in the higher n, k, ℓ direction.

Jumpstarting amplitudes: two-dimensional kinematics

- It is useful to consider $(2n)$ external momenta embedded in a two dimensional sub-space, which reduces the superconformal group $SU(2,2|4)$ to $SL(2|2) \times SL(2|2)$,

$$\mathcal{Z}_{2i-1} = (\lambda_i^{1+}, 0, \lambda_i^{2+}, 0, \chi_i^{1+}, 0, \chi_i^{2+}, 0), \qquad \mathcal{Z}_{2i} = (0, \lambda_i^{1-}, 0, \lambda_i^{2-}, 0, \chi_i^{1-}, 0, \chi_i^{2-}).$$

  The only non-vanishing four-brackets are $\langle 2i{-}1\, 2j{-}1\, 2k\, 2l\rangle = \langle ij\rangle^{+}\langle kl\rangle^{-}$, and we can define odd and even cross-ratios $u_{i,j}^{\pm} := \frac{\langle i j{+}1\rangle^{\pm}\langle i{+}1 j\rangle^{\pm}}{\langle ij\rangle^{\pm}\langle i{+}1 j{+}1\rangle^{\pm}}$.

- For superamplitude, R-invariants also factorize into odd and even parts, $[*\, 2i{-}1\, 2j{-}1\, 2k\, 2l] = (*\, i\, j)[*\, k\, l]$ where $(*\, i\, j) := \frac{\delta^{0|2}(\langle\langle *\, i\, j\rangle\rangle^{+})}{\langle *\, i\rangle^{+}\langle i\, j\rangle^{+}\langle j\, *\rangle^{+}}$ and similarly for $[*\, k\, l]$, which all satisfy $(a\, b\, c) - (a\, b\, d) + (a\, c\, d) - (b\, c\, d) = 0$. In this notation, the NMHV tree is $R_{2n,1}^{\text{tree}} = \frac{1}{2}\sum_{i,j}(*\, i\, j)\,([i\, j{-}1\, j] - [i{-}1\, j{-}1 j])$.

- The natural collinear limit in 2d is a triple-collinear limit, $R_{2n} \to f(g^2)R_{2n-2}$, and it is convenient to rescale the BDS-subtracted amplitude so that it has natural k-preserving and decreasing limits, $R_{2n,k} := e^{(n-2)f_1(g^2)+k f_2(g^2)}\tilde{R}_{2n,k}$, where

$$f_1(g^2) = -\Gamma_{\text{cusp}}^2\frac{\pi^4}{9} + \mathcal{O}(\Gamma_{\text{cusp}}^3), \quad f_2(g^2) = -\Gamma_{\text{cusp}}\frac{\pi^2}{3} + \Gamma_{\text{cusp}}^2\frac{7\pi^4}{30} + \mathcal{O}(\Gamma_{\text{cusp}}^3).$$

- In the even sector, the $\bar{Q}$ equation in 2d is ($\lambda^{\pm}_{n+1}=\lambda^{\pm}_n+\epsilon\lambda^{\pm}_1, \chi^{\pm}_{n+1}=\chi^{\pm}_n+\epsilon\chi^{\pm}_1$),

$$\bar{Q}^A_a \tilde{R}_{2n,k}=\Gamma_{\text{cusp}}\int d^{1|2}\lambda^+_{n+1}\int d^{0|1}\lambda^-_{n+1}(\tilde{R}_{2n+2,k+1}-R^{\text{tree}}\tilde{R}_{2n,k})+\text{cyclic.}$$

- One can easily write down N^{2}MHV tree, including the degenerate terms,

$$R^{\text{tree}}_{2n,2}=\frac{1}{2}\sum_{i<j<k<l<i}(*\,i\,j)(*\,k\,l)\left([i\,j{-}1\,j]-[i{-}1\,j{-}1j]\right)\left([k\,l{-}1\,l]-[k{-}1\,l{-}1l]\right)+\ldots,$$

from which we can get one-loop NMHV, and the two-loop MHV amplitude,

$$\tilde{R}^{\text{2-loop}}_{2n,0}=-\sum_{i<j<k<l<i}\log\langle ik\rangle\log\langle jl\rangle\log u^-_{i-1,k-1}\log u^-_{j-1,l-1}-2\sum_{i<j<k<i}\log\langle ij\rangle\log\langle jk\rangle$$
$$\times\left(\log u^-_{j-1,k-1}\log u^-_{k-1,i,i-1,j}+\text{cyclic}\right)-\sum_{i<j<i}\log^2\langle ij\rangle\log u^-_{i-1,j-1}\log(1-u^-_{i-1,j-1}).$$

Nothing prevents us from getting n-point two-loop NMHV and three-loop MHV. Main complexity of loop amplitudes in 2d already hidden inside tree amplitudes?

Outline of a derivation

- A heuristic derivation expresses the RHS of $\bar{Q}$ equation in terms of a fermion excitation inserted on each edge of the Wilson loop (See also Dave's talk [Bullimore Skinner 2011]),

$$\bar{Q}^A_{\dot{\alpha}}\langle W_n\rangle \propto g^2 \oint dx_{\dot{\alpha}\alpha}\langle(\psi^A + F\theta^A + \ldots)^\alpha W_n\rangle,$$

which was calculated in explicit examples by Feynman diagrams [Caron-Huot 2011].

- The key new ingredient: there is only one fermionic excitation of null edges with given quantum numbers. The Operator Product Expansion [Alday Gaiotto Maldacena Sever Vieira 2010] allows us to extract the excited n-gon Wilson loop from an $(n+1)$-gon in collinear limit,

$$\frac{1}{A_n^{\text{BDS}}}\bar{Q}\langle W_{n,k}\rangle = \frac{g^2}{F(g^2)}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R_{n+1,k+1}(\tau,\epsilon) + \text{cyclic}.$$

Given that BDS ansatz is one-loop exact, we obtain the $\bar{Q}$ of BDS,

$$\langle W_{n,k}\rangle\bar{Q}\frac{1}{A_n^{\text{BDS}}} = -\Gamma_{\text{cusp}}R_{n,k}\,\text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}R^{\text{tree}}_{n+1,1}(\tau,\epsilon) + \text{cyclic}.$$

- Both τ-integrals diverge, and the combination is finite provided $g^2/F(g^2) = \Gamma_{\text{cusp}}$. A crucial test of the above derivation is to check the prefactor is indeed Γ_{cusp}.

Outline of a derivation

- The first non-trivial check of the prefactor $\Gamma_{\rm cusp}$ is the two-loop NMHV hexagon,

$$R_{6,1}^{\rm tree}+\Gamma_{\rm cusp}R_{6,1}^{\text{1-loop}}+\Gamma_{\rm cusp}^2 R_{6,1}^{\text{2-loop}}+\ldots = R_{6,1}^{\prime\,\rm tree}+g^2 R_{6,1}^{\prime\,\text{1-loop}}+g^4(R_{6,1}^{\prime\,\text{2-loop}}-\frac{\pi^2}{3}R_{6,1}^{\prime\,\text{1-loop}})+\ldots.$$

The fermion excitation can be labeled by a momentum, p, and we can read off

$$f(\epsilon,p)=\int_0^\infty d\tau\,\tau^{i\frac{p}{2}}\,d^{0|3}\chi_6\,R_{6,1}(\epsilon,\tau).$$

For twist-one fermion insertion, OPE predicts $\lim_{\epsilon\to 0} f(\epsilon,p)=\log\epsilon\times\gamma(p)+C(p)$, where $\gamma(p)$ is the dispersion relation known for any couplings by integrability [Basso 2010].

- From $R_{6,1}^{\text{2-loop}}$, we obtain $\gamma(p)$ to order $\Gamma_{\rm cusp}^2$, and we find agreement including π^2's!

$$\gamma(p)=\Gamma_{\rm cusp}\,(\psi_+-\psi(1))-\frac{\Gamma_{\rm cusp}^2}{8}\left(\psi_+''+4\psi_-'(\psi_- -\frac{1}{p})+6\zeta(3)\right).$$

The cancelation of $\log\epsilon$ in total-τ integral is guaranteed: the zero-momentum fermion is the Goldstone fermion for the symmetry breaking, thus $\gamma(0)=0$.

Summary and outlook

- The all-loop S-matrix in planar $\mathcal{N}=4$ SYM is invariant under a suitably deformed Yangian symmetry at the quantum level, and is fully determined by it.
- We derive new, elegant equations based on the quantum-corrected symmetry, and test them extensively against e.g. results of loop amplitudes and OPE.
- We expect the equations to provide a powerful engine for computation of multi-loop amplitudes, and insights into the integrability of the theory.
- Open questions
  - Getting the actual functions, from symbol, or directly from the integral?
  - Corrected Yangian invariants from non-chiral Wilson loops [Beisert SH Schwab Vergu 2012]?
  - Construct quantum-corrected transfer matrices for the Yangian symmetry? Strong coupling tests of the equations? Relations to TBA, Y-system?
  - At least in 2d kinematics, can we imagine to compute certain amplitudes to all loops? Even as functions of the coupling constant?