The three-loop form factor in $\mathcal{N} = 4$ super Yang-Mills theory

Tobias Huber

Universität Siegen

In collaboration with
Thomas Gehrmann and Johannes Henn

Based on arXiv:1112.4524

Amplitudes 2012, DESY Hamburg. Friday, March 9th, 2012

Outline

- Intro, $\mathcal{N} = 4$ SYM
- Form factor definition, result to two loops
- Calculation at three-loops
  - Result, exponentiation
  - Unitarity cuts
  - Momentum routing invariances
- Form factor in $D > 4$, UV behaviour
- Conclusion

Introduction, motivation

- Form factors (FF) are closely related to scattering amplitudes (SA)
  - Factorisation of planar amplitudes [Bern,Dixon,Smirnov'05]

    "infrared divergent part" $\times$ "infrared finite remainder"

  - Infrared finite remainder
    - "Hard" function in QCD terminology
  - Infrared divergent part
    - given by a product of form factors
    - exponentiates and has a simple universal form
    - For 4- and 5-point SAs exponentiation carries over to finite part
    - Consequence of dual conformal symmetry
    - Relation to FFs makes it possible to give operator definition of finite remainder

Introduction, motivation

- SAs in $\mathcal{N} = 4$ SYM have many special properties
  - Dual conformal symmetry and all that
- How much of the simplicity in $\mathcal{N} = 4$ SAs carries over to FFs?
- Form factors are simpler than SAs
  - No ratios of Mandelstam variables (like s/t)
  - Only trivial scale dependence
- Form factors are more complicated than SAs
  - Non-planar diagrams
  - Unitarity approach:
    Sensitive to subleading double trace terms of SAs
- Goal: Extend van Neerven's calculation from 1986 to three loops

$\mathcal{N} = 4$ SYM

- Particle content of $\mathcal{N} = 4$ SYM
  - spin-1 gauge field A_μ
  - left Weyl fermions $\lambda^a_\alpha, \quad a = 1, \ldots, 4$
  - Real scalars $X^i, \quad i = 1, \ldots, 6$
  - They form the $\mathcal{N} = 4$ *Gauge Multiplet* $(A_\mu \ \lambda^a_\alpha \ X^i)$
- Under rotations $SU(4)_R$ in superspace
  - A_μ is a singlet
  - λ^a_α is a **4**
  - X^i are a rank 2 anti-symmetric **6**

  $\Longrightarrow$ Denote them by $\phi_{AB} = \phi^a_{AB} T_a \ , \qquad A, B = 1, \ldots, 4$

$\mathcal{N} = 4$ SYM

- Lagrangian density

$$\mathcal{L} = \mathrm{tr}\Big\{ -\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \Big\}$$

- C_{iab}: Constants related to the Clifford Dirac matrices for $SO(6)_R \sim SU(4)_R$
- $\mathcal{N} = 4$ SYM has vanishing β-function

Definition of the form factor

- Introduce bilinear Operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Properties of $\mathcal{O}$
  - $\mathcal{O}$ is a component of the stress-energy supermultiplet of $\mathcal{N} = 4$ SYM
  - $\mathcal{O}$ is a colour singlet and has zero anomalous dimension
- Definition of the form factor

$$\mathcal{F}_{\mathcal{S}} = \langle \phi^a_{34}(p_1)\phi^b_{34}(p_2)\,\mathcal{O}\rangle = \mathrm{Tr}(T^a T^b) F_{\mathcal{S}}$$

  - $\phi^a_{34}(p_i)$: on-shell states in the adjoint representation
  - Only scale is $(p_1 + p_2)^2 \equiv q^2$
- Work in dimensional regularisation, $D = 4 - 2\epsilon$
- 't Hooft coupling: $a = \frac{g^2\,N}{8\pi^2}\,(4\pi)^\epsilon\,e^{-\epsilon\gamma_E}$
- Perturbative expansion of the FF, $x = \mu^2/(-q^2 - i\eta)$

$$F_S = 1 + a\,x^\epsilon\,F_S^{(1)} + a^2\,x^{2\epsilon}\,F_S^{(2)} + a^3\,x^{3\epsilon}\,F_S^{(3)} + \mathcal{O}(a^4)$$

Definition of the form factor

- Up to $L = 3$ loops, have $F_S^{(L)} \propto N^L$
- Dependence on N is exact
- Possible colour structures at three loops

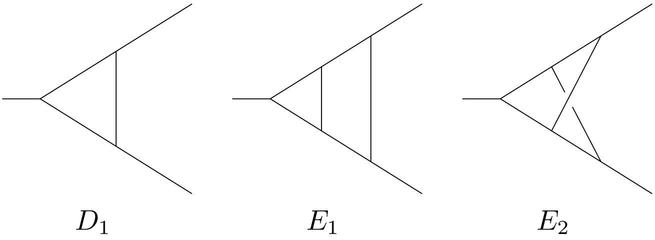
$$A = f^{abg} f^{bcg} f^{cdh} f^{edi} f^{efi} f^{fah} = (N^2 - 1)N^3$$
$$B = f^{abg} f^{bch} f^{cdg} f^{dei} f^{eif} f^{fha} = -\frac{1}{2}(N^2 - 1)N^3$$
$$C = f^{abg} f^{bch} f^{cdi} f^{deg} f^{ehf} f^{fia} = 0$$

- Normalise to $$\sum_{a,b} \delta^{ab} \mathrm{Tr}(T^a T^b) = N^2 - 1$$

- This changes at four loops due to $(d_{abcd})^2$
  C. f. computation of QCD β-function at four loops

[Larin,van Ritbergen,Vermaseren'97]

Form factor at one and two loops



- Form factor to two loops [van Neerven'86]

$$F_S = 1 + a\,x^{\epsilon}\,R_{\epsilon} \cdot 2\,D_1 + a^2\,x^{2\epsilon}\,R_{\epsilon}^2 \cdot [4\,E_1 + E_2]$$

- Result is remarkably simple
- Non-planar diagram due to “double trace terms” $\times$ “colour algebra”

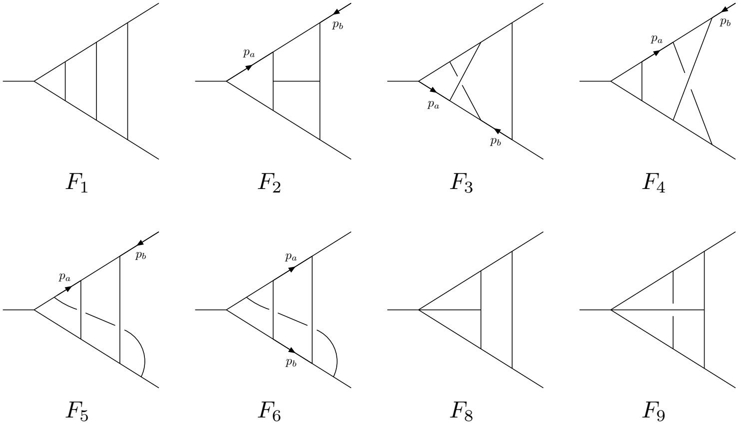
Form factor at one and two loops

$$
\begin{aligned}
F_S^{(1)} &= R_\epsilon \cdot 2\, D_1 \\
&= -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3}\,\epsilon + \frac{47\pi^4}{1440}\,\epsilon^2 + \epsilon^3\left(\frac{31\zeta_5}{5} - \frac{7\pi^2\zeta_3}{36}\right) + \epsilon^4\left(\frac{949\pi^6}{120960} - \frac{49\zeta_3^2}{18}\right) \\
&+ \epsilon^5\left(-\frac{329\pi^4\zeta_3}{4320} - \frac{31\pi^2\zeta_5}{60} + \frac{127\zeta_7}{7}\right) + \epsilon^6\left(\frac{49\pi^2\zeta_3^2}{216} - \frac{217\zeta_3\zeta_5}{15} + \frac{18593\pi^8}{9676800}\right) \\
&+ \mathcal{O}(\epsilon^7)\,,
\end{aligned}
$$

$$
\begin{aligned}
F_S^{(2)} &= R_\epsilon^2 \cdot [4\, E_1 + E_2] \\
&= +\frac{1}{2\epsilon^4} - \frac{\pi^2}{24\epsilon^2} - \frac{25\zeta_3}{12\epsilon} - \frac{7\pi^4}{240} + \epsilon\left(\frac{23\pi^2\zeta_3}{72} + \frac{71\zeta_5}{20}\right) + \epsilon^2\left(\frac{901\zeta_3^2}{36} + \frac{257\pi^6}{6720}\right) \\
&+ \epsilon^3\left(\frac{1291\pi^4\zeta_3}{1440} - \frac{313\pi^2\zeta_5}{120} + \frac{3169\zeta_7}{14}\right) \\
&+ \epsilon^4\left(-66\zeta_{5,3} + \frac{845\zeta_3\zeta_5}{6} - \frac{1547\pi^2\zeta_3^2}{216} + \frac{50419\pi^8}{518400}\right) + \mathcal{O}(\epsilon^5)\,.
\end{aligned}
$$

- Leading coefficient at L loops is $1/\epsilon^{2L}$
- Each diagram has uniform transcendentality (UT) in its ϵ-expansion

Result at three loops



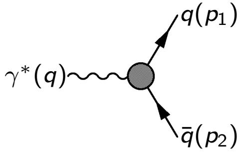
$$F_S = 1 + a\,x^{\epsilon}\,R_{\epsilon} \cdot 2\,D_1 + a^2\,x^{2\epsilon}\,R_{\epsilon}^2 \cdot [4\,E_1 + E_2]$$
$$+\, a^3\,x^{3\epsilon}\,R_{\epsilon}^3 \cdot [8\,F_1 - 2\,F_2 + 4\,F_3 + 4\,F_4 - 4\,F_5 - 4\,F_6 - 4\,F_8 + 2\,F_9]$$

- Also at three loops the result is remarkably simple.
  Only a few scalar diagrams contribute.

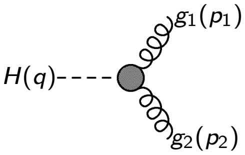
Result at three loops

- The ϵ-expansions of all integrals are known from the three-loop quark and gluon form factor in QCD [Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09] [Gehrmann,Glover,Ikizlerli,Studerus,TH'10] [Tödtli'09]
- Reduce amplitude to master integrals using IBP. [Chetyrkin,Tkachov'81] [Laporta'01][Anastasiou,Lazopoulos'04][Smirnov'08][Studerus'09]
- Compute masters (${}_pF_q$, Mellin Barnes ...) [Gehrmann,Heinrich,Studerus,TH'05] [Heinrich,Maître,TH'07][Heinrich,Kosower,Smirnov,TH'09][Lee,Smirnov,Smirnov'10]

- Application of quark form factor in QCD
  - DIS, Drell-Yan process $q\bar{q} \to W^{\pm}, Z^0, \gamma^*$
  - Two-parton contribution to $e^+e^- \to$ jets
  - Determination of α_s and its running at ILC



- Gluon form factor in QCD:
  - From effective $H\,gg$ vertex.
    Integrate out t-quark
  - $\mathcal{L}_{eff} = -\frac{C_1(\alpha_s)}{4v} H\, G^{\mu\nu}\, G_{\mu\nu}$
  - Application in Higgs production $gg \to H$



Result at three loops

- Back to $\mathcal{N} = 4$ SYM
- Again, each diagram has UT in its ϵ-expansion

$$F_S^{(3)} = R_\epsilon^3 \cdot [8\,F_1 - 2\,F_2 + 4\,F_3 + 4\,F_4 - 4\,F_5 - 4\,F_6 - 4\,F_8 + 2\,F_9]$$
$$= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right)$$
$$- \frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right)$$
$$+ \epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)$$

- Leading transcendentality (LT) principle [Kotikov,Lipatov,Onishchenko,Velizhanin'04]
  - Specify QCD quark, gluon FF to a SUSY theory ($C_A = C_F = 2T_F,\ n_f = 1$)
  - LT pieces of quark, gluon, and $\mathcal{N} = 4$ SYM form factor coincide!
  - Holds at $L = 1, 2, 3$ loops
  - Holds in all coefficients up to transcendental weight 8 (!!)

Logarithm of the form factor

- Logarithm of the form factor

$$
\begin{aligned}
\ln(F_S) &= \ln\left(1 + a\,x^{\epsilon}\,F_S^{(1)} + a^2\,x^{2\epsilon}\,F_S^{(2)} + a^3\,x^{3\epsilon}\,F_S^{(3)}\right) \\
&= a\,x^{\epsilon}\,F_S^{(1)} + a^2\,x^{2\epsilon}\left[F_S^{(2)} - \frac{1}{2}\left(F_S^{(1)}\right)^2\right] + a^3\,x^{3\epsilon}\left[F_S^{(3)} - F_S^{(1)}F_S^{(2)} + \frac{1}{3}\left(F_S^{(1)}\right)^3\right]
\end{aligned}
$$

- At most double poles, expected from exponentiation

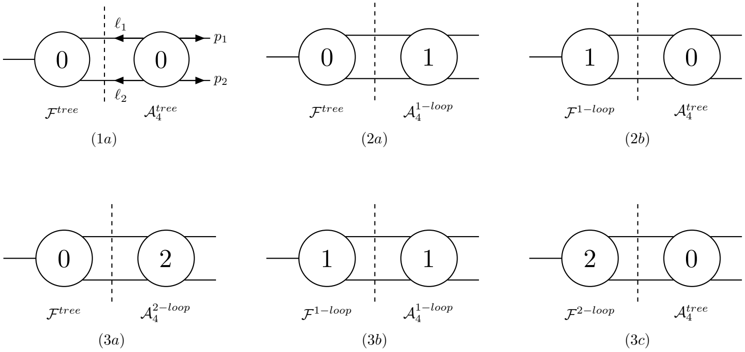
$$
\begin{aligned}
F_S^{(1)} &= -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3}\,\epsilon + \frac{47\pi^4}{1440}\,\epsilon^2 + \ldots \\
F_S^{(2)} - \frac{1}{2}\left(F_S^{(1)}\right)^2 &= \frac{\pi^2}{24\epsilon^2} + \frac{\zeta_3}{4\epsilon} + 0 + \epsilon\left(\frac{39\zeta_5}{4} - \frac{5\pi^2\zeta_3}{72}\right) + \ldots \\
F_S^{(3)} - F_S^{(1)}F_S^{(2)} + \frac{1}{3}\left(F_S^{(1)}\right)^3 &= -\frac{11\pi^4}{1620\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{5\pi^2\zeta_3}{54} - \frac{2\zeta_5}{3}\right) - \frac{13\zeta_3^2}{9} - \frac{193\pi^6}{25515} + \ldots
\end{aligned}
$$

- Poles give correct values of cusp and collinear anomalous dimension
- Finite piece does not exponentiate

Unitarity cuts

[Bern,Dixon,Dunbar,Kosower'94]

- Two-particle cuts to three loops



Scattering amplitudes to two loops

- Tree-level amplitude

$$\mathcal{A}_4^{tree} = g^2\mu^{2\epsilon} \sum_{\sigma\in S_4/Z_4} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})A_{4;1;1}^{tree}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))$$

- One-loop amplitude

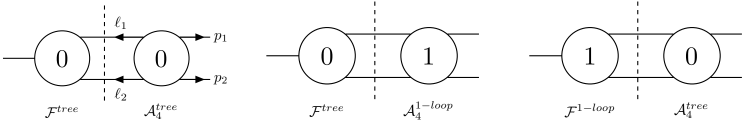
$$\mathcal{A}_4^{1-loop} = g^4\mu^{4\epsilon} \sum_{\sigma\in S_4/Z_4} N\,\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})A_{4;1,1}^{1-loop}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))$$
$$+ g^4\mu^{4\epsilon} \sum_{\sigma\in S_4/Z_2^3} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}})\mathrm{Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})A_{4;1,3}^{1-loop}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))$$

- Two-loop amplitude

[Bern,Rozowsky,Yan'97]

$$\mathcal{A}_4^{2-loop} = g^6\mu^{6\epsilon} \sum_{\sigma\in S_4/Z_4} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})\times$$
$$\times\left(N^2\,A_{4;1,1}^{2-loop,LC}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) + A_{4;1,1}^{2-loop,SC}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))\right)$$
$$+ g^6\mu^{6\epsilon} \sum_{\sigma\in S_4/Z_2^3} N\,\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}})\mathrm{Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}})A_{4;1,3}^{2-loop}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))\,.$$

Unitarity cuts



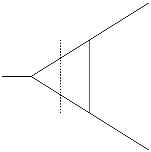
(1a) (2a) (2b)

- One loop, *cut (1a)*

$$\mathcal{F}_S^{1-loop}\Big|_{\mathrm{cut(1a)}} = \int \sum_{P_1,P_2} \frac{d^Dk}{(2\pi)^D}\, \frac{i}{\ell_2^2}\, \mathcal{F}_S^{tree}(-\ell_1,-\ell_2)\, \frac{i}{\ell_1^2}\, \mathcal{A}_4^{tree}(\ell_2,\ell_1,p_1,p_2)\Big|_{\ell_1^2=\ell_2^2=0}$$

$$= g^2\mu^{2\epsilon}\, N\, s_{12}\, \mathrm{Tr}(T^aT^b) \int \frac{d^Dk}{(2\pi)^D}\, \frac{i}{\ell_2^2}\, \frac{i}{\ell_1^2} \left(\frac{-i}{(p_1+\ell_1)^2} + \frac{-i}{(p_2+\ell_2)^2}\right)\Big|_{\ell_1^2=\ell_2^2=0}$$

$$= -2\, g^2\mu^{2\epsilon}\, N\, s_{12}\, \mathrm{Tr}(T^aT^b)\, \underbrace{\int \frac{d^Dk}{i(2\pi)^D}\, \frac{1}{k^2(k+p_1)^2(k-p_2)^2}}_{D_1}\Big|_{\mathrm{cut\,(1a)}}$$

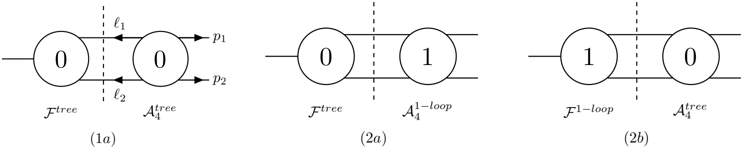


D_1

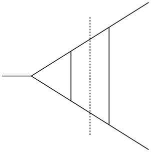
$$F_S^{1-loop} = g^2 N \mu^{2\epsilon}(-q^2)\, 2D_1$$

- No *D*-dim. subleties
  no spinor helicity formalism

Unitarity cuts

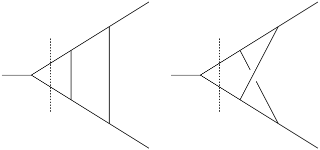


- Two loops, *cut (2b)*
  - Detects only E_1
  - $F_S^{2-loop} = g^4 N^2 \mu^{4\epsilon}(-q^2)^2\, 4\, E_1\Big|_{\text{cut (2b)}}$



E_1

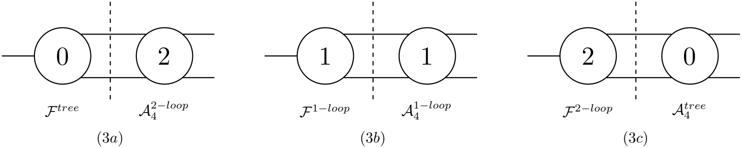
- Two loops, *cut (2a)*
  - Detects E_1 and E_2
  - $F_S^{2-loop} = g^4 N^2 \mu^{4\epsilon}(-q^2)^2 \left[4\, E_1 + E_2\right]$



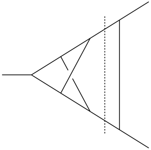
E_1 E_2

- Non-planar diagram due to "double trace terms" $\times$ "colour algebra"

Unitarity cuts

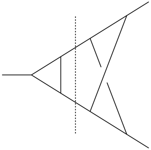


- Three loops, *cut (3c)*
  - $\mathcal{F}_S^{3-loop}\Big|_{\text{cut(3c)}} =$
    $g^6\,\mu^{6\epsilon}\,N^3\,(-q^2)^3\left[8\,F_1 + 2\,F_3^*\right]\Big|_{\text{cut(3c)}}$



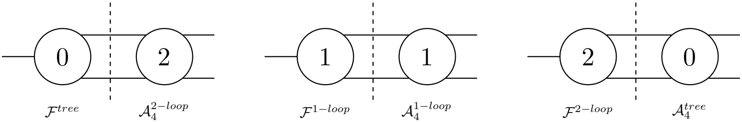
F_3^*

- Three loops, *cut (3b)*
  - $F_S^{3-loop}\Big|_{\text{cut(3b)}} =$
    $g^6\,\mu^{6\epsilon}\,N^3\,(-q^2)^3\left[8\,F_1 + 2\,F_4^*\right]\Big|_{\text{cut(3b)}}$



F_4^*

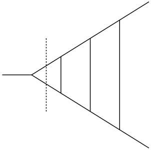
Unitarity cuts



(3a) (3b) (3c)

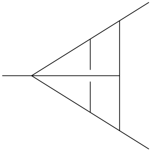
- Three loops, *cut (3a)*
  - $F_S^{3-loop}\Big|_{\mathrm{cut(3a)}} = g^6\,\mu^{6\epsilon}\,N^3\,(-q^2)^2 \times$

$$\left[8(-q^2)F_1 - 2F_2 + 2(-q^2)\left(F_3^* + F_4^*\right) + 4F_5^*\right]\Big|_{\mathrm{cut(3a)}}$$



F_1

- Three-particle cuts
  - Study generalised unitarity cuts [Britto,Cachazo,Feng'04]
  - Cut all or all but one propagator
  - Cross-check two-particle cuts
  - Detect further integrals such as F_9



F_9

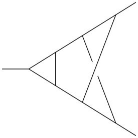
- Find perfect consistency between all cut constraints

Momentum routing invariances

- How to switch to a basis in which all integrals have UT?

- Parametrisation of F_4^*

$$\{k_1\,,\,k_2\,,\,k_3\,,\,k_1-k_2\,,\,k_1-k_3\,,\\ k_1-q\,,\,k_1-k_2-p_2\,,\,k_3-q\,,\,k_2-p_1\}$$



F_4^*

- *Integrand* remains invariant under

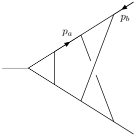
$$k_1 \to q - k_1\,, \qquad k_2 \to p_1 - k_2\,, \qquad k_3 \to q - k_3$$

- Numerator

$$(k_1-p_1)^2 \quad \to \quad (k_1-p_2)^2 \quad = \quad k_1^2 + (k_1-q)^2 - (k_1-p_1)^2 - q^2$$

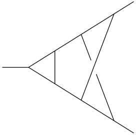
- Yields

$$F_4 = -\frac{1}{2}\,q^2\,F_4^* + F_{a2}$$



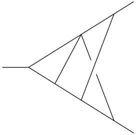
F_4

$= -\frac{1}{2}\,q^2$



F_4^*

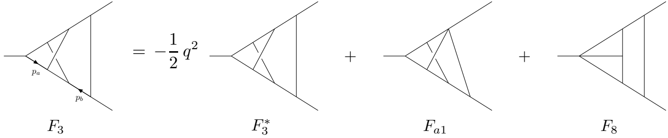
$+$



F_{a2}

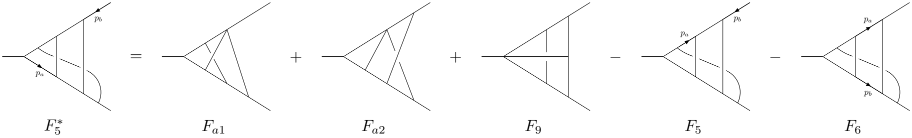
Momentum routing invariances

- More momentum routing invariances



$$F_3 = -\frac{1}{2}\, q^2\, F_3^* + F_{a1} + F_8$$

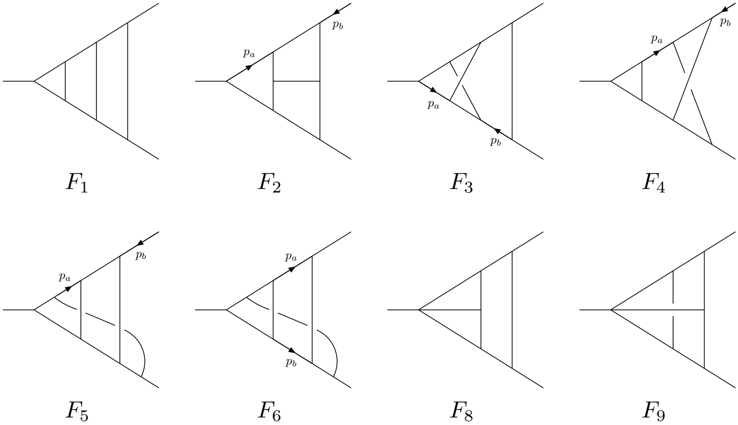
- And momentum conservation



$$F_5^* = F_{a1} + F_{a2} + F_9 - F_5 - F_6$$

- F_{a1} and F_{a2} invisible to certain cuts, cancel in final result

Result at three loops



$$F_S = 1 + a\, x^{\epsilon}\, R_{\epsilon} \cdot 2\, D_1 + a^2\, x^{2\epsilon}\, R_{\epsilon}^2 \cdot [4\, E_1 + E_2]$$
$$+ a^3\, x^{3\epsilon}\, R_{\epsilon}^3 \cdot [8\, F_1 - 2\, F_2 + 4\, F_3 + 4\, F_4 - 4\, F_5 - 4\, F_6 - 4\, F_8 + 2\, F_9]$$

- UT basis much simpler than that of conventionally chosen master integrals

Result at three loops

- Form factor in term of conventionally chosen masters

$$
\begin{aligned}
F_S^{(3)} = R_\epsilon^3 \Bigg[&+ \frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \\
&- \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} \\
&- \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} - \frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
&- \frac{16(2D-7)(5D-18)(52D^2-485D+1128)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} - \frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
&- \frac{128(2D-7)(3D-8)(91D^2-821D+1851)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
&- \frac{128(2D-7)(1497D^3-20423D^2+92824D-140556)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} + \frac{4(D-3)}{D-4} B_{8,1} + \frac{64(D-3)^3}{(D-4)^3} B_{6,1} \\
&+ \frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} - \frac{16(3D-10)(3D-8)(144D^2-1285D+2866)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
&+ \frac{128(2D-7)(177D^2-1584D+3542)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} + \frac{4(D-3)}{D-4} C_{8,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \\
&+ \frac{64(2D-5)(3D-8)(D-3)(2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} B_{4,1} \Bigg]
\end{aligned}
$$

- Simpler basis also for QCD form factor?

UV properties in higher dimensions

- Form factor in $\mathcal{N}=4$ is finite in $D=4$
- Q: In which dimension $D_c(L)$ (critical dimension) does it first develop UV divergences?
- Bound on $D_c(L)$ (power counting f. supergraphs, backgr. field method)

  [Grisaru,Siegel'82][Marcus,Sagnotti'85]

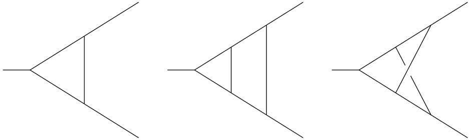
$$D_c(L) \geq 4 + \frac{2(\mathcal{N}-1)}{L}\,, \qquad L > 1$$

- Here: $\mathcal{N}=3$ (SUSYs that can be realized off-shell)

$$D_c(L) \geq 4 + \frac{4}{L}\,, \qquad L > 1$$

- Lower bound on $D_c(L)$
  - Saturated?
  - Too conservative?

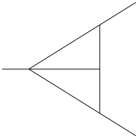
UV properties in higher dimensions



D_1 E_1 E_2

- One loop
  - Have $D_c(L = 1) = 6$
- Two loops
  - Bound: $D_c(L = 2) \geq 4 + 4/L = 6$
  - Bound saturated at two loops, have $D_c(L = 2) = 6$
  - Leading $1/\epsilon^2$ pole given by planar ladder diagram

- Diagrams like E_3 do not appear:
  Worse UV properties



E_3

UV properties in higher dimensions

- Three loops
  - Bound: $D_c(L=3) \geq 4 + 4/L = 16/3$
  - Investigate form factor's UV properties in $D = 16/3$
  - Rewrite form factor in a *non*-UT basis

$$F_S^{3-loop} \propto (-q^2)\ \underbrace{[8\,F_1 + 2\,F_3^* + 2\,F_4^*]}_{\text{finite by power counting}}\ \underbrace{-2\,F_2 + 4\,F_5^* - 2\,F_9}_{\text{Cancel in UV limit}}$$

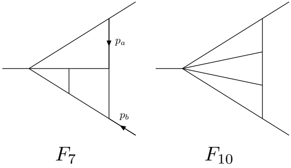
  - Alternatively,

$$F_S^{3-loop} \propto (-q^2)\ \underbrace{[8\,F_1 + 2\,F_3^* + 2\,F_4^*]}_{\text{finite}}\ \underbrace{-2\,(F_2 - F_7^*)}_{\text{Cancel in UV limit}}\ + 2\underbrace{(2\,F_5^* - F_7^* - F_9)}_{\text{Cancel in UV limit}}$$

  - No subdivergences in $D = 16/3$

  $\Longrightarrow$ Form factor UV finite in $D = 16/3$, have $D_c(L=3) = 6$

- Diagrams like F_7 or F_{10} do not appear: Worse UV properties



UV properties in higher dimensions

- Three-loop form factor shows improved UV behaviour
- Investigate UV properties in $D = 6$
- Mass regulator, $D = 6 - 2\epsilon$

$$2D_1^{\mathrm{UV}} = S_\Gamma \left[m^2\right]^{-\epsilon} \left\{-\frac{1}{\epsilon} - \frac{\pi^2}{6}\,\epsilon - \frac{7\pi^4}{360}\,\epsilon^3 + \mathcal{O}(\epsilon^5)\right\} ,$$

$$4E_1^{\mathrm{UV}} + E_2^{\mathrm{UV}} = S_\Gamma^2 \left[m^2\right]^{-2\epsilon} \left\{\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} + \left[\frac{1}{2} + \frac{\pi^2}{6} - \frac{1}{5}\,a_\Phi\right] + \mathcal{O}(\epsilon)\right\} ,$$

$$8F_1^{\mathrm{UV}} + 2F_3^{*\,\mathrm{UV}} + 2F_4^{*\,\mathrm{UV}} = S_\Gamma^3 \left[m^2\right]^{-3\epsilon} \left\{-\frac{1}{6\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left[\frac{\zeta_3}{3} - \frac{\pi^2}{12} - \frac{13}{9} + \frac{1}{5}\,a_\Phi\right] + \mathcal{O}(\epsilon^0)\right\}$$

- Momentum regulator, $D = 6 - 2\epsilon$

$$2D_1^{\mathrm{UV}} = S_\Gamma \left(-q^2\right)^{-\epsilon} \left\{-\frac{1}{\epsilon} - 2 - 4\,\epsilon + (2\zeta_3 - 8)\,\epsilon^2 + \mathcal{O}(\epsilon^3)\right\} ,$$

$$4E_1^{\mathrm{UV}} + E_2^{\mathrm{UV}} = S_\Gamma^2 \left(-q^2\right)^{-2\epsilon} \left\{\frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \left[\frac{53}{6} - \zeta_3\right] + \mathcal{O}(\epsilon)\right\} ,$$

$$8F_1^{\mathrm{UV}} + 2F_3^{*\,\mathrm{UV}} + 2F_4^{*\,\mathrm{UV}} = S_\Gamma^3 \left(-q^2\right)^{-3\epsilon} \left\{-\frac{1}{6\epsilon^3} - \frac{3}{2\epsilon^2} + \frac{1}{\epsilon}\left[\frac{4\zeta_3}{3} - \frac{79}{9}\right] + \mathcal{O}(\epsilon^0)\right\}$$

- Leading poles agree, subleading ones do not (as expected)
- All leading poles given by $L-$loop planar ladder diagram

UV properties in higher dimensions

- $\log(F_{\mathcal{S}})$ in the UV limit
  - Has only simple poles
  - Is identical in mass and momentum regulator scheme

$$\ln(F_{\mathcal{S}}^{\mathrm{UV}}) \stackrel{D=6-2\epsilon}{=} -\frac{\alpha}{\epsilon} + \frac{\alpha^2}{\epsilon}\frac{1}{2} + \frac{\alpha^3}{\epsilon}\left(\frac{\zeta_3}{3} - \frac{17}{18}\right) + \mathcal{O}(\alpha^4, \epsilon^0)$$

$$\text{with} \quad \alpha = -q^2 \frac{g^2 N}{(4\pi)^3}$$

- Comparison to scattering amplitudes: $D_c(L) \geq 4 + 6/L \qquad (L > 1)$
  - Scattering amplitudes are still better behaved than form factors
- Reasons
  - Dual conformal symmetry
  - Specific counterterms in higher dimensions. Take $D = 6$:
    $\mathrm{tr}\,(\phi^2)$ can mix at one loop with $g^2 \,\Box\, \mathrm{tr}\,(\phi^2)$

Conclusion

- We extended the computation of the scalar form factor in $\mathcal{N} = 4$ SYM to three loops
- We used (generalised) unitarity cuts to derive the result in terms of only a handful(!) planar and non-planar integrals
- Exponentiation behaviour verified: correct cusp and collinear AD
- Leading transcendentality pieces agree with those from QCD
- Different choices of basis make different properties such as UT or UV manifest
- Form factor shows improved UV behaviour: $D_c(L) = 6$ up to $L = 3$.