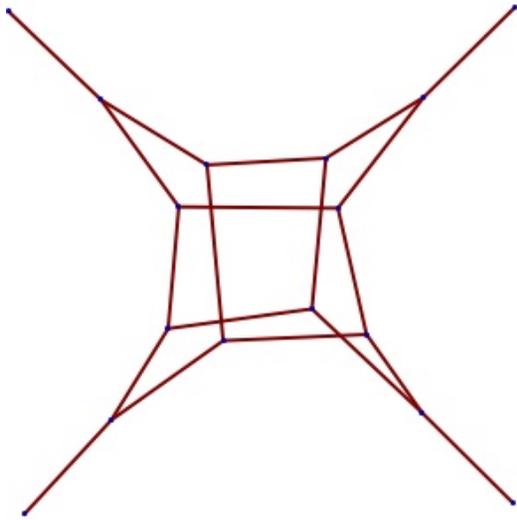


Non-Planar SYM at Five Loops



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Work in collaboration with:
Z.Bern, J.J.Carrasco, L.Dixon, R.Roiban

Motivation

- 5-loop $N=4$ SYM steppingstone to 5-loop $N=8$ supergravity
- Calculation of 5-loop $N=8$ UV divergence (in $24/5$ dim.) gives prediction on 7-loop divergence in $D=4$.
- Need UV divergences of $N=4$ SYM
 - Compare with $N=8$ SG
 - Double trace structure
 - Single-trace NNLC
- Find BCJ structure at 5-loops
- Most important question: is $N=8$ SG sufficiently well-behaved to be a UV finite theory ?



Scheme of calculation

$\mathcal{N}=4$ super-Yang-Mills in $D=4$
(10-D $\mathcal{N}=1$ SYM dimensionally reduced to D dim)

Related via KLT
and BCJ

$\mathcal{N}=8$ supergravity in $D=4$
(11-D $\mathcal{N}=1$ SG dimensionally reduced to D dim)

Unitarity based
perturbative tools

5-loop
amplitudes
↓
divergences

Outline

- Motivation
- Review UV status of $N=4$ SYM and $N=8$ SG
- How we calculated 5 loops SYM
 - Unitarity, maximal cuts
 - Two-particle cuts, box cuts
 - Duality between color and kinematics at cut level
- 5-loop result and UV properties
- Conclusions

UV properties $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SG

In $D=4$ dimensions:

see *Dixon's talk*

- $\mathcal{N}=4$ SYM ultraviolet finite *Mandelstam*
- $\mathcal{N}=8$ SG: conventional superspace power counting forbids $L=1,2$ divergences *Green, Schwarz, Brink, Howe and Stelle, Marcus and Sagnotti*
- Three-loop divergence ruled out by calculation: *Bern, Carrasco, Dixon, HJ, Kosower, Roiban, (2007), Bern, Carrasco, Dixon, HJ, Roiban (2008)*
- $L < 7$ loop divergences ruled out by counterterm analysis, using $E_{7(7)}$ symmetry and other methods, but a $L=7$ divergence is still possible *Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Björnsson, Green, Bossard, Howe, Stelle, Vanhove, Kallosh, Ramond, Lindström, Berkovits, Grisar, Siegel, Russo, and more....*

In $D > 4$ dimensions:

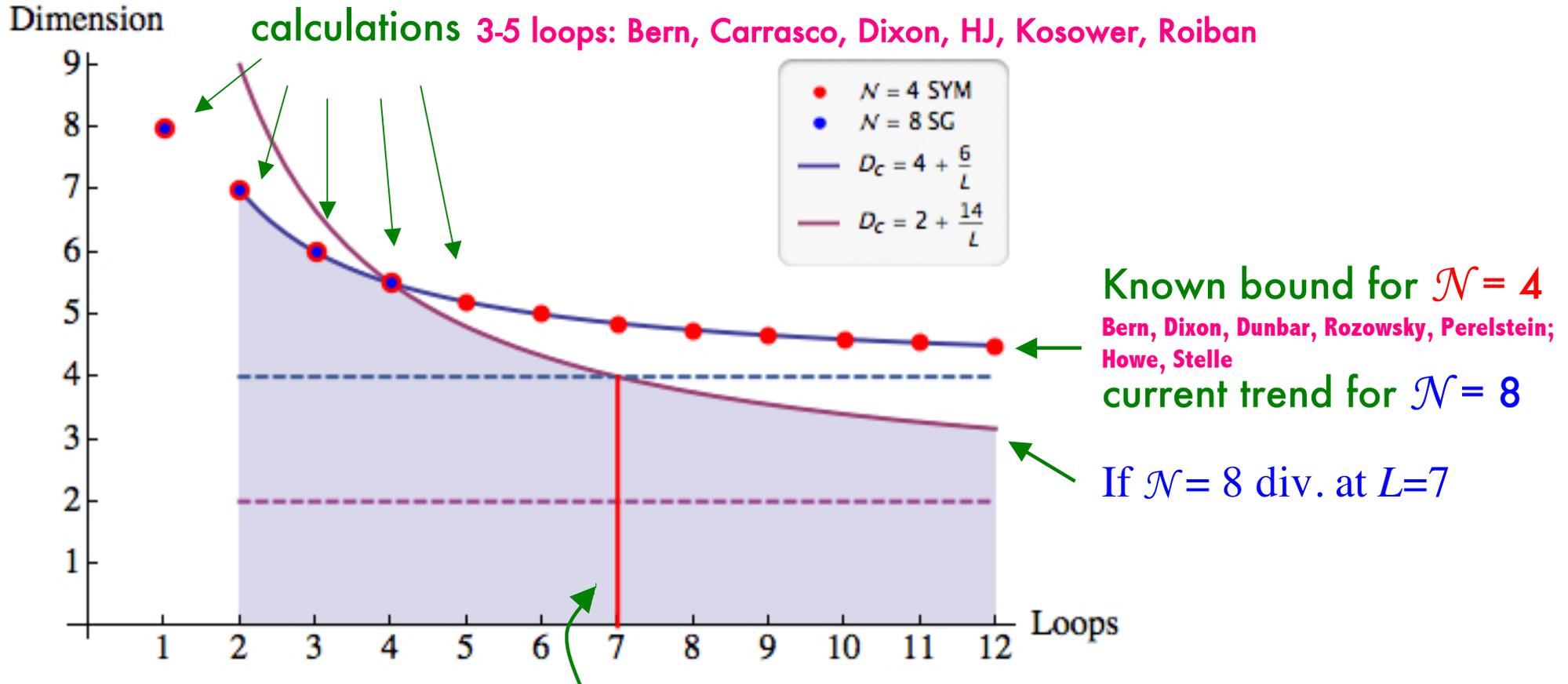
Through four loops $\mathcal{N}=8$ SG and $\mathcal{N}=4$ SYM diverge in exactly the same dimension:

$$D_c = 4 + \frac{6}{L} \quad (L > 1)$$

Marcus and Sagnotti; Bern, Dixon, Dunbar, Perelstein, Rozowsky; Bern, Carrasco, Dixon, HJ, Kosower, Roiban

UV divergence trend

Plot of critical dimensions of $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM

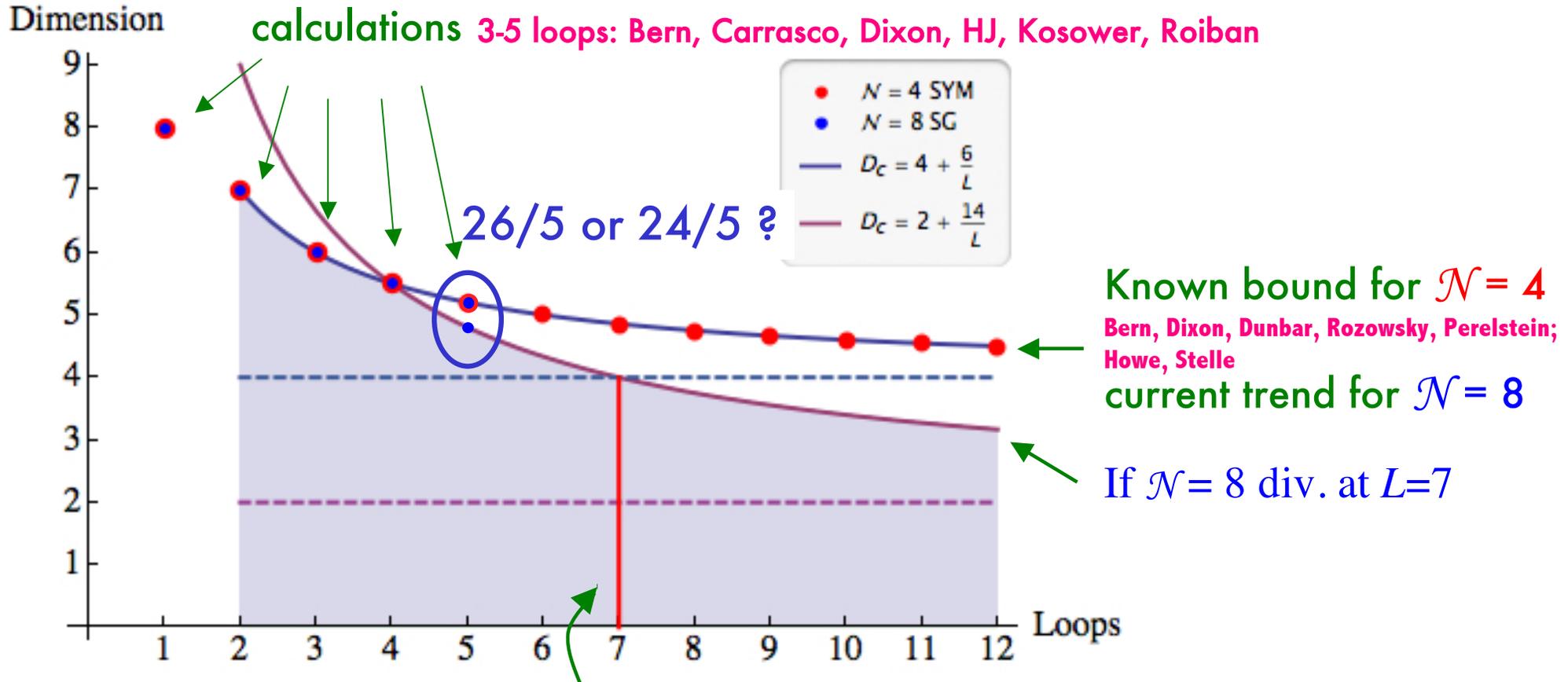


$L = 7$ lowest loop order for possible $D = 4$ divergence

Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger;
 Björnsson, Green, Bossard, Howe, Stelle, Vanhove Kallosh, Ramond, Lindström, Berkovits,
 Grisar, Siegel, Russo, and more....

UV divergence trend

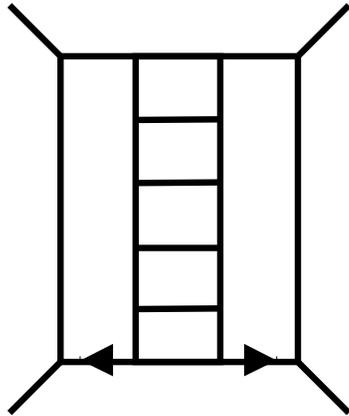
Plot of critical dimensions of $\mathcal{N} = 8$ SUGRA and $\mathcal{N} = 4$ SYM



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7-loop feature accessible at 5 loops



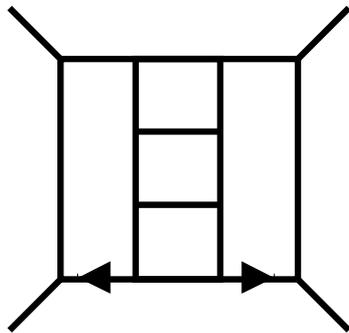
rung rule numerator $\sim ((l_1 + l_2)^2)^4$

gravity numerator $\sim [((l_1 + l_2)^2)^4]^2$

log div. integral $\sim \int d^{4 \times 7} l \frac{[((l_1 + l_2)^2)^4]^2}{(l^2)^{22}} \sim \int \frac{d^{28} l}{l^{28}}$

Can a local gauge transformation remove the offending numerator ?

No, maximal cut is gauge invariant, shows $[(2l_1 \cdot l_2)^4]^2$ is there in any rep.



rung rule numerator $\sim ((l_1 + l_2)^2)^2$

gravity numerator $\sim [((l_1 + l_2)^2)^2]^2$

log div. integral $\sim \int d^{D \times 5} l \frac{[((l_1 + l_2)^2)^2]^2}{(l^2)^{16}} \sim \int \frac{d^{5D} l}{l^{24}}$

Similar analysis: Björnsson, Green; Vanhove

Conclusion: when in doubt, calculate! see Bern's talk on N=4 SG

How nonplanar 5-loops is calculated

Overcoming 5 loop complexity

At 5 loops factorial growth start to become an issue

- cubic diagram topologies: $\sim 400-900$ (not counting triangles, bubbles)
- possible numerator terms: $\sim 10\,000$ (conservative)
- maximal cuts $\sim 500\,000$ (conservative)

Divide and conquer approach

- Use maximal cut method in D dimensions
 - Exploit known power counting
- Obtain compact analytical cut expressions through
 - Two-particle cuts
 - Box cuts
 - Color-kinematics duality: non-planar cuts from planar ones
- Automate calculation

With the N=4 SYM amplitude in our hands

- analyze its structure, UV behavior
- find representation where color-kinematics duality is manifest
- get N=8 supergravity

see Dixon's talk

Maximal Cuts

Idea: reconstruct amplitude piece by piece Bern, Carrasco, HJ, Kosower (2007)

similar to one-loop technology: BCF, Forde, OPP, Badger, etc.

and multi-loop leading singularity: Cachazo, Skinner; Arkani-Hamed et al.

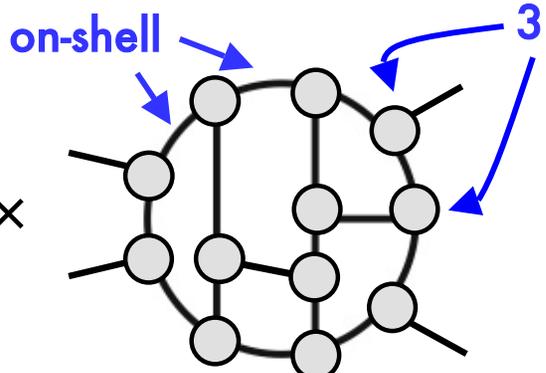
and current two-loop progress: Mastrolia, Ossola; Kosower, Larsen; Badger

see talks by Kosower, Ossola

Used to construct 3,4-loop N=4 SYM and N=8 SG previously (08-09)

Bern, Carrasco, Dixon, HJ, Roiban

put maximum number of propagator on-shell → simplifies calculation

$$N_i = \frac{1}{stA_4^{\text{tree}}} \times \text{[Diagram]} + O(l^2) \text{ corrections}$$


Detects terms: $l_i \cdot l_j$, $(l_i + l_j)^2$, $(l_i + l_j + l_k + \dots)^2$, ...

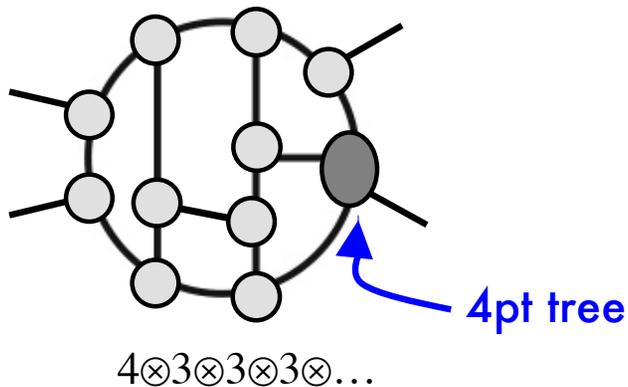
Blind to: l_i^2 , $l_i^2 (l_j \cdot l_k)$, $(l_i + l_j)^2 - 2l_i \cdot l_j$, ...

Near-Maximal Cuts

Bern, Carrasco, HJ and Kosower

- systematically release cut conditions \rightarrow get missing contact terms

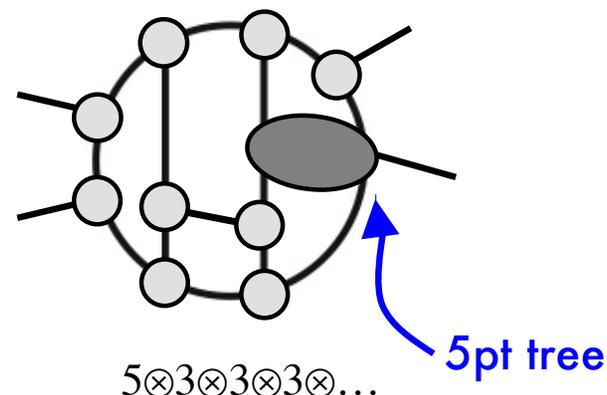
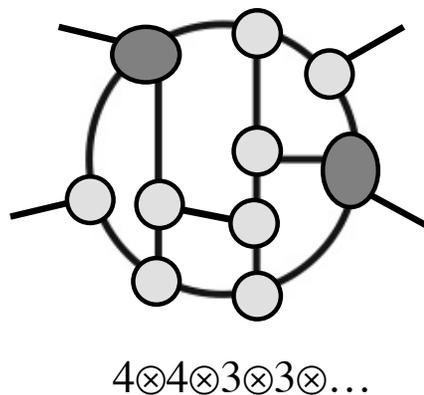
remove 1 cut line: next-to-maximal cuts



Detects: $l_i^2, l_i^2 (l_j \cdot l_k), \dots$

Blind to: $l_i^2 l_j^2, \dots$

remove 2 cut lines: next-to-next-to-maximal cuts

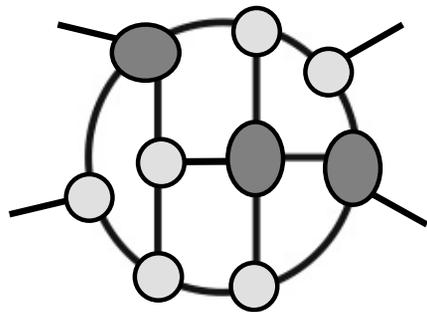


Detects: $l_i^2 l_j^2, \dots$

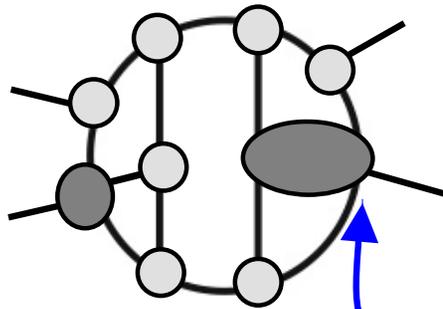
Blind to: $l_i^2 l_j^2 l_k^2, \dots$

Near-Maximal Cuts cont.

remove 3 cut lines: (next-to)³-maximal cut

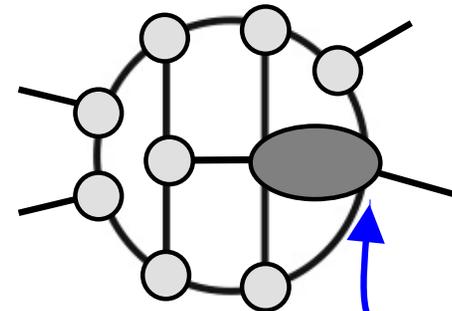


$4 \otimes 4 \otimes 4 \otimes 3 \otimes \dots$



$5 \otimes 4 \otimes 3 \otimes 3 \otimes \dots$

5pt tree



$6 \otimes 3 \otimes 3 \otimes 3 \otimes \dots$

6pt tree

Detects: $l_i^2 l_j^2 l_k^2, \dots$ **Blind to:** $l_i^2 l_j^2 l_k^2 l_m^2, \dots$

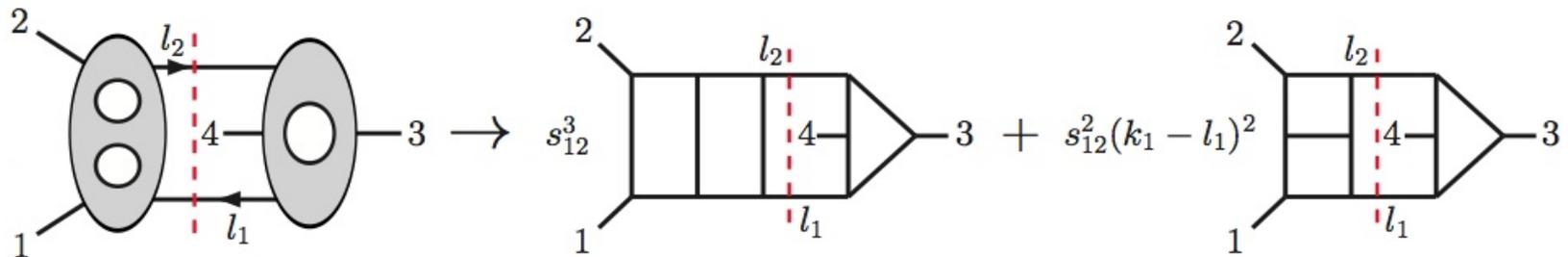
- By power counting $D_c = 4 + \frac{6}{L}$ numerators should take the form

$$N = stA^{\text{tree}} \left(sP_1^{(6)}(l_i) + tP_2^{(6)}(l_i) \right)$$

- All such terms are detected on the (next-to)³-maximal cuts

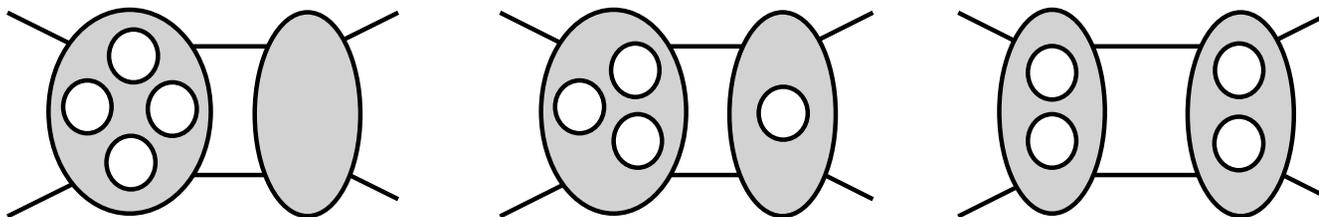
Two-particle cut

In $\mathcal{N}=4$ SYM, 2-particle cuts of the form $2 \rightarrow 2 \rightarrow 2$ are trivial, e.g.



- Works for non-planar cuts and in D-dimensions
 - Cut expressions are recycled from lower loops, thus very compact
- (closely related to the **rung rule** in the planar case [Bern, Rozowsky, Yan](#))

At 5 loops one obtains...

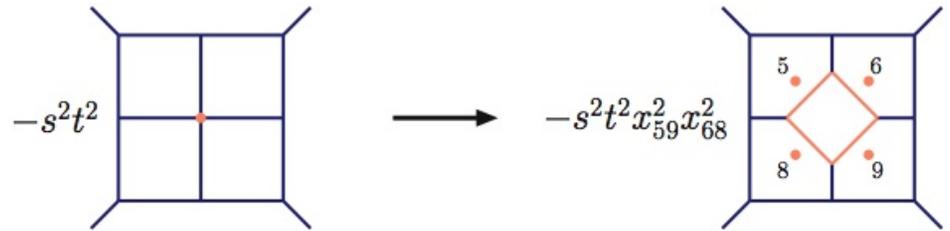


...cutting these further gives maximal and near-maximal cuts

Box cut (box substitution rule)

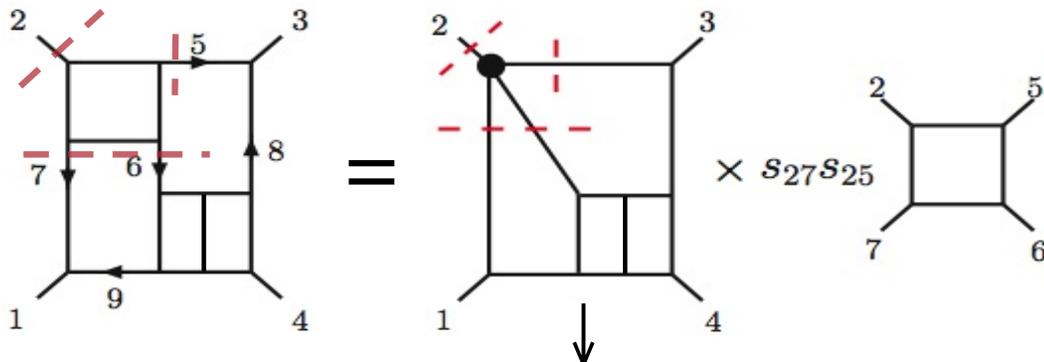
Observation in 0705.1864 [hep-th] inspired a "box-substitution rule"

Bern, Carrasco, HJ and Kosower



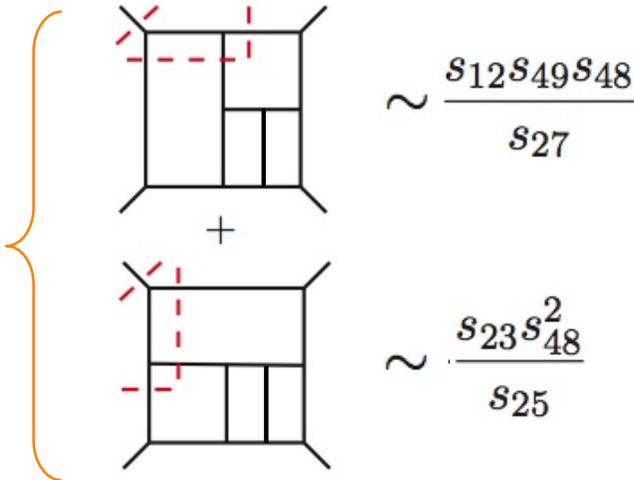
- Can be applied to any diagram containing a 4-pt "box" subdiagram

Bern, Carrasco, Dixon, HJ, Roiban



- Works in D dim, non-planar
- Recycles lower loop cuts
- Back-of-the-envelope calculations of non-trivial N 's

Known 4-loop diagrams

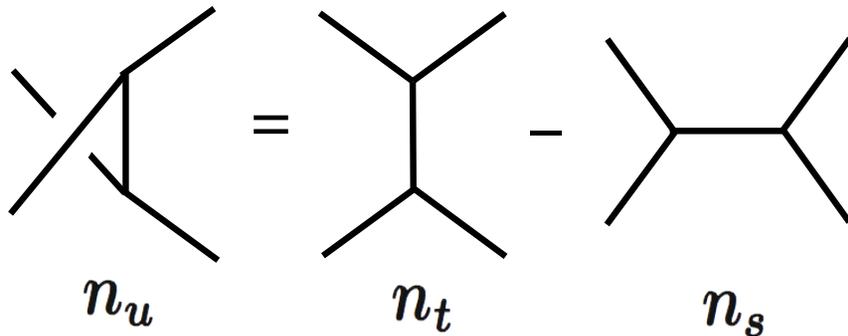


$$\begin{aligned}
 N &= s_{25}s_{27} \left(\frac{s_{12}s_{49}s_{48}}{s_{27}} + \frac{s_{23}s_{48}^2}{s_{25}} \right) \\
 &= (s_{12}s_{25}s_{49}s_{48} + s_{23}s_{27}s_{48}^2)
 \end{aligned}$$

similar rules: Cachazo, Skinner

Color-kinematics relations for cuts

We use the Jacobi-like numerator identity at the level of the cuts

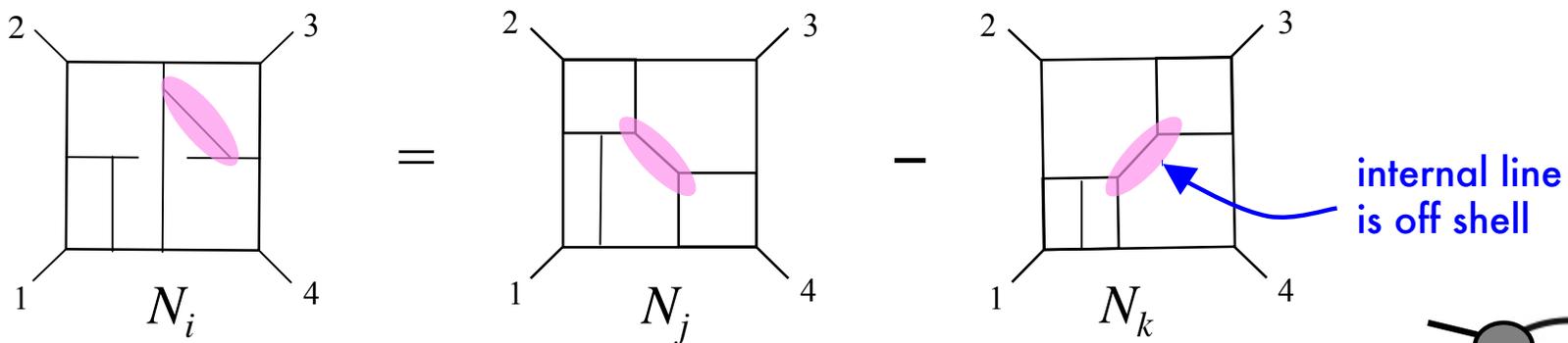


Bern, Carrasco, HJ

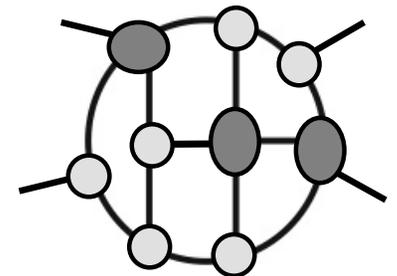
$$\mathcal{A}_4^{\text{tree}}(1, 2, 3, 4) = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

see talks by Broedel, Dixon, O'Connell

Diagram numerators are dependent, so are the color ordered cuts



- Obtain non-planar cuts from planar ones
- Valid in D dimensions (for generic gauge theories)
- Input is planar \Rightarrow compact expressions

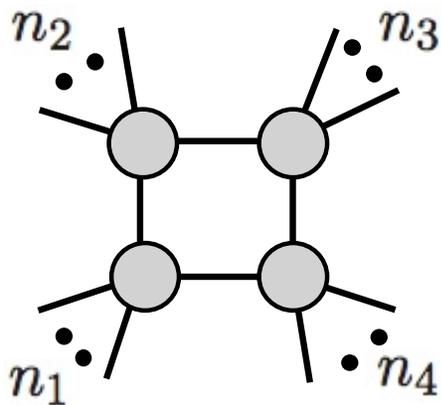


Color-kinematics cut basis

Good bookkeeping saves a lot of work:

- half a million color-ordered cuts \rightarrow stored as 25000 (5%) cuts via Kleiss-Kujf and BCJ relations.

One-loop example:



Color orders: $\frac{1}{2^4} (n_1 - 1)! (n_2 - 1)! (n_3 - 1)! (n_4 - 1)!$

KK basis: $(n_1 - 2)! (n_2 - 2)! (n_3 - 2)! (n_4 - 2)!$

BCJ basis: $(n_1 - 3)! (n_2 - 3)! (n_3 - 3)! (n_4 - 3)!$

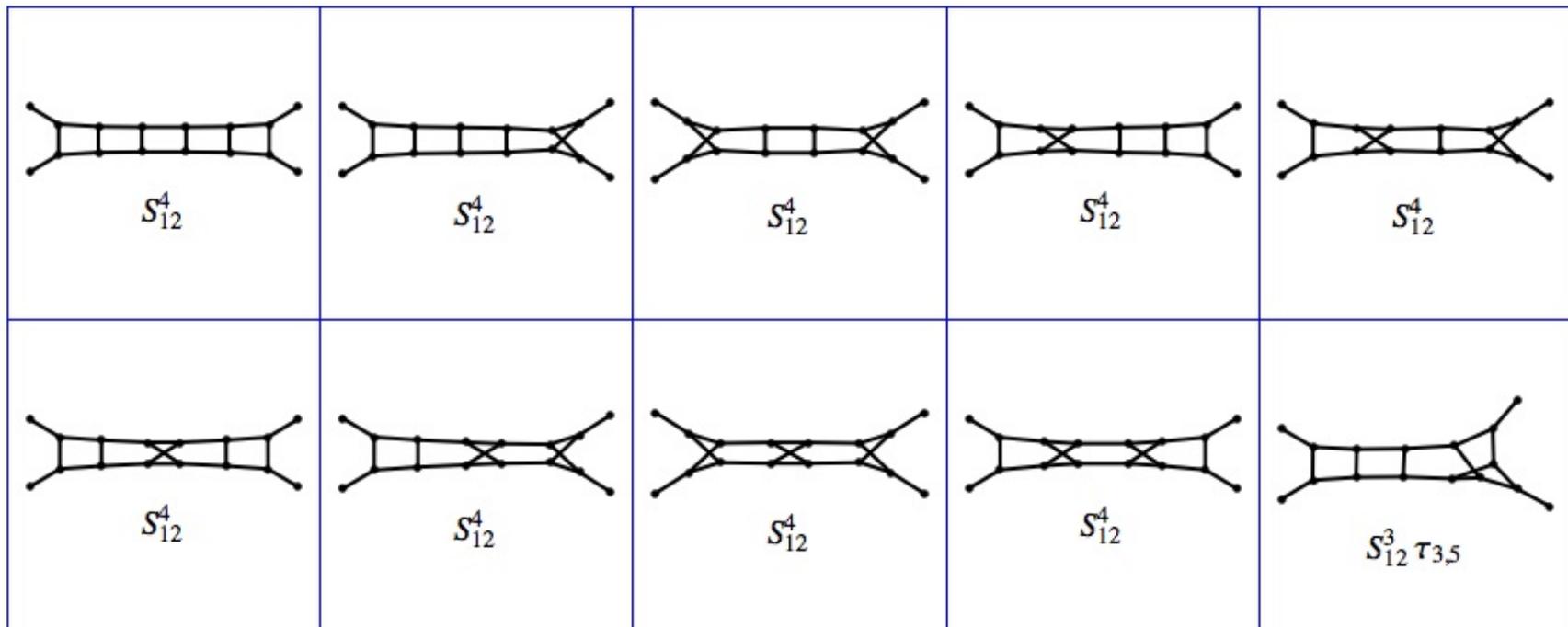
Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove
similar to: Boels, Isermann

The 5-loop amplitude

Amplitude diagrams I

Some examples of the 435 diagrams...

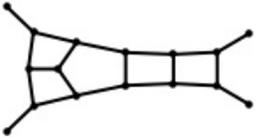
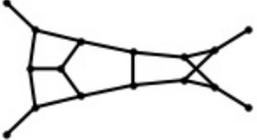
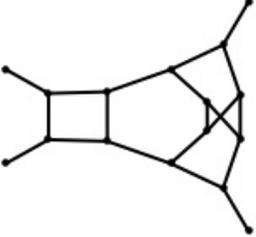
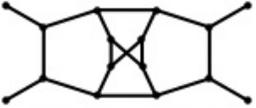
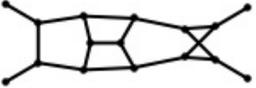
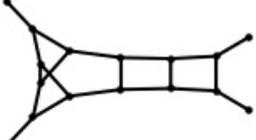
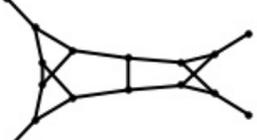
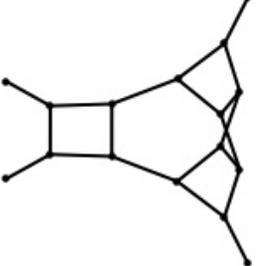
The simplest ones: ladder diagrams



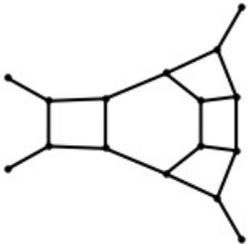
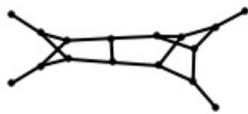
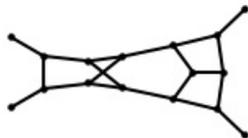
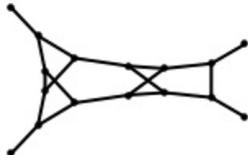
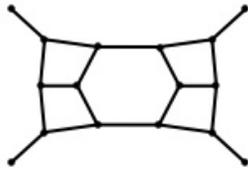
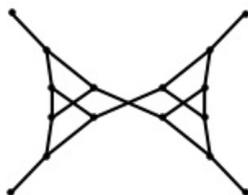
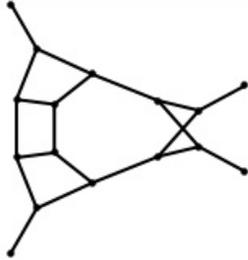
Color structures f^{abc} s are obtained from diagram topology

$$\tau_{i,j} = 2l_i \cdot l_j$$

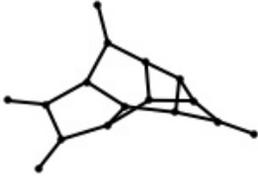
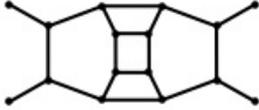
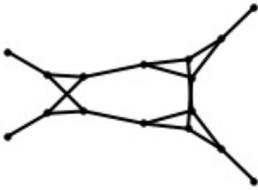
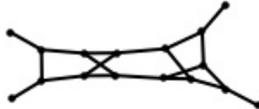
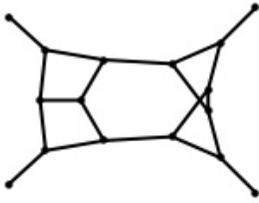
Amplitude diagrams II

 $\frac{1}{2} S_{12}^3 (\tau_{3,5} + \tau_{4,15})$	 $\frac{1}{2} S_{12}^3 (\tau_{3,5} + \tau_{4,15})$	 $\frac{1}{2} S_{12}^2 (\tau_{3,5}^2 + \tau_{4,13}^2)$	 $\frac{1}{2} S_{12}^2 (\tau_{10,20}^2 + \tau_{13,18}^2)$	 $\frac{1}{2} S_{12}^3 (\tau_{6,19} + \tau_{13,20} - 2 S_{12})$
 $\frac{1}{2} S_{12}^3 (\tau_{3,5} + \tau_{4,15} + l_5^2 + l_{15}^2)$	 $\frac{1}{2} S_{12}^3 (\tau_{3,5} + \tau_{4,15} + l_5^2 + l_{15}^2)$	 $\frac{1}{2} S_{12}^3 (\tau_{6,19} + \tau_{13,20} + l_6^2 + l_{13}^2 - 2 S_{12})$	 $\frac{1}{2} S_{12}^3 (\tau_{6,19} + \tau_{13,20} + l_{19}^2 + l_{20}^2 - 2 S_{12})$	 $\frac{1}{2} S_{12}^2 (\tau_{3,5}^2 + \tau_{4,13}^2 + 2 l_5^2 l_{13}^2)$

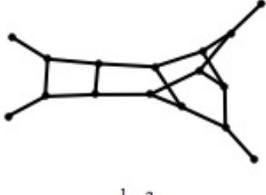
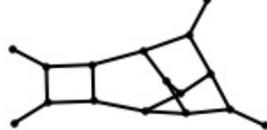
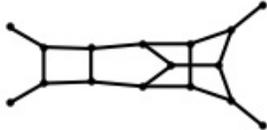
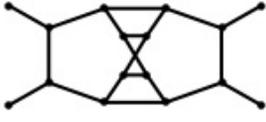
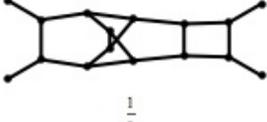
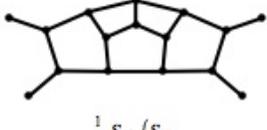
Amplitude diagrams III

 $\frac{1}{2} S_{12}^2 (\tau_{3,5}^2 + \tau_{4,13}^2 + 2 l_3^2 l_{13}^2)$	 $S_{12}^3 (\tau_{3,5} + l_5^2) - \frac{5}{2} l_3^2 l_{13}^2 l_{15}^2 S_{12}$	 $-\frac{1}{2} S_{12} \tau_{1,20}$ $(2 l_6^2 (\tau_{1,5} - \tau_{1,20} + S_{12}) + 2 \tau_{1,6} \tau_{1,18})$	 $-\frac{1}{2} S_{12}^2$ $(S_{12} (\tau_{3,15} + \tau_{4,5} + 2 S_{12}) + 2 (l_8^2 + l_9^2) (l_7^2 + l_{10}^2))$	 $-\frac{1}{2} S_{12}^2$ $(S_{12} (\tau_{3,15} + \tau_{4,5} + 2 S_{12}) + 2 (l_8^2 + l_9^2) (l_7^2 + l_{10}^2))$
 $-\frac{1}{4} S_{12}^2$ $(4 l_{13}^2 (-\tau_{1,5} - \tau_{2,5} - 2 \tau_{4,5} + l_5^2 + S_{12}) - 4 \tau_{4,13}^2)$	 $\frac{1}{2} S_{12}^2$ $(S_{12} (\tau_{1,17} + \tau_{2,5} - \tau_{3,5} - \tau_{4,17}) - \tau_{2,5} \tau_{3,5} - \tau_{1,17} \tau_{4,17} + 2 S_{12}^2)$	 $\frac{1}{2} S_{12}^2$ $(S_{12} (\tau_{1,17} + \tau_{2,5} - \tau_{3,5} - \tau_{4,17}) - \tau_{2,5} \tau_{3,5} - \tau_{1,17} \tau_{4,17} + 2 S_{12}^2)$	 $\frac{1}{64} S_{12}$ $(64 S_{12} (\tau_{1,6} + \tau_{2,6} + l_{19}^2) (l_5^2 - \tau_{3,5}) - 8 l_7^2 (l_5^2 - 20 l_{12}^2) l_{13}^2)$	 $\frac{1}{2} S_{12}^2$ $(l_5^2 (\tau_{1,13} + \tau_{2,13} + 2 l_{13}^2) + l_{13}^2 (\tau_{1,5} + \tau_{2,5}) + \tau_{3,5}^2 + \tau_{4,13}^2)$

Amplitude diagrams IV

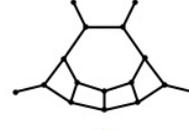
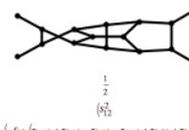
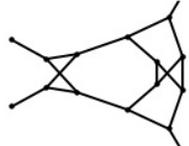
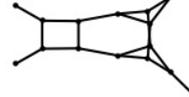
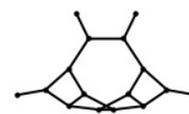
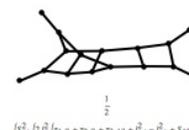
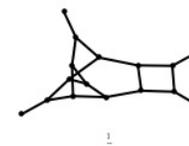
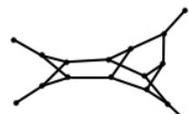
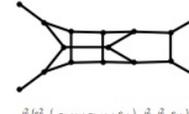
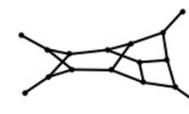
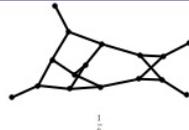
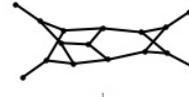
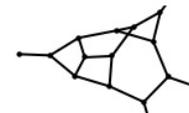
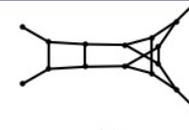
 $-\frac{1}{2} S_{12}^2 (\ell_{13}^2 (\tau_{3,8} + \tau_{4,8} - 3 S_{12}) + S_{12} (2 \tau_{13,19} - 2 \ell_{19}^2 + \ell_{20}^2) + \ell_6^2 (2 \ell_{19}^2 - 2 \ell_{20}^2 + S_{12}))$	 $\frac{1}{4} S_{12} (-4 \ell_6^2 (\ell_5^2 (-\tau_{2,7} + S_{12} + S_{13}) - (\tau_{3,18} + \tau_{4,18}) (S_{12} - \tau_{2,20})) - 4 \tau_{1,6} \tau_{1,18} \tau_{1,20})$	 $\frac{1}{8} (8 S_{12}^2 (\ell_5^2 (-\tau_{1,12} + \tau_{2,12} + \tau_{3,12})) - \tau_{3,5} \tau_{4,12} + \ell_{12}^2 S_{12}) + 8 \ell_5^2 \ell_{10}^2 \ell_{14}^2 (S_{12} + S_{13}))$	 $\frac{1}{2} S_{12}^2 (\ell_6^2 (-2 \tau_{3,5} - 2 \ell_{13}^2 + S_{12}) + 2 \ell_5^2 (\tau_{1,6} + \tau_{2,6} + \ell_{19}^2) + 2 \tau_{3,5} \tau_{4,6} + \ell_{12}^2 S_{12})$	 $\frac{1}{2} (S_{12}^2 (2 S_{12} (\tau_{10,13} + \tau_{18,20}) + \tau_{10,13}^2 + \tau_{18,20}^2 + 2 S_{12}^2) + 4 (\ell_5^2 + \ell_6^2) (\ell_{13}^2 \ell_{18}^2 + \ell_{10}^2 \ell_{20}^2) S_{23})$
 $\frac{1}{2} S_{12} (S_{12} (S_{12} (2 \tau_{6,19} + \ell_{20}^2 - 2 S_{12}) + 2 \ell_{13}^2 (S_{12} - \ell_{20}^2)) - \ell_6^2 (2 (\ell_{19}^2 - \ell_{20}^2) S_{12} + \ell_7^2 \ell_{19}^2))$	 $\frac{1}{16} S_{12} (8 S_{12} (-2 S_{12} (\tau_{3,13} + \tau_{4,5}) + \tau_{3,13}^2 + \tau_{4,5}^2 + 2 S_{12}^2) + \ell_5^2 \ell_{13}^2 (19 (\ell_7^2 + \ell_{17}^2) + 16 S_{12}))$	 $-\frac{1}{2} S_{12} (3 \ell_7^2 \ell_{10}^2 (\tau_{1,15} + \tau_{2,15} + \tau_{4,15} + \ell_6^2 - \ell_{14}^2 - \ell_{17}^2 + \ell_{20}^2 - S_{12}) + 2 S_{12}^2 (\tau_{3,15} + S_{12}))$	 $\frac{1}{2} S_{12}^2 (2 (\tau_{1,6} + \tau_{2,6} + \ell_{19}^2) (\ell_5^2 - \tau_{3,5}) + \ell_6^2 (S_{12} - 2 \ell_{13}^2) + \ell_{12}^2 S_{12})$	 $\frac{1}{2} (\ell_5^2 \ell_{17}^2 S_{13} (\tau_{2,13} - \tau_{1,12}) + S_{12}^3 (\tau_{1,17} + \tau_{2,5} - \tau_{3,5} - \tau_{4,17}) - S_{12}^2 (\tau_{2,5} \tau_{3,5} + \tau_{1,17} \tau_{4,17}) + 2 S_{12}^4)$

Amplitude diagrams V

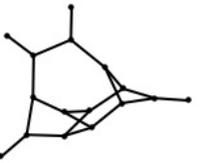
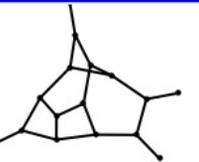
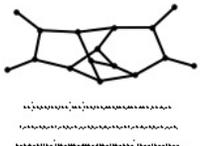
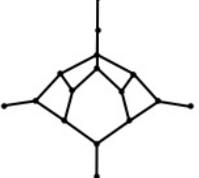
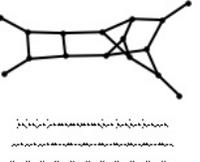
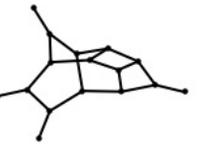
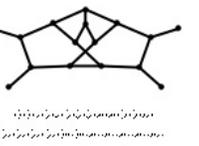
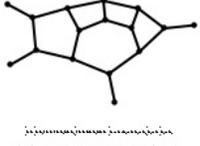
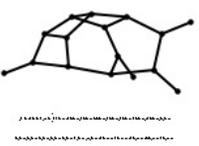
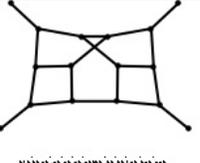
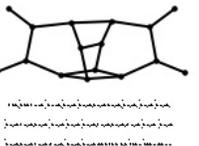
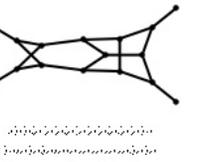
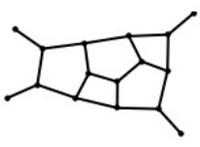
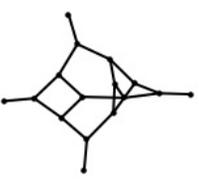
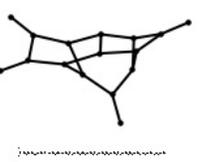
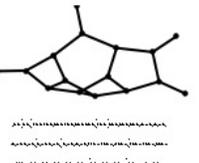
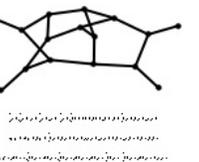
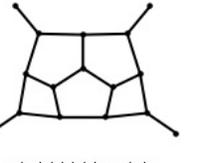
 $\frac{1}{4} (4 S_{12}^2 (\ell_{13}^2 (\tau_{1,5} + \tau_{2,5} + 2\tau_{4,5} - S_{12}) + \tau_{4,13}^2) + \ell_5^2 \ell_{13}^2 (3 \ell_{16}^2 S_{12} - 4 S_{13} (S_{12} + S_{13})))$	 $-\frac{1}{4} S_{12}^2 (2 S_{12} (2\tau_{1,7} + 2\tau_{2,6} + \tau_{3,5} + \tau_{3,6} + \tau_{3,7} + \tau_{3,13} + \ell_6^2 - \ell_7^2) + 4(\tau_{1,6} - \tau_{1,7})\tau_{3,5} + 4(\tau_{2,6} - \tau_{2,7})\tau_{3,13})$	 $\frac{1}{2} (2 S_{12}^2 (\tau_{3,5} (\tau_{4,6} - \ell_6^2) + \ell_{12}^2 S_{12} + (\ell_6^2 + \ell_8^2 - \ell_{20}^2) \ell_5^2) + \ell_7^2 \ell_{13}^2 (2 \ell_6^2 (S_{12} - S_{13}) + \ell_{10}^2 S_{12}))$	 $\ell_5^2 (S_{12}^2 (\tau_{1,6} + \tau_{2,6} + \ell_{19}^2) - 2 \ell_6^2 \ell_8^2 (S_{12} + 2 S_{13})) + S_{12}^2 \tau_{3,5} (\tau_{4,6} - \ell_6^2) + \ell_5^2 \ell_{13}^2 \ell_6^2 (2 S_{12} + S_{13})$	 $S_{12}^2 (\ell_5^2 (-(\tau_{1,12} + \tau_{2,12} + \tau_{3,12} + \ell_{12}^2)) + \tau_{3,5} (\tau_{1,12} + \tau_{2,12} + \tau_{3,5} - \ell_6^2) + (\ell_{12}^2 + \ell_{13}^2) S_{12})$
 $\frac{1}{2} S_{12}^2 (S_{12} (\tau_{1,6} - \tau_{1,12} + \tau_{2,6} - \tau_{2,12} + \tau_{3,5} + \tau_{3,12} - \tau_{4,6} - \tau_{4,13} + \tau_{5,17} + \tau_{13,17}) + 2(\tau_{1,12} - \tau_{1,6})\tau_{3,5} + 2(\tau_{2,6} - \tau_{2,12})\tau_{4,13} - 2 S_{12}^2)$	 $\frac{1}{2} (S_{12}^2 (2 S_{12} (\tau_{10,13} + \tau_{18,20}) + \tau_{10,13}^2 + \tau_{18,20}^2 + 2 S_{12}^2) - (\ell_{13}^2 \ell_{18}^2 + \ell_{10}^2 \ell_{20}^2) \ell_5^2 (4 S_{12} + 5 S_{23}) - \ell_6^2 (\ell_{13}^2 \ell_{18}^2 + \ell_{10}^2 \ell_{20}^2) (4 S_{12} + 5 S_{23}))$	 $\frac{1}{2} (S_{12}^2 (S_{12} (\tau_{6,19} + \tau_{13,20} + \ell_{19}^2 + \ell_{20}^2 - 2 S_{12}) + \ell_{13}^2 (S_{12} - 2 \ell_{20}^2)) + \ell_6^2 (4 \ell_{10}^2 \ell_{20}^2 (S_{12} + 2 S_{13}) + S_{12}^2 (S_{12} - 2 \ell_{19}^2)) + 4 \ell_6^2 \ell_{13}^2 \ell_{19}^2 (S_{12} + 2 S_{13}))$	 $\frac{1}{8} (2 S_{12}^2 (2 S_{12} (-2(\tau_{3,5} - \tau_{3,14} + \tau_{5,14}) + 2 \ell_5^2 - \ell_6^2 + \ell_{12}^2 + \ell_{15}^2) + 4 \ell_{15}^2 (\tau_{3,13} + \ell_{13}^2) - 4 \tau_{3,5} \tau_{4,15}) + 8 \ell_5^2 \ell_6^2 \ell_{10}^2 S_{13} + 40 \ell_5^2 \ell_6^2 \ell_{18}^2 (S_{12} + 2 S_{13}))$	 $\frac{1}{2} S_{12} (S_{12} (S_{12} (-\tau_{1,5} - \tau_{2,5} + \tau_{3,5} + \tau_{4,5} + \tau_{10,20} - \tau_{18,19}) + (\tau_{1,5} + \tau_{2,5})\tau_{10,20} + (\tau_{3,5} + \tau_{4,5})\tau_{18,19} - 2 \ell_{18}^2 \ell_{20}^2 - 2 S_{12}^2) - 2 \ell_5^2 (S_{12} \tau_{10,19} + \ell_{18}^2 \ell_{20}^2))$

So far: the 50 simplest diagrams. Here are some more...

Amplitude diagrams VI

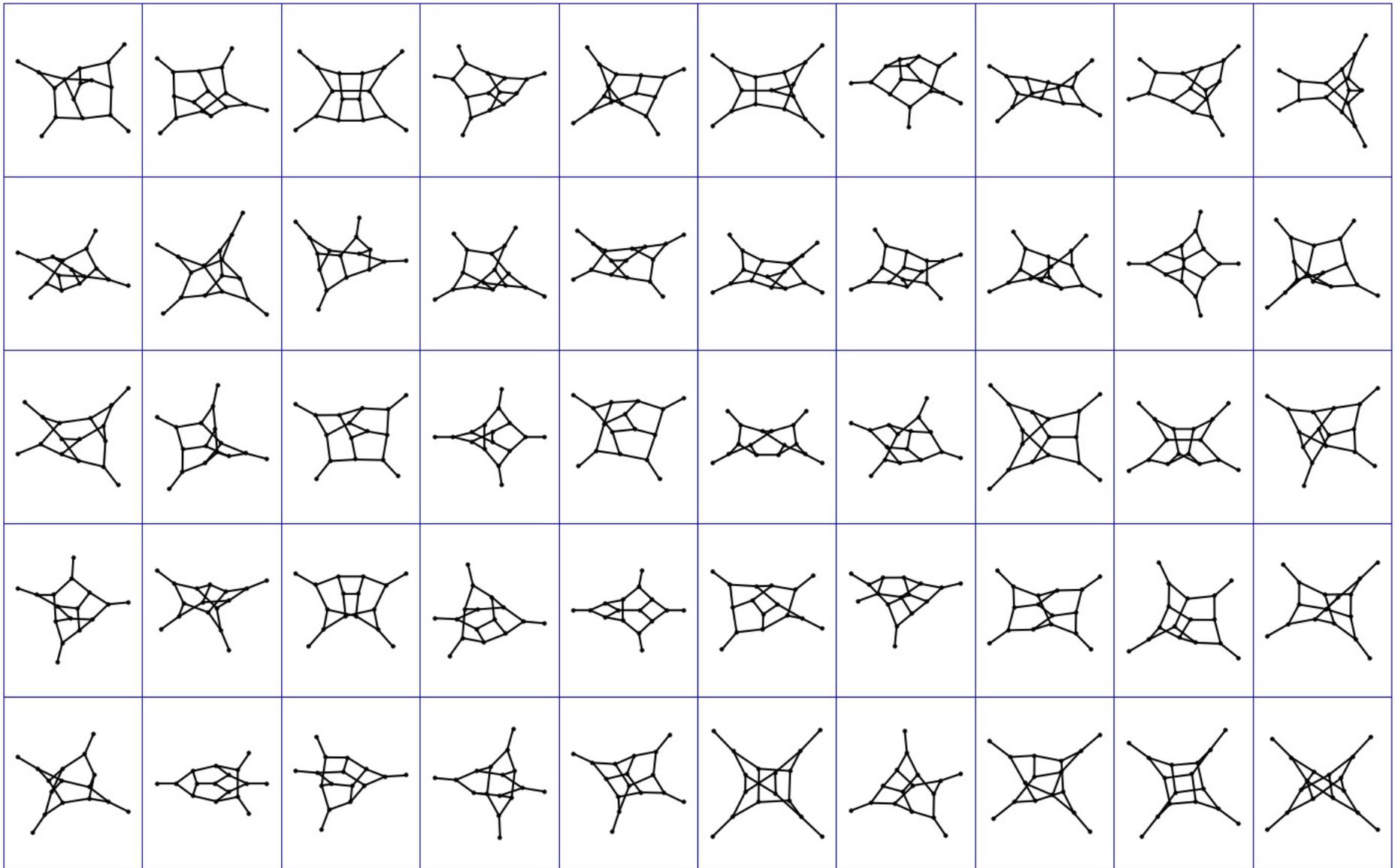
 $\frac{1}{2} (2 \tilde{q}_2^2 (\tilde{q}_2^2 (71,6 + 72,6 - \tilde{q}_2 + \tilde{q}_6) - \tilde{q}_6 \tilde{q}_3 513) - \tilde{q}_2^2 (\tilde{q}_2^2 (-273,5 - 2 \tilde{q}_2^2 + 512) + 2 \tilde{q}_2^2 (71,7 + 72,7 + 73,7) - 273,5 \tilde{q}_6))$	 $\frac{1}{2} (\tilde{q}_6 512 (71,19 + 72,19) - (512 + 513) 74,10 (74,10 + 512 + 513) - 2 \tilde{q}_6 \tilde{q}_6 512 - \tilde{q}_2^2 (512 + 513)^2 - (512 + 513) 74,6 \tilde{q}_6 \tilde{q}_6 - \tilde{q}_2^2 \tilde{q}_6 \tilde{q}_2 512)$	 $\frac{1}{2} (\tilde{q}_2^2 (2 \tilde{q}_2^2 (71,6 + 72,6 - \tilde{q}_2 + \tilde{q}_6) - 273,5 (71,6 + 72,6 + \tilde{q}_6) - \tilde{q}_2 512) - 2 \tilde{q}_2^2 \tilde{q}_2^2 (\tilde{q}_2^2 512 + 512 + \tilde{q}_2^2 + 512 513))$	 $\frac{1}{2} (-512 (2 \tilde{q}_2^2 (73,10 + 74,11) + 3 \tilde{q}_2^2 (\tilde{q}_2^2 10 + \tilde{q}_2^2 11) + \tilde{q}_2 10 + \tilde{q}_2^2 11 + 2 \tilde{q}_2^2) - 2 \tilde{q}_6 \tilde{q}_1 (512 + 513)^2 - 2 \tilde{q}_2^2 \tilde{q}_6 \tilde{q}_1 (3 \tilde{q}_2^2 + 523))$	 $\frac{1}{2} (-512 (71,10 + 72,10 - 73,10 - 74,10 + 75,20 + 77,19 - 74,18 - 75,17 + 77,19 + 78,20) + (74,10 - 72,10) 77,19 + (73,10 - 71,10) 78,20 + 2 \tilde{q}_6 512 - 4 \tilde{q}_2^2) + 2 (\tilde{q}_1^2 + \tilde{q}_6^2 + \tilde{q}_2^2 (\tilde{q}_6 + \tilde{q}_2)) \tilde{q}_2^2$	 $\tilde{q}_2^2 (\tilde{q}_2^2 (-73,5 + \tilde{q}_2 - 512)) + (\tilde{q}_2^2 - \tilde{q}_2^2) (71,17 + 512) + \tilde{q}_6 (71,9 - 71,13 + 72,9 - 72,13 - 73,13 + \tilde{q}_2^2 - \tilde{q}_6 - 512) - 72,10 73,16 - \tilde{q}_2^2 \tilde{q}_2^2$	 $\frac{1}{2} (512 (71,20 (2 \tilde{q}_2^2 (74,19 - 73,19) + \tilde{q}_2 10 + \tilde{q}_2^2 19 + 2 \tilde{q}_2^2) - 2 \tilde{q}_1 \tilde{q}_6 (-71,18 + 72,18 - 73,5 + 74,5 + 275,18 + 2513) - 2 \tilde{q}_6 \tilde{q}_6 (512 + 513)^2)$
 $\frac{1}{2} (\tilde{q}_2^2 (\tilde{q}_2^2 (512 + 2513) (-71,14 + 72,14 + 512 + 2513) - 2 \tilde{q}_2^2 (71,12 + 72,12 + 73,12)) + \tilde{q}_2^2 (273,5 (71,12 + 72,12 + 73,5 - \tilde{q}_2^2) - (\tilde{q}_2^2 + \tilde{q}_3^2) 512))$	 $\frac{1}{2} (\tilde{q}_2^2 (\tilde{q}_2^2 (71,13 + 72,13 + 2 \tilde{q}_2^2 - 512) + 2 (\tilde{q}_2^2 + 2513) 512 + 2 \tilde{q}_2^2 73,13))$	 $\frac{1}{2} (-2 \tilde{q}_2^2 (\tilde{q}_2^2 (71,12 + 72,12 + 73,12) + \tilde{q}_2^2 \tilde{q}_2^2 (512 + 513 512 + 513)) + \tilde{q}_2^2 (273,5 (71,12 + 72,12 + 73,5 - \tilde{q}_2^2) - \tilde{q}_2^2 512) - 2 \tilde{q}_2^2 \tilde{q}_2^2 (\tilde{q}_2^2 + 2513 512 + 2 \tilde{q}_2^2))$	 $\frac{1}{2} \tilde{q}_2^2 (\tilde{q}_2^2 (-71,7 - 71,17 - 72,7 - 72,17 - 2 (73,7 + 73,17) + 2 \tilde{q}_2^2) + \tilde{q}_2^2 (71,7 + 71,17 + 72,7 + 72,17 + 2 (74,7 + 74,17) - 2 \tilde{q}_2^2 (73,13 + 74,5) + \tilde{q}_2^2 13 + \tilde{q}_2^2 + 2 \tilde{q}_2^2))$	 $\frac{1}{2} (-2 \tilde{q}_6 \tilde{q}_1 (512 + 513) (-273,5 + 274,5 + 5512 + 5513) - \tilde{q}_2^2 (2 \tilde{q}_2^2 (73,10 + 74,11) + 3 \tilde{q}_2^2 (\tilde{q}_2^2 10 + \tilde{q}_2^2 11) + \tilde{q}_2 10 + \tilde{q}_2^2 11 + 2 \tilde{q}_2^2) - 2 \tilde{q}_6 \tilde{q}_1 (2 \tilde{q}_2^2 + 5523))$	 $\frac{1}{2} (\tilde{q}_2^2 (2 \tilde{q}_2^2 (71,7 + 72,7 + 73,13 + \tilde{q}_6 - \tilde{q}_6 + 512) - \tilde{q}_2^2 (273,6 + 73,14 - 74,6) - 512) - 2512 73,5 - 2512 73,16 + 271,14 73,5 + 272,14 73,5 - 2 \tilde{q}_2^2 \tilde{q}_2^2 512 - \tilde{q}_6 512 - \tilde{q}_6 512 - \tilde{q}_6 512) - \tilde{q}_2^2 (\tilde{q}_6 \tilde{q}_2^2 (512 - 3512) + 2 \tilde{q}_2^2)$	 $\frac{1}{2} (2 \tilde{q}_2^2 (-71,13 + 71,14 - 72,13 + 72,14 + 273,14 - \tilde{q}_2^2 + \tilde{q}_2^2 + 512) - 2 \tilde{q}_2^2 (73,14 + 512) + 273,5 (71,13 + 72,13 - 73,5 - \tilde{q}_2^2) - \tilde{q}_2^2 512) - \tilde{q}_2^2 \tilde{q}_6 \tilde{q}_4 512 + \tilde{q}_2^2 \tilde{q}_6 \tilde{q}_4 (512 + 2513))$
 $-\frac{1}{8} 512 (\tilde{q}_2^2 (4 \tilde{q}_2^2 (2 (-71,13 + 71,14 - 72,13 + 72,14 + 273,14 + 512) - 2 \tilde{q}_2^2 + \tilde{q}_2^2) - \tilde{q}_2^2 (73,14 + 512) - 4 \tilde{q}_2^2) + 2 (\tilde{q}_2^2 (71,5 + 72,5 - 73,5) - (71,13 + 72,13 - 73,5) 73,5) - \tilde{q}_2^2 512))$	 $\frac{1}{2} \tilde{q}_2^2 (\tilde{q}_2^2 (71,11 + 72,11 + 73,17 + 74,17 - 2710,19) + 512 (-271,5 - 272,5 + 273,5 + 274,5 - 710,17 + 711,19) - (71,5 + 72,5) (71,5 + 72,5 + 710,17) - (73,5 + 74,5) (73,5 + 74,5 + 711,19) - 2 \tilde{q}_2^2)$	 $\frac{1}{2} (2 \tilde{q}_2^2 (\tilde{q}_2^2 (71,6 + 72,6) - \tilde{q}_2^2 (\tilde{q}_2^2 + 2513 512 + 2 \tilde{q}_2^2)) - 2 \tilde{q}_2^2 (\tilde{q}_2^2 (71,7 + 72,7) - \tilde{q}_2^2 (\tilde{q}_2^2 + 2513 512 + 2 \tilde{q}_2^2)) - \tilde{q}_2^2 (512 73,5 + 512 73,6 + 512 73,7 + 73,13 (512 - 2 (71,7 + 72,7)) - 271,6 73,5 + 272,6 73,5 + \tilde{q}_2 512 - \tilde{q}_2^2 512))$	 $\frac{1}{4} (2 \tilde{q}_2^2 (\tilde{q}_2^2 (-512 (71,6 - 71,9 - 72,6 - 72,9 + 73,11 - 74,18 + 76,9 - 76,18 + 79,11) + 513 (-271,6 + 271,9 + 272,6 - 272,9 + 73,8 - 73,11 + 74,10 + 74,18 + 76,10 + 76,18 + 79,9 - 79,11) + \tilde{q}_2^2) + \tilde{q}_2^2 (71,13 + 72,13) + 2 \tilde{q}_2^2 (\tilde{q}_2^2 (71,5 + 72,5) - \tilde{q}_2^2 13 + \tilde{q}_2^2 + \tilde{q}_2^2 13))$	 $\frac{1}{2} (2 \tilde{q}_2^2 (\tilde{q}_2^2 (73,14 + 512) + \tilde{q}_2^2 (71,5 - 71,15 + 72,5 - 72,15 + 73,5 - 73,15 + 512) + 73,5 (71,12 + 72,12 + 73,5 - \tilde{q}_2^2) + \tilde{q}_2^2 512) - \tilde{q}_2^2 (\tilde{q}_2^2 (71,12 + 72,12 + 73,12) + \tilde{q}_2^2) - 2 \tilde{q}_2^2 \tilde{q}_6 (2 \tilde{q}_2^2 + 523))$	 $\tilde{q}_2^2 (-71,20 + 72,19 + 512) - \tilde{q}_6 \tilde{q}_6 513 - \frac{1}{2} 512 (\tilde{q}_2^2 (71,5 + 71,15 - 71,20 + 72,5 - 72,6 - 72,19 - 76,11 + 76,15 - 76,20 + 711,19) + 2512 (71,5 + 71,15 - 271,20 - 710,20) + 72,5 (72,5 - 272,19 + 711,19) - 2 (71,5 + 72,5) (71,5 - 72,5 72,19))$	 $\frac{1}{16} (8 \tilde{q}_2^2 (\tilde{q}_2^2 (71,14 + 72,14 - 73,5 + 73,14) + 2 \tilde{q}_2^2 - \tilde{q}_2^2 + 2 \tilde{q}_2^2 + 2 \tilde{q}_2^2 - 2 \tilde{q}_2^2 - \tilde{q}_2^2) + 16 \tilde{q}_2^2 (\tilde{q}_2^2 (71,7 + 72,7 + 73,7 - \tilde{q}_2^2) + (71,15 + 72,15) 73,5) - 2 \tilde{q}_2^2 512 (4 \tilde{q}_2^2 + 513) + 4 (\tilde{q}_2^2 + \tilde{q}_2^2) \tilde{q}_2^2) + 8 \tilde{q}_2^2 \tilde{q}_6 (\tilde{q}_6 (513 + 523) - \tilde{q}_2^2))$
 $\frac{1}{4} (\tilde{q}_2^2 (\tilde{q}_2^2 (11 \tilde{q}_6 (512 + 2513) - 2 (2 \tilde{q}_2^2 (73,14 + 513) - 513 (71,14 + 72,14) - \tilde{q}_2^2) + \tilde{q}_6 (8 \tilde{q}_2^2 + 9513)) + 4 \tilde{q}_2^2 74,12) + 2 \tilde{q}_2^2 (-\tilde{q}_2^2 (273,14 + 512) - 273,5 74,12 + \tilde{q}_2^2 512) + \tilde{q}_2^2 \tilde{q}_6 (4 \tilde{q}_2^2 (512 + 2523) - \tilde{q}_6 (4 \tilde{q}_2^2 + 513) - 2 \tilde{q}_2^2))$	 $\frac{1}{8} (\tilde{q}_2^2 (\tilde{q}_2^2 (-512 (72,16 + 74,11 + 74,16 + 711,16 + 513) + 513 (71,16 - 72,16 - 74,9 + 74,11 - 79,16 + 711,16 + 513)) + 2 \tilde{q}_2^2 (71,6 + 72,6) - 2 \tilde{q}_6 \tilde{q}_2^2) - 2 \tilde{q}_2^2 \tilde{q}_2^2 512 + 17 \tilde{q}_6 \tilde{q}_2^2 512) + 4 \tilde{q}_2^2 (-\tilde{q}_2^2 73,13 + \tilde{q}_2^2 \tilde{q}_6 (513 + 523) + \tilde{q}_2^2) + \tilde{q}_2^2 (71,6 + 72,6) 73,5)$	 $-\frac{1}{8} 512 (\tilde{q}_2^2 (\tilde{q}_2^2 (2 (71,20 + 72,20) - \tilde{q}_2^2) + 512 (4 \tilde{q}_2^2 (-71,6 + 71,20 - 272,7 - 76,20 - 77) + \tilde{q}_2^2 (73,18 - 74,18 - \tilde{q}_2^2) + 71,20 (71,6 + 72,5 - \tilde{q}_2^2) - 71,6 72,7 + 71,5 (-71,6 + 71,20 + 76,20)) + (-4 \tilde{q}_2^2 \tilde{q}_2^2 (2 (71,8 + 513) + 72,7) - 271,5 71,20 (71,6 + \tilde{q}_2^2) - 271,6 71,20 (72,5 - \tilde{q}_2^2))$	 $\frac{1}{2} (\tilde{q}_2^2 (\tilde{q}_2^2 (71,20 + 72,20 + 73,13 + 512) - 2 \tilde{q}_6 (\tilde{q}_2^2 + 2513 512 + 2 \tilde{q}_2^2) - 2 \tilde{q}_6 \tilde{q}_2^2 - \tilde{q}_2^2 \tilde{q}_6 (513 + 523)) + \tilde{q}_2^2 (2 \tilde{q}_2^2 (-73,5 + 73,6 + 74,6 + \tilde{q}_6 + 512) - 2512 71,14 - 2512 72,14 - 2512 73,5 + 2512 73,7 + 271,14 73,5 + 272,14 73,5 - \tilde{q}_6 512 - \tilde{q}_6 512 - \tilde{q}_6 512))$	 $\frac{1}{12} (16 \tilde{q}_2^2 (-73,13 + 74,13 - 73,19 + 73,20 + \tilde{q}_6 - \tilde{q}_6) - 8 \tilde{q}_2^2 (2 \tilde{q}_2^2 (71,19 - 71,20 + 72,19 - 72,20 + 2 \tilde{q}_6 - 2 \tilde{q}_6) + 4 (71,19 + 72,19) 73,13 - 4 (71,20 + 72,20) 74,13) - 4 \tilde{q}_2^2 512 (8 \tilde{q}_2^2 (72,7 - 71,16) - 4 \tilde{q}_2^2 (74,5 - 73,5) + 3 (\tilde{q}_6 - \tilde{q}_6) \tilde{q}_2^2) + 2 \tilde{q}_2^2 \tilde{q}_4 513 (71,7 + 71,16 - 72,7 - 72,16))$	 $\tilde{q}_2^2 (\tilde{q}_2^2 (512 + 513) (71,5 + 512) - \tilde{q}_6 \tilde{q}_6 (512 + 513) (71,5 + 512) - \tilde{q}_2^2 (512 + 513) (71,19 + 512 + 513) - (71,19 + 72,7 + 73,7 + 513) - \tilde{q}_6 \tilde{q}_6 (513 (271,19 + 72,7 + 72,19 + 73,7 + 513) + 512 (71,19 + 272,7 + 72,19 + 373,7 + 2513))$	 $\frac{1}{4} \tilde{q}_2^2 (-2 \tilde{q}_2^2 (71,6 + 71,12 + 72,6 + 72,12) + 2 \tilde{q}_2^2 (71,15 + 71,17 + 72,15 + 72,17) + 273,5 (73,5 + 73,13 - 74,5 - 74,13) + 512 (-271,7 - 271,16 + 272,19 + 272,20 - 73,12 + 273,13 + 73,17 - 274,5 - 74,6 + 74,15 + 75,7 + 75,16 + 71,19 + 71,20) - \tilde{q}_2^2 - \tilde{q}_2^2 13 - \tilde{q}_2^2 13 + 471,7 73,5 + 472,19 73,13 + 473,16 74,5 + 472,20 74,13)$

Amplitude diagrams VIII

Only 300 more to go...

Amplitude diagrams IX



....here are 50 of the most complicated remaining diagrams

UV properties

Planar $\mathcal{N}=4$ SYM 5-loop UV divergence

Planar UV divergence in $D=26/5$,

Bern, Carrasco, Dixon, HJ, Roiban (2009)

divergence $\propto N_c^5 \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}]$

$$12ul^2 \text{ (Diagram 1)} + 6u(l_1 + l_2)^2 \text{ (Diagram 2)} = \frac{114}{5}u \text{ (Diagram 3)}$$

With IBP relations expression collapses to a single scalar integral

One integral: by positivity non-zero, $D_c = 26/5$ as expected

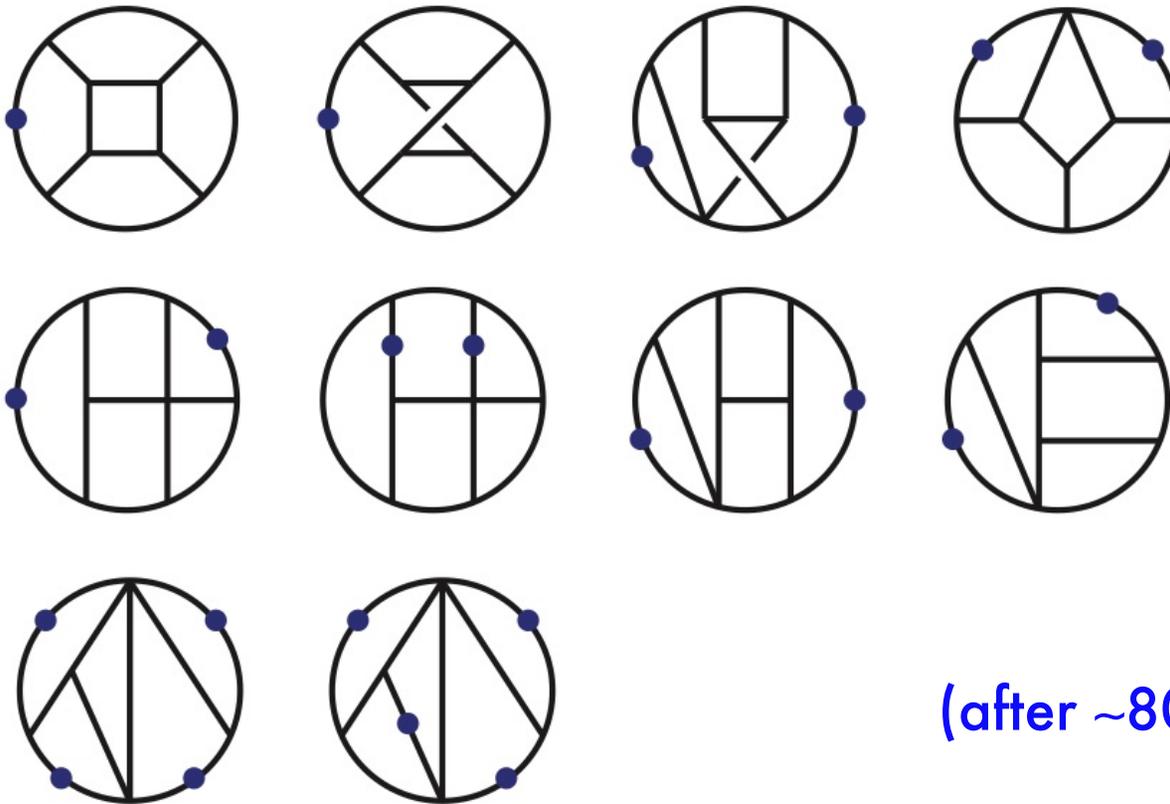
Non-planar $\mathcal{N}=4$ SYM 5-loop UV divergence I

Non-Planar UV divergence in $D=26/5$: **Preliminary!**

divergence $\propto N_c^3 \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}]$

expressible in terms of 10 vacuum integrals

(not independent,
ongoing work)



(after ~ 8000 IBP relations)

Non-planar $\mathcal{N}=4$ SYM 5-loop UV divergence II

Non-Planar UV divergence in $D=26/5$:

$$\begin{aligned} \text{Components: } & N_c^1 \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] \\ & N_c^4 \text{Tr}[T^{a_1} T^{a_2}] \text{Tr}[T^{a_3} T^{a_4}] \\ & N_c^2 \text{Tr}[T^{a_1} T^{a_2}] \text{Tr}[T^{a_3} T^{a_4}] \\ & N_c^0 \text{Tr}[T^{a_1} T^{a_2}] \text{Tr}[T^{a_3} T^{a_4}] \quad \text{Vanish!} \end{aligned}$$

Double traces and single-trace NNLC finite in $D=26/5$,
only single-trace LC and NLC are divergent

similar to 4 loops: *see Dixon's talk*

Summary / Outlook

- The 5-loop divergence is the most important tangible calculation on the question of UV finiteness of N=8 SG.
- This calculation will provide critical information on the amplitude structure: $24/5$ vs. $26/5$ will correspond to
$$A_4^{(L=5)} \sim \partial^8 R^4 \times (\text{intgrl.}) \quad \rightarrow L = 7 \text{ divergent}$$
$$A_4^{(L=5)} \sim \partial^{10} R^4 \times (\text{intgrl.}) \quad \rightarrow L = 7 \text{ finite}$$
- As a first step, we have obtained the 5-loop N=4 SYM amplitude and UV divergences
- Next step is to use the duality between color and kinematics to obtain the gravity amplitude. This task is greatly simplified with a representation of the N=4 SYM amplitude in our hands.
- Stay tuned...