Worldsheet Arena:
Lorentz invariance versus T-duality

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Overview

Motivation:

- Recent progress on generalized geometry and non-geometry
  - $O(D,D)$ structures on $TM \otimes T^*M$
  - ”manifolds” with a structure group involving T-dualities
  - Duality chain of fluxes: $H_{\mu\nu\rho} \rightarrow f_{\mu\nu}{}^{\rho} \rightarrow Q_{\mu}{}^{\nu\rho} \rightarrow R^{\mu\nu\rho}$
- Development of double field theory
  - doubling of torus coordinates: $X^\mu \rightarrow Y = (X^\mu, \widetilde{X}_\mu)$.
  - Bosonic part of the supergravity action written in a T-duality invariant form

cf. Peter Patalong's talk

The basic question addressed in this talk is what is the appropriate worldsheet formalism to investigate these theories.

Reminder: T-duality

The spectrum of closed strings on a circle of radius R:

$$M^2 = \frac{m^2}{R^2} + n^2 R^2$$

exhibits T-duality invariance:

$$R \leftrightarrow \frac{1}{R} \qquad\qquad m \leftrightarrow n$$

The generalization to a D dimensional torus T^D with constant metric g and b–field is described by the Buscher's rule

$$g + b \leftrightarrow \tilde{g}^{-1} + \tilde{\beta} = (g+b)^{-1}$$

Worldsheet action

On the level of the sigma model action,

$$S = \int d^2\sigma\, \partial_L X^T E(X)\, \partial_R X \ , \qquad E(X) = g(X) + b(X) \ ,$$

with the left/right-moving derivatives

$$\partial_L = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1) \ , \qquad \partial_R = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1) \ ,$$

Under Lorentz transformations on the worldsheet these derivatives transform as follows:

$$\begin{pmatrix} \partial_0 \\ \partial_1 \end{pmatrix} \rightarrow \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \partial_L \rightarrow e^{+\alpha}\, \partial_L \\ \partial_R \rightarrow e^{-\alpha}\, \partial_R \end{cases}$$

Hence the worldsheet action is Lorentz invariant.

Three wordsheet approaches to T-duality

There are essentially three approaches to describe T-duality:

1. Buscher's approach
2. Tseytlin's approach
3. Hull's approach

Let us review each of them briefly...

Buscher's approach

The torus isometries are promoted to local symmetries

$$X(\sigma) \to X(\sigma) + \xi(\sigma) \ , \qquad V_a(\sigma) \to V_a(\sigma) - \partial_a \xi(\sigma) \ ,$$

by the introduction of gauge fields V_a:

$$S = \int d^2\sigma \left\{ D_L X^T E \, D_R X + \tilde{X}^T F \right\} , \qquad \begin{aligned} D_a X &= \partial_a X + V_a \\ F &= \partial_R V_L - \partial_L V_R \end{aligned}$$

T-duality is just a gauge choice:

choice I: $\xi \stackrel{!}{=} 0$: $\quad S = \int d^2\sigma \, \partial_L X^T E \, \partial_R X \ , \qquad E = g + b \ ,$

(E.o.M. of $\widetilde{X}$ $\quad \Rightarrow \quad$ V is pure gauge: $V_a(\sigma) = \partial_a \xi(\sigma)$.)

choice II: $X \stackrel{!}{=} 0$: $\quad S = \int d^2\sigma \, \partial_L \tilde{X}^T \tilde{E} \, \partial_R \tilde{X} \ , \qquad \tilde{E} = (g + b)^{-1} \ .$

Tseytlin's approach ($g = \mathbb{1}, b = 0$)

Tseytlin starts from a double Floreanini–Jackiw chiral boson action

$$S = \int d^2\sigma \Big\{ -\frac{1}{\sqrt{2}}\, \partial_1 X_L{}^T \partial_L X_L + \frac{1}{\sqrt{2}}\, \partial_1 X_R{}^T \partial_R X_R \Big\}$$

By a change of basis $X_{L,R} = (X \pm \widetilde{X})/\sqrt{2}$ this takes the form

$$S = \frac{1}{2} \int d^2\sigma \Big\{ -\partial_0 X^T \partial_1 \widetilde{X} - \partial_0 \widetilde{X}^T \partial_1 X - (\partial_1 X)^2 - (\partial_1 \widetilde{X})^2 \Big\}$$

Upon using certain boundary conditions the E.o.M.'s read:

$$\partial_0 X = \partial_1 \tilde{X} \,, \qquad \partial_0 \tilde{X} = \partial_1 X$$

These can be thought of as the Hamilton equations where the canonical momenta P are promoted to dual coordinates $\widetilde{X}$ as $P = \partial_1 \widetilde{X}$.

Tseytlin's approach

The action generalizes to:

$$S = \int d^2\sigma \left\{ -\frac{1}{2}\partial_1 Y^T \eta\, \partial_0 Y - \frac{1}{2}\partial_1 Y^T \mathcal{H}\, \partial_1 Y \right\}, \qquad Y = \begin{pmatrix} X \\ \tilde{X} \end{pmatrix},$$

where for general constant metric g and b-field:

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}.$$

- Since η is the $O(D,D)$ invariant metric and $\mathcal{H} \in O(D,D)$ the generalized metric $O(D,D)$ T-duality covariance is manifest.
- But (manifest) Lorentz invariance is lost.

Hull's approach

In Hull's approach one doubles the torus coordinates:

$$X \qquad \longrightarrow \qquad Y = \begin{pmatrix} X \\ \tilde{X} \end{pmatrix}$$

and writes down the $O(D, D)$ covariant and Lorentz invariant action in the doubled world:

$$S = \frac{1}{2} \int d^2\sigma \, \partial_L Y^T \mathcal{H} \, \partial_R Y$$

In order to reduce the number of worldsheet D.o.F.'s a self-duality constraint is enforced by hand

$$dY = \eta^{-1} \mathcal{H} * dY$$

Basic questions

1. Are these different approaches related? If yes, how?
   - Can one find a Lorentz invariant and $O(D,D)$ covariant description on the worldsheet that builds in the constraint naturally?
2. How to defined the quantum theory?
   - Can the worldsheet D.o.F.'s keep the same?
3. How to go beyond constant backgrounds?

Quantum theory (constant backgrounds):

Buscher's gauge theory can be defined at the quantum level by the Faddeev-Poppov gauge fixed path integral

$$Z = \int \mathcal{D}[X, \tilde{X}, V, \tilde{B}, \textsc{b}, \textsc{c}] \, \sqrt{\det E} \; \exp iS \;,$$

$$S = \int d^2\sigma \left\{ D_L X^T E \, D_R X + \tilde{X}^T F + \tilde{B}^T G + \textsc{b}^T \delta_{\textsc{c}} G \right\}$$

- The gauge fixing condition $G \stackrel{!}{=} 0$ is implemented by a Lagrange multiplier $\tilde{B}$.
- The $(\textsc{b}, \textsc{c})$ ghosts are determined by the variation of the gauge fixing condition $\delta_\xi G$ with the gauge parameter ξ replaced by $\textsc{c}$.

Buscher's and Rocek-Tseytlin's gauges

The non-gauged path integral reads:

$$Z = \int \mathcal{D}[X] \sqrt{\det E} \, \exp i \int \partial_L X^T E \, \partial_R X \qquad (1)$$

The Buscher's gauge $X \overset{!}{=} 0$ leads to:

$$Z = \int \mathcal{D}[\tilde{X}] \sqrt{\det \tilde{E}} \, \exp i \int \partial_L \tilde{X}^T \tilde{E} \, \partial_R \tilde{X} \qquad (2)$$

The Rocek-Tseytlin's gauge $V_1 \overset{!}{=} 0$ gives:

$$\begin{aligned} Z &= \int \mathcal{D}[\ldots] \exp i \int \Big\{ -\frac{1}{2} \partial_1 Y^T \mathcal{H} \, \partial_1 Y - \frac{1}{2} \partial_1 Y^T \eta \, \partial_0 Y + \mathrm{B}^T \partial_1 \mathrm{C} \Big\} \\ &= \int \mathcal{D}[Y] (\det \partial_1) \, \exp i \int \Big\{ -\frac{1}{2} \partial_1 Y^T \mathcal{H} \, \partial_1 Y - \frac{1}{2} \partial_1 Y^T \eta \, \partial_0 Y \Big\} \end{aligned} \qquad (3)$$

A Lorentz invariant gauge

A Lorentz invariant gauge fixing $V_L \stackrel{!}{=} 0$ gives

$$Z = \int \mathcal{D}[\ldots] \exp i \int \left\{ \partial_L X^T E\, \partial_R X + (\partial_L X^T E + \partial_L \tilde{X}^T) V_R - \partial_L \mathrm{c}^T \mathrm{B} \right\} \quad (4)$$

Hence V_R enforces the constraint: $\partial_L X^T E + \partial_L \tilde{X}^T \stackrel{!}{=} 0$.

Using this constraint and some partial integrations we can rewrite this in the Hull's $O(D, D)$ covariant form:

$$Z = \int \mathcal{D}[\ldots] \exp i \int \left\{ \frac{1}{2} \partial_L Y^T \mathcal{H}\, \partial_R Y + (\partial_L X^T E + \partial_L \tilde{X}^T) V_R - \partial_L \mathrm{c}^T \mathrm{B} \right\}$$

Hence the constraint that Hull enforced "by hand" is here fundamental to arrive at the form with the generalized metric.

Cancellation of the chiral bosons

By a change of variables, $\tilde{X} \to \tilde{X} - E^T X$, the path integral can be written as:

$$Z = \int \mathcal{D}[\ldots] \exp i \int \left\{ \partial_L X^T E \, \partial_R X + \partial_L \tilde{X}^T V_R - \partial_L \mathrm{c}^T \mathrm{B} \right\}$$

- The fields $(\widetilde{X}, V_R)$ describes chiral bosons.
- Their contribution is precisely cancelled by the ghosts (B, C).

This is not surprising: As we started from a consistent gauge theory, one would not expect to obtain a sick theory hence by a gauge fixing.

Quantum theory (non-constant backgrounds):

Have another look at the Lorentz invariantly gauged fixed path integral:

$$Z = \int \mathcal{D}[\ldots] \exp i \int \Big\{ \partial_L X^T E \, \partial_R X + G_L^T \, V_R - \partial_L \mathrm{C}^T \mathrm{B} \Big\}$$

The constraint enforced by V_R,

$$G_L^T = \partial_L X^T E + \partial_L \tilde{X}^T \stackrel{!}{=} 0$$

can be thought of as a gauge fixing condition of: $\delta_{\tilde{\xi}} \tilde{X} = \tilde{\xi}$.

Preserving this gauge symmetry, this theory may be generalized to

$$Z = \int \mathcal{D}[\ldots] \exp i \int \Big\{ \partial_L X^T E(X) \, \partial_R X + G_L^T \, V_R - \delta_{\mathrm{C}} G_L^T \mathrm{B} \Big\}$$

with $G_L^T = \partial_L X^T K(Y) + \partial_L \tilde{X}^T L(Y)$.

Different guises

There are various representations for the path integral

$$Z = \int \mathcal{D}[\ldots] \exp i \int \Big\{ \partial_L X^T E(X)\, \partial_R X + G_L^T\, V_R - \delta_{\rm C} G_L^T\, {\rm B} \Big\}$$

with $G_L^T = \partial_L X^T K(Y) + \partial_L \tilde{X}^T L(Y)$:

The matrices K and L are defined up to matrix functions ρ:

$$K \to K\, \rho^{-1}\,, \quad L \to L\, \rho^{-1}\,, \quad V_R \to \rho\, V_R\ .$$

By the transformation, $V_R \to V_R + \kappa(Y)\, \partial_R X + \lambda(Y)\, \partial_R \tilde{X}$, the classical action can be rewritten as

$$S_{\rm cl.} = \int \Big\{ \partial_L Y^T \begin{pmatrix} E + K\, \kappa & K\, \lambda - \mu \\ L\, \kappa + \mu & L\, \lambda \end{pmatrix} \partial_R Y + \partial_L Y^T \begin{pmatrix} K \\ L \end{pmatrix} V_R \Big\}$$

A constant matrix μ can be introduced by a double partial integration.

O(D,D) invariance

We can bring the constrained classical action to the form:

$$S_{\text{cl.}} = \int \Big\{ -\frac{1}{2}\,\partial_L Y^T \eta\, \partial_R Y + \partial_L Y^T \begin{pmatrix} K \\ L \end{pmatrix} V_R \Big\}\,, \qquad \begin{pmatrix} K \\ L \end{pmatrix} = \begin{pmatrix} E \\ \mathbb{1} \end{pmatrix}$$

This is invariant under constant $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(D,D)$ mappings:

$$Y \to Y' = M^{-T} Y\,, \quad \begin{pmatrix} K \\ L \end{pmatrix} \to \begin{pmatrix} K' \\ L' \end{pmatrix} = M \begin{pmatrix} K \\ L \end{pmatrix}\,, \quad V'_R = V_R$$

Keeping $L = L' = \mathbb{1}$ fixed induces a fractional linear transformation

$$E(Y) \to E'(Y') = \big(\alpha E(Y) + \beta\big)\big(\gamma E(Y) + \delta\big)^{-1}\,, \quad V_R \to \big(\gamma E(Y) + \delta\big)\, V_R$$

O(D,D) invariance

An equivalent form

$$S_{\text{cl.}} = \int \left\{ \frac{1}{2} \partial_L Y^T \mathcal{H}(Y) \partial_R Y + \partial_L (Y)^T \begin{pmatrix} K \\ L \end{pmatrix} V_R \right\} ,$$

involves the generalized metric

$$\mathcal{H}(Y) = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix}$$

which transforms under $M \in O(D,D)$ as

$$Y \to Y' = M^{-T} Y , \qquad \mathcal{H}(Y) \to \mathcal{H}'(Y') = M \mathcal{H}(Y) M^T$$

Twisted torus

Consider a three torus T^3 with unit radii, i.e. $T_z:\ z \to z+1$, with:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad b = z\,\omega\,, \quad \omega = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The combination $E(z) = g + b(z)$ is not invariance under all torus periodicities:

$$T_z:\ E \to E' = E(z+1) = E + \omega$$

This can be compensated by an $O(3,3)$ transformation:

$$(\alpha\,E') + \beta)(\gamma\,E' + \delta)^{-1} \stackrel{!}{=} E \quad \Rightarrow \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -\omega \\ 0 & \mathbb{1} \end{pmatrix}$$

In terms of the doubled coordinates we have

$$T_z:\ Y \to M^{-T}\,Y + \begin{pmatrix} e_z \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} Y + \begin{pmatrix} e_z \\ 0 \end{pmatrix}$$

Conclusion

1. We showed that various worldsheet approaches to T-duality can be understood as different gauges of Buscher's gauge theory.

2. In particular, we have obtained a Lorentz invariant and $O(D,D)$ covariant description by Lorentz invariant gauge choice, $V_L = 0$.

   The remaining gauge field component V_R reduces the number of D.o.F.'s, $Y = (X, \widetilde{X})$, by half (this is conventionally done by hand).

3. We given a path integral definition of this theory and checked that it does not suffer from extra chiral boson modes.

4. Interpreting the constraint enforced by V_R as a gauge condition for the transformation, $\delta \widetilde{X} = \tilde{\xi}$, we are able to go beyond constant backgrounds.