A planar four-loop form factor and cusp anomalous dimension in QCD

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J. Henn, A. Smirnov, V. Smirnov and M. Steinhauser, arXiv:1604.03126

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- Perspectives

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Let Γ_q^μ be the photon-quark vertex function.

The scalar form factor is

$$F_q(q^2) \;\; = \;\; -\frac{1}{4(1-\epsilon)q^2} \mathrm{Tr}\left(\not{p}_2\, \Gamma_q^\mu \not{p}_1\, \gamma_\mu\right) \; ,$$

where $D = 4 - 2\epsilon$, $q = p_1 + p_2$ and p_1 (p_2) is the incoming (anti-)quark momentum.

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The large-N_c asymptotics of $F_q(q^2) \to$ planar Feynman diagrams.

Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A. V. Smirnov, V. A. Smirnov and M. Steinhauser'09,
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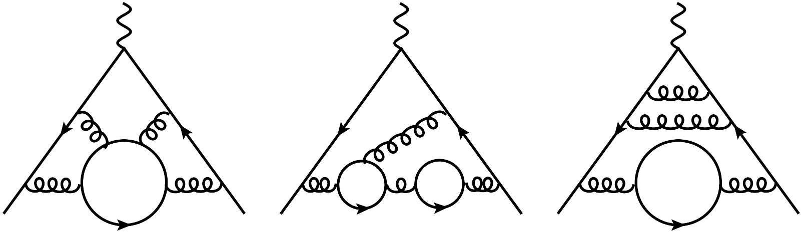
Analytic results for the three-loop master integrals up to weight 8
[R. N. Lee and V. A. Smirnov'10]
motivated by a future four-loop calculation.

The fermionic corrections ($\sim n_f$) to F_q in the large-N_c limit, to the four-loop order.



Numerical four-loop calculations
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Results for some individual integrals in an analytical form
[A. von Manteuffel, E. Panzer & R. M. Schabinger'15]

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Our result is the fermionic contribution to $F_q^{(4)}$ in the large-N_c limit.

$$\frac{1}{\epsilon^7}\left[\frac{1}{12}N_c^3 n_f\right] + \frac{1}{\epsilon^6}\left[\frac{41}{648}N_c^2 n_f^2 - \frac{37}{648}N_c^3 n_f\right] + \frac{1}{\epsilon^5}\left[\frac{1}{54}N_c n_f^3 + \frac{277}{972}N_c^2 n_f^2 + \left(\frac{41\pi^2}{648} - \frac{6431}{3888}\right)N_c^3 n_f\right] + \frac{1}{\epsilon^4}\left[\left(\frac{215\zeta_3}{108} - \frac{72953}{7776} - \frac{227\pi^2}{972}\right)N_c^3 n_f + \frac{11}{54}N_c n_f^3 + \left(\frac{5}{24} + \frac{127\pi^2}{1944}\right)N_c^2 n_f^2\right] + \frac{1}{\epsilon^3}\left[\left(\frac{229\zeta_3}{486} - \frac{630593}{69984} + \frac{293\pi^2}{2916}\right)N_c^2 n_f^2 + \left(\frac{2411\zeta_3}{243} - \frac{1074359}{69984} - \frac{2125\pi^2}{1296} + \frac{413\pi^4}{3888}\right)N_c^3 n_f + \left(\frac{127}{81} + \frac{5\pi^2}{162}\right)N_c n_f^3\right] + \frac{1}{\epsilon^2}\left[\left(-\frac{41\zeta_3}{81} + \frac{29023}{2916} + \frac{55\pi^2}{162}\right)N_c n_f^3 + \left(\frac{11684\zeta_3}{729} - \frac{41264407}{419904} - \frac{155\pi^2}{72} + \frac{2623\pi^4}{29160}\right)N_c^2 n_f^2 + \left(-\frac{537625\zeta_3}{11664} - \frac{599\pi^2\zeta_3}{486} + \frac{12853\zeta_5}{180} + \frac{155932291}{839808} - \frac{27377\pi^2}{69984} - \frac{1309\pi^4}{7290}\right)N_c^3 n_f\right] + \ldots$$

The coefficients of the cusp and collinear anomalous dimensions

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with $x \in \{\text{cusp}, q\}$.

$$\begin{aligned}
\gamma^0_{\text{cusp}} &= 4\,,\\
\gamma^1_{\text{cusp}} &= \left(-\frac{4\pi^2}{3}+\frac{268}{9}\right)N_c - \frac{40 n_f}{9}\,,\\
\gamma^2_{\text{cusp}} &= \left(\frac{44\pi^4}{45}+\frac{88\zeta_3}{3}-\frac{536\pi^2}{27}+\frac{490}{3}\right)N_c^2\\
&\quad+\left(-\frac{64\zeta_3}{3}+\frac{80\pi^2}{27}-\frac{1331}{27}\right)N_c n_f - \frac{16 n_f^2}{27}\,,\\
\gamma^3_{\text{cusp}} &= \left(-\frac{32\pi^4}{135}+\frac{1280\zeta_3}{27}-\frac{304\pi^2}{243}+\frac{3463}{81}\right)N_c n_f^2 + \left(\frac{128\pi^2\zeta_3}{9}+224\zeta_5\right.\\
&\quad\left.-\frac{44\pi^4}{27}-\frac{16252\zeta_3}{27}+\frac{13346\pi^2}{243}-\frac{60391}{81}\right)N_c^2 n_f + \left(\frac{64\zeta_3}{27}-\frac{32}{81}\right)n_f^3 + \ldots
\end{aligned}$$

$$\begin{aligned}
\gamma_q^0 &= -\frac{3N_c}{2}, \quad \gamma_q^1 = \left(\frac{\pi^2}{6}+\frac{65}{54}\right)N_c n_f + \left(7\zeta_3 - \frac{5\pi^2}{12} - \frac{2003}{216}\right)N_c^2, \\
\gamma_q^2 &= \left(-\frac{\pi^4}{135} - \frac{290\zeta_3}{27} + \frac{2243\pi^2}{972} + \frac{45095}{5832}\right)N_c^2 n_f + \left(-\frac{4\zeta_3}{27} - \frac{5\pi^2}{27} + \frac{2417}{1458}\right) \\
&\quad + N_c^3\left(-68\zeta_5 - \frac{22\pi^2\zeta_3}{9} - \frac{11\pi^4}{54} + \frac{2107\zeta_3}{18} - \frac{3985\pi^2}{1944} - \frac{204955}{5832}\right), \\
\gamma_q^3 &= N_c^3\left[\left(-\frac{680\zeta_3^2}{9} - \frac{1567\pi^6}{20412} + \frac{83\pi^2\zeta_3}{9} + \frac{557\zeta_5}{9} + \frac{3557\pi^4}{19440} - \frac{94807\zeta_3}{972}\right.\right. \\
&\quad \left.\left. + \frac{354343\pi^2}{17496} + \frac{145651}{1728}\right)n_f\right] + \left(-\frac{8\pi^4}{1215} - \frac{356\zeta_3}{243} - \frac{2\pi^2}{81} + \frac{18691}{13122}\right)N_c n_f^3 \\
&\quad + \left(-\frac{2}{3}\pi^2\zeta_3 + \frac{166\zeta_5}{9} + \frac{331\pi^4}{2430} - \frac{2131\zeta_3}{243} - \frac{68201\pi^2}{17496} - \frac{82181}{69984}\right)N_c^2 n_f^2 + \ldots
\end{aligned}$$

We reproduce results up to three loops
[A. Vogt'01; C.F. Berger'02; S. Moch, J.A.M. Vermaseren & A. Vogt'04,05; P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov & M. Steinhauser'09; T. Becher & M. Neubert'09; T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli & C. Studerus'10]

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All the other four-loop terms in γ^3_{cusp} and γ^3_q are new.

All planar four-loop on-shell form-factor integrals with $p_1^2 = p_2^2 = 0$, with $q^2 \equiv p_3^2 = (p_1 + p_2)^2$

$$
\begin{aligned}
F a_1, \ldots, a_{18} \quad = \quad & \int \cdots \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4}{(-(k_1+p_1)^2)^{a_1}(-(k_2+p_1)^2)^{a_2}(-(k_3+p_1)^2)^{a_3}} \\
& \times \frac{1}{(-(k_4+p_1)^2)^{a_4}(-(k_1-p_2)^2)^{a_5}(-(k_2-p_2)^2)^{a_6}(-(k_3-p_2)^2)^{a_7}} \\
& \times \frac{1}{(-(k_4-p_2)^2)^{a_8}(-k_1^2)^{a_9}(-k_2^2)^{a_{10}}(-k_3^2)^{a_{11}}(-k_4^2)^{a_{12}}} \\
& \times \frac{1}{(-(k_1-k_2)^2)^{-a_{13}}(-(k_1-k_3)^2)^{-a_{14}}(-(k_1-k_4)^2)^{-a_{15}}} \\
& \times \frac{1}{(-(k_2-k_3)^2)^{-a_{16}}(-(k_2-k_4)^2)^{-a_{17}}(-(k_3-k_4)^2)^{-a_{18}}} \, .
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\end{aligned}
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At most 12 indices can be positive.

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Henn: use uniform transcendental (UT) bases.

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

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Henn (2013): turn to a new basis where DE take the form

$$\partial_i f(\epsilon, x) = \epsilon\, A_i(x) f(\epsilon, x)\,.$$

In the case of two scales, i.e. with one variable in the DE, i.e. $n = 1$.

$$f'(\epsilon, x) = \epsilon \sum_k \frac{a_k}{x - x^{(k)}} f(\epsilon, x) \,.$$

where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

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For example, if $x_k = 0, -1, 1$ then results are expressed in terms of HPLs [E. Remiddi & J.A.M. Vermaseren]

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t) \mathrm{d}t\,,$$

where $f(\pm 1; t) = 1/(1 \mp t)$, $f(0; t) = 1/t$

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- Select basis integrals that have constant leading singularities [F. Cachazo'08] which are multidimensional residues of the integrand. (Replace propagators by delta functions).
- If you have almost reached the ε-form, make a small final rotation of the current basis.
  See, e.g., [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs'14]

We obtain (KZ) differential equations with respect to $x = p_2^2/p_3^2$

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Solving these equations in terms of HPL with letters 0 and 1.

Asymptotic behaviour at the points $x = 0$ and $x = 1$

$$f(x,\epsilon) \overset{x\to 0}{=} \left[1 + \sum_{k\geq 1} p_k(\epsilon) x^k\right] x^{\epsilon a} f_0(\epsilon)\,,$$

$$f(x,\epsilon) \overset{x\to 1}{=} \left[1 + \sum_{k\geq 1} q_k(\epsilon)(1-x)^k\right] (1-x)^{-\epsilon b} f_1(\epsilon)\,,$$

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there is no singularity and it corresponds to propagator-type integrals which are known
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To obtain analytical results for the 99 one-scale MI, we perform (with the help of the HPL package [D. Maître'06]) matching at the point $x = 0$.

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Two alternative descriptions of the asymptotic expansion: with DE and with with expansion by regions [M. Beneke and V. Smirnov'98]

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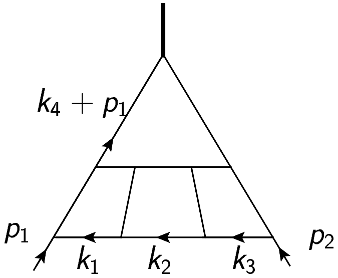
An example of our result



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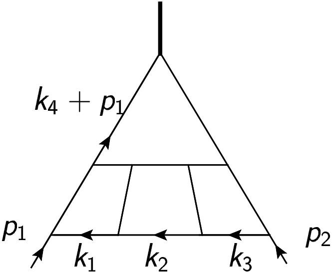

$$I_{12} = \int \ldots \int \prod_{j=1}^{4} \mathrm{d}^D k_j \, \frac{(k_4^2)^2}{k_1^2 k_2^2 k_3^2 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_1 - k_4)^2}$$
$$\times \frac{1}{(k_2 - k_4)^2 (k_3 - k_4)^2 (k_1 + p_1)^2 (k_4 + p_1)^2 (k_4 - p_2)^2 (k_3 - p_2)^2}$$

$$
\begin{aligned}
=&\frac{1}{576\varepsilon^8}+\frac{1}{216}\pi^2\frac{1}{\epsilon^6}+\frac{151}{864}\zeta_3\frac{1}{\epsilon^5}+\frac{173}{10368}\pi^4\frac{1}{\epsilon^4}+\left[\frac{505}{1296}\pi^2\zeta_3+\frac{5503}{1440}\zeta_5\right]\frac{1}{\epsilon^3}+\\
&+\left[\frac{6317}{155520}\pi^6+\frac{9895}{2592}\zeta_3^2\right]\frac{1}{\epsilon^2}+\left[\frac{89593}{77760}\pi^4\zeta_3+\frac{3419}{270}\pi^2\zeta_5-\frac{169789}{4032}\zeta_7\right]\frac{1}{\epsilon}\\
&+\left[\frac{407}{15}s_{8a}+\frac{41820167}{653184000}\pi^8+\frac{41719}{972}\pi^2\zeta_3^2-\frac{263897}{2160}\zeta_3\zeta_5\right]+\mathcal{O}(\epsilon)\,,
\end{aligned}
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