30 years, 715 integrals, and 1 dessert

or:

Bosonic contributions to the 2-loop Zbb vertex

Tord Riemann (15711 Königs Wusterhausen, Germany)

Work done together with:

I. Dubovyk (U. Hamburg) + **J. Usovitsch** (Humboldt-U. Berlin)

A. Freitas (U. Pittsburg), **J. Gluza** (Silesian U. Katowice) → Talk at LL2016

https://indico.desy.de/conferenceDisplay.py?ovw=True&confId=12010

talk held at “Loops and Legs 2016”, April 24-29, 2016, Leipzig, Germany

Outline

Numerical approach to the 2-loop Z-boson width

We aim at calculating the last missing contributions to the Z-boson parameters:
The electroweak bosonic 2-loop corrections to $Z \to \bar{b}b$
depending on s, M_Z, M_W, M_H, m_t, at $s/M_Z^2 = 1 + i\epsilon$.

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$d\sigma/d\cos\theta\ (e^+e^- \to \bar{b}b)$

Close to the Z-boson peak:

$$\frac{d\sigma}{d\cos\theta} \sim G_F^2 \left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2 \left[(a_e^2+v_e^2)(a_b^2+v_b^2)(1+\cos^2\theta)+(2a_e v_e)(2a_b v_b)(2\cos\theta)\right]$$

Symmetric integration over $\cos\theta$

$$\sigma_T = \int_{-1}^{1} d\cos\theta \frac{d\sigma}{d\cos\theta}$$

$$\sim \left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2 G_F(a_e^2+v_e^2)\ \mathbf{G_F(a_b^2+v_b^2)} \qquad \sim \Gamma_e\ \Gamma_b$$

Anti-symmetric integration over $\cos\theta$

$$A_{F-B} = \frac{\left[\int_0^1 d\cos\theta - \int_{-1}^0 d\cos\theta\right]\frac{d\sigma}{d\cos\theta}}{\sigma_T}$$

$$\sim \frac{2a_e v_e}{a_e^2+v_b^2}\ \frac{\mathbf{2a_b v_b}}{\mathbf{a_b^2+v_b^2}} \qquad \sim A_e\ \mathbf{A_b}$$

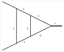
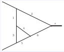
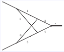
Correct renormalization uses S-matrix notations: $\mathcal{M} \sim R/(s-M_Z^2+iM_Z\Gamma_Z)$ + Taylor expansion.

Outline – What to be calculated?

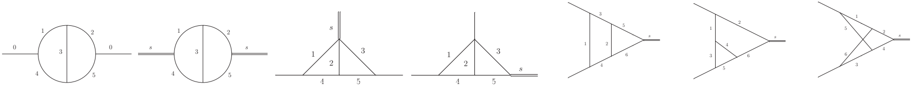
The Z-boson width

$$\Gamma(z \to \bar{b}b) \sim |\mathcal{M}_{1-loop} + \mathcal{M}_{2-loop} + \cdots|^2 + \cdots$$

$$\mathcal{M}_{2-loop} \sim \cdots + \bar{u}\, \mathbf{V}^{Zb\bar{b}}{}_{\mu}\, u\, \epsilon^{\mu} + \cdots$$

$$\mathbf{V}^{Zb\bar{b}}_{\mu} \sim \cdots +$$  $+$  $+$  $+ \cdots$

We calculate the bosonic integrals for the 2-loop diagrams for the vertex $\mathbf{V}^{Zb\bar{b}}_{\mu}$



The vector and axial vector components are, with $k^2 = M_Z^2$:

$$g_{\mathrm{V}}^{b}(k^2) = \frac{1}{2(2-D)k^2}\,\mathrm{Tr}[\gamma^{\mu}\, \not{p}_1\, \mathbf{V}^{Zb\bar{b}}_{\mu}\, \not{p}_2],$$

$$g_{\mathrm{A}}^{b}(k^2) = \frac{1}{2(2-D)k^2}\,\mathrm{Tr}[\gamma_5\, \gamma^{\mu}\, \not{p}_1\, \mathbf{V}^{Zb\bar{b}}_{\mu}\, \not{p}_2],$$

Outline .....

$\sin^2\theta_{\rm eff}^{\rm b}$

$$\Re e \, \frac{g_V^b}{g_A^b} \equiv 1 - 4|Q_b| \sin^2\theta_{\rm eff}^{\rm b}$$

$\sin^2\theta_{\rm eff}^{\rm b}$ and $\Delta\kappa_b$

$$\begin{aligned} \sin^2\theta_{\rm eff}^{\rm b} &= \frac{1}{4|Q_b|}\left(1 - \Re e \, \frac{g_V^b}{g_A^b}\right) \\ &= \left(1 - \frac{M_W^2}{M_Z^2}\right)(1 + \Delta\kappa_b) \\ &= 0.281 \pm 0.016 \quad \text{LEP1 2005 [6]} \end{aligned}$$

$\Delta\kappa_b$ and A_b

$$\begin{aligned} A_{\rm b} &= \frac{2\Re e \, \frac{g_V^b}{g_A^b}}{1 + (\Re e \, \frac{g_V^b}{g_A^b})^2} \\ &= \frac{1 - 4|Q_b| \sin^2\theta_{\rm eff}^{\rm b}}{1 - 4|Q_b| \sin^2\theta_{\rm eff}^{\rm b} + 8Q_b^2\left(\sin^2\theta_{\rm eff}^{\rm b}\right)^2} \\ &= 0.899 \pm 0.013 \quad \text{LEP1 2005 [6]} \end{aligned}$$

LEP1 $\to$ LEPEWWG-extended [6], using ZFITTER [7, 8, 9]
See also the ZFITTER webpage `http://sanc.jinr.ru/users/zfitter/`.

Higher order references for the Zbb vertex

1-loop contributions – Akhundov:1985,Bernabeu:1987,Beenakker:1988,Jegerlehner:1988,Diakonos:1988 [10, 11, 12, 13, 14]

Fermionic electroweak two-loop corrections – Awramik:2008 [15]

$\mathcal{O}(\alpha\alpha_{\rm s})$ **QCD corrections** – Djouadi:1987,Djouadi:1988,Kniehl:1990,Kniehl:1991,Djouadi:1993, Czarnecki:1996,Harlander:1998 [16, 17, 18, 19, 20, 21, 22]

partial h.o. corr's. of order $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2)$ – Avdeev:1994,Chetyrkin:1995 [23, 24]

partial h.o. corr's. of order $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^3)$ – Schroeder:2005,Chetyrkin:2006,Boughezal:2006 [25, 26, 27]

partial h.o. corr's. of order $\mathcal{O}(\alpha^2\alpha_{\rm t})$ **and** $\mathcal{O}(\alpha_{\rm t}^3)$ – vanderBij:2000,Faisst:2003 [28, 29]

Standard Model prediction of $M_{\rm W}$ **from the Fermi constant** G_μ – Awramik:2003 [30].

Further references

Awramik:2006 [31], Freitas:2013 [32], Freitas:2014h [33], Awramik:2002 [34], Onishchenko:2002[35], Freitas:2002 [36], Bardin:1988 [37].

Present official numerical values: See Particle Data Group 2014 [38].

Lacking: Bosonic 2-loop vertices – have more scales than the fermionic ones

30 years ago: 1-loop corrections to $Z \to l^+ l^-$

Around 1980 it was clear that the Glashow/Salam/Weinberg model might become the Standard Model.
$\to$ More elaborated loop calculations calculations became meaningful.

Sufficient knowlegde for 1-loop calculations

– Euler Dilog $\mathrm{Li}_2(x)$
– plus Passarino Veltman 1978 [39]
– plus 't Hooft + Veltman 1978 [40]

Among the first substantial projects was:
Complete EWRC for massless external particles: $2 \to 2$*, including* $ee \to ff$ *and* $Z \to ff$*:*
1981 – Bardin + Christova + Fedorenko [41] – no expanded numerics

Triggered partly by detecting $Z \to l^+ l^-$ with UA1, UA2, there were several calculations of Z-decay into leptons (light fermions):
1983 – Consoli + Lo Presti + Maiani [42]
1984 – Fleischer + Jegerlehner [43]

30 years ago: 1-loop corrections to $Z \to \bar{b}b$

The $Z \to bb$ was not included. Specifics: m_t-dependent vertex contributions
We at Dubna – Akhundov + Bardin + Riemann – observed we can get the additional terms by a combination of 2 known pieces:

$$\mathcal{M}(Z \to qq)|_{m_q=0} \quad + \quad \mathcal{M}(Z \to q_1 q_2)|_{m_1 \neq m_2}$$

$\mathcal{M}(Z \to qq)|_{m_q=0}$ – was known from Bardin [41].
$\mathcal{M}(Z \to q_1 q_2)|_{m_1 \neq m_2}$ – was also known: Mann + Riemann 1982 [44]:

So we combined the two, plus a recalculation of the whole
$\to$ We had 2 independent calculations

First complete EWRC for $Z \to \bar{b}b$

30 years ago: 1985 – Akhundov Bardin Riemann August 1985 JINR-E2-85-617 + Nucl. Phys. B [45]

When LEP 1 approached with a potential to observe $Z \to bb$:

Oct 1987 – Bernabeu + Pich + Santamaria [11],
Jan 1988 – Beenakker + Hollik [12],
Jan 1988– Jegerlehner [13],
July 1988 – Diakonos + Wetzel [14]

We invented the language of ρ_Z and κ_Z for the raditive corrections

The Fortran program **ZRATE** became the first piece of the electroweak standard model library of **ZFITTER** [7, 8, 9]
The official ZFITTER Webpage: `http://sanc.jinr.ru/users/zfitter/`

20 years after: 1-loop corrections to $Z \to \bar{b}b$ in Gfitter: 2007

ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2-85-617

A.A.Akhundov*, D.Yu.Bardin, T.Riemann

ELECTROWEAK ONE-LOOP CORRECTIONS
TO THE DECAY
OF THE NEUTRAL VECTOR BOSON

Submitted to "Nuclear Physics"

* Institute of Physics of the Academy of Sciences of the Azerbaijan SSR, Baku, USSR

1985

APPENDIX

Using the expressions of ref./9/ for X, $\mathcal{F}_{1i}$, $\mathcal{F}_{2i}$ one gets according to eqs. (16) and (17) the following expressions for ρ_i and k_i in the approximation $m_f^2 \ll M_Z^2$:

$$\rho_i = 1 + \frac{\alpha}{4\pi(1-R)} \left\{ Z(-1) + Z_F(-1) - W(0) + \frac{5}{8} R(1+R) - \frac{11}{2} - \frac{9}{4(1-R)} \ln R + u_i \right\}, \quad (A.1)$$

$$k_i = 1 - \frac{\alpha}{4\pi(1-R)} \left\{ \frac{R}{1-R} \left[W(-1) - Z(-1) \right] - M(-1) + \frac{1}{2} u_i - \frac{(1-R)^2}{R} Q_i^2 \left[V_1(-M_Z^2, M_Z^2) + \frac{3}{2} \right] \right\}, \quad (A.2)$$

$$u_i = \frac{1}{2R} \left[1 - 6|Q_i|(1-R) + 12 Q_i^2 (1-R)^2 \right] \cdot \left[V_1(-M_Z^2, M_Z^2) + \frac{3}{2} \right] +$$

$$+ \left[1 - 2R - 2|Q_i|(1-R) \right] \cdot \left[V_1(-M_Z^2, M_W^2) + \frac{3}{2} \right] + 2R \left[V_2(-M_Z^2, M_W^2, M_W^2) + \frac{3}{2} \right]. \quad (A.3)$$

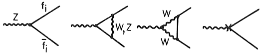


Fig.1. Born and one-loop electroweak contributions in the unitary gauge to the partial width Γ_i^{ew} of the decay $Z \to \bar{f}_i f_i$.

From Gfitter C++ source code
15 June 2008
Haller + Hoecker + Goebel

```
case kCharm:
   uff = ( 0.25/m_R*(1.0 - 6.0*TMath::Abs(Charge)*(1.0-m_R)
                     + 12.0*Charge*Charge*(1.0-m_R)*(1.0-m_R))*GetVertex().GetV1ZZ()
             + (0.5 - m_R - TMath::Abs(Charge)*(1.0-m_R))*GetVertex().GetV1ZW()
             + m_R*GetVertex().GetV2ZWW() );
   break;
```

e.3144f

The bosonic Zbb topologies

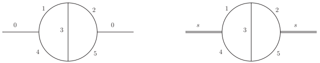


Figure 1 : The two self-energy topologies: topology 1 (a) and topology 2 (b).

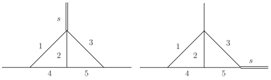


Figure 2 : Two vertex topologies: topology 3 (a) and topology 4 (b).

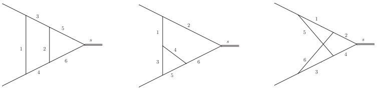


Figure 3 : Three vertex topologies: topology 5 (a), topology 6 (b) and topology 7 (c).

Mellin-Barnes (MB) integrals + sector decomposition

We aimed at using the MB-approach

Non-planar: **AMBRE 3** – Dubovyk 2015 [2, 46].

Planar: **AMBRE 2.2** – Kajda + Dubovyk 2007–2010-2015 [3, 47] [48, 49, 50]

Distinguish planar/non-planar automatically: **PlanarityTest** – Bielas + Dubovyk 2013 [4, 51]

The resulting MB-representations have at most 8-dim MB-integrals → numerically.

In general, it looks difficult to make 8-dim MB-integrals in the *Minkowskian* with Numerics.

So we decided to use both MB-methods and secdec-methods, each where it is appropriate.

Sector decomposition + Mellin-Barnes integrals

Need of working out numerical treatment of MB-integrals in the Minkowskian space-time

MBnumerics/MB – Usovitsch, Dubovyk 2016 [52] (yet unpublished, work in progress, a bit involvement of TR)

Integrations with **CUHRE** – CUBA, Hahn 2004 [53]

Crucial alternative: Sector decomposition
Fiesta 3 – Smirnov, Tentyukov, Smirnov 2008-2013 [54, 55]
SecDec 3 – Carter Heinrich Borowka 2010-2015 [56, 57]

Further, we made also some use of
NICODEMOS – A. Freitas 2014 [58, 59], a numerical approach with explicit subtractions

Electroweak bosonic 2-loop vertices

Zbb vertices

Parameters:
$s = M_Z^2$ and masses: M_Z, M_W, M_H, m_t, while $m_b = 0$
$\rightarrow$ 2-loop 3-point tensor-integrals with up to 3 different masses
$\rightarrow$ **up to 3 dimensionless scales**.

715 integrals needed for κ_b

We decided to try a brute-force approach: Perform as minimal reduction work as possible
– i.e. practically no reduction
$\rightarrow$ may be improved..
The only modification from the “naive” list was to arrange for tensor ranks to be not higher than $R = 3$.

Ayres Freitas sent us the table of integrals.

Aim: 2 independent calculations with accuracy of 8 digits each

$\rightarrow$ not yet achieved, so preliminary
We first made Euklidean kinematics: finally easy.
Proves correctness of MB-representations.
Experience:
Minkowskian: Many integrals are easy, but for several cases we got no or very bad results

$\rightarrow$ MBnumerics – See the following examples.

Six integral classes

– Integrals I(0H0W0txZ) – s, M_Z – 100 integrals; **no idea with secdec**; good mood with MB
– Integrals I(0HxWxt0Z) – s, M_W, m_t – **some IR-div are truly difficult** for both methods; finally

Few examples with AMBRE/MB/MBnumerics

1-scale *planar* integral with 2 massive lines $I_{14}(\mathbf{0H0W0txZ})$

See also: Aglietti + Bonciani 2004 [60]

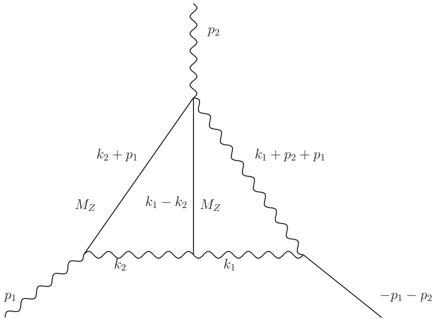


Figure 4 : The integral I_{14}^Z

1-scale *planar* integral with 2 massive lines $I_{14}(\mathbf{0H0W0txZ})$ (ii)

We have taken the integral $I_{14}(\mathrm{0H0W0txZ})$ as an example because its MB-representation consists of only one term:

$$I_{14}(\mathrm{0H0W0txZ}) =$$

$$\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-\mathbf{s}}{\mathbf{M_Z^2}}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}.$$

As an example, we consider the *scalar* case $I_{14,s}(\mathrm{0H0W0txZ})$ of the integral topology of $I_{14}(\mathrm{0H0W0txZ})$ see figure 4.
The integral is Figure 4a in [60] and the answer given there in equation (210)–(212) is:

$I_{14}(\mathrm{0H0W0txZ})$

$$I_{14}(\mathrm{0H0W0txZ}) = M_Z^2 F_0^{17} = -\frac{1}{x}\left[\zeta_2 H(0,1,x) - 2\,H(0,1,0,-1,x) + 2\,H(0,r,r,0,x)\right],$$

1-scale *planar* integral with 2 massive lines $I_{14}(\mathbf{0H0W0txZ})$ (iii)

The r indicates the appearance of

$$g(r,x) = \frac{1}{\sqrt{x(4-x)}}$$
$$x = -\frac{s}{M_Z^2} \to -1 - i\epsilon.$$

We have determined the HPLs and GPLs at $s = -\frac{s}{M_Z^2} \to -1 - i\epsilon$ with **HPL4num_v104.m** (Riemann 2004 [61]):

$$H(0,1,-1-i\,\epsilon) = \mathrm{Li}_2(-1-i\epsilon)$$
$$H(0,1,0,-1,-1-i\,\epsilon) = \frac{3}{40}\zeta_2^2$$

1-scale *planar* integral with 2 massive lines $I_{14}(\mathbf{0H0W0txZ})$ (iv)

$$
\begin{aligned}
H(0,r,r,0,-1-i\,\epsilon) &= \pi^4/50 + 2\ln[1/2+\sqrt{5}/2]^4 + \ln[1/2+\sqrt{5}/2]^2\ln[3/2+\sqrt{5}/2]^2 \\
&\quad -1/24\ln[3/2+\sqrt{5}/2]^3\ln[2889+1292\sqrt{5}] \\
&\quad +\Big[-(4/3)\pi\ln[1/2+\sqrt{5}/2]^3 + 2\pi\ln[1/2+\sqrt{5}/2]^2\ln[3/2+\sqrt{5}/2] \\
&\quad +1/6\pi\ln[1/2(7-3\sqrt{5})]\ln[2/(3+\sqrt{5})]^2 + 2/5\pi\zeta_3\Big]\,i. \\
&= 1.9481818206800487447 + 1.5105492544652315709\,i
\end{aligned}
$$

Table 1 : Minkowskian numerics for the scalar integral I^{Z}_{14a} at $s = M_Z^2$. The sign difference is due to different metrics used in [60].

Program	Real Part	Imaginary Part
MBintegrate,Cuhre,F77	-2.137588	-3.021098
MBnumerics,Cuhre,Math	-2.1375883865794 300s	-3.0210985089304
Math, HPL, GPL	2.1375883865794979282441073006 7	3.021098508930463141762

From Gluza at LL2016: MB Tools at HEPforge

Mellin-Barnes representations in HEP - developments

https : //mbtools.hepforge.org/

- "Automatized analytic continuation of Mellin-Barnes integrals", Michał Czakon, CPC, 2006 $\longrightarrow$ MB.m, MBasymptotics.m
- "On the Resolution of Singularities of Multiple Mellin-Barnes Integrals", A.V. Smirnov, V.A. Smirnov $\longrightarrow$ MBresolve.m
- "AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals", JG, Krzysztof Kajda, Tord Riemann, CPC, 2007 $\longrightarrow$ AMBRE.m
- "Some Remarks on Non-planar Feynman Diagrams", K. Bielas, I. Dubovyk, JG, T. Riemann, APPB, 2013 $\longrightarrow$ PlanarityTest.m
- $\longrightarrow$ barnesroutines.m : a tool by David Kosower

MB Tools Webpage: `https://mbtools.hepforge.org/`
The HepForge projects: `https://www.hepforge.org/projects`

From J. Gluza at LL2016: AMBRE + numerics

Planar integrals: AMBRE 2 – Make the loop representations one after the other (loop-by-loop)
Non-planar integrals: AMBRE 3 – Make the multi-loop representation globally
Need a tool for calculations with Minkowskian kinematics

Types of numerical improvements

BASIC METHODS

I. Specific integration methods for oscillating integrands
II. Contour deformations
III. Contour shifts

- I. and II. limited, though can be quite effective for 1-dim cases (reduction of n-dim MB "bad" integrals into 1-dim cases?)
- III. **is new, effective for n-dim MB integrations.**

Auxiliary tricks.

(a) Mappings ($\rightarrow$ MB.m, here);
(b) Variables transformations, e.g. $\{z_1, z_2\} \rightarrow \{z_1 - z_2, z_2\}$ ($\rightarrow$ here);
(c) Contour rotations ($\rightarrow$ A. Freitas, here);
(d) Stationary phase method; steepest decent ($\rightarrow$ Phd by Z. Peng [D. Kosower], here)

15/30

Comment: contour deformations

For 1-dim. MB-integrals: Many – sometimes – mighty tools are available
Among them:

Sophisticated contour deformations

Not yet explored

Contour deformations for n-dimensional MB-integrals

Difficult, due to potential crossing of poles.
Global deformations – equal for all n variables – are possible but not very flexible: A. Freitas 2010 [62]

Comment: contour shifts

For 100 integrals: The on-shell problem

MB-integrands depend exclusively on

$$(-\frac{s}{M_Z^2})^z = (-1)^z$$

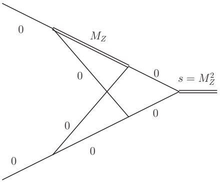
Contour deformation does not help, because of the (-1).

Contour shifts are a mighty tool.
Contour shifts . . . Any time when a pole is crossed by a shift . . .
introduce MB-integrals of lower dimension
Repeated shifts $\rightarrow$ may arrive at **many** 1-dimensional integrals plus numerically small "rests" like $(0 \pm 20) \times 10^{-12}$ or so

Mathematica program **MBsums** (Ochman, Riemann 2015) [63]

Derive single or multiple sums via Cauchy theorem and hope for summing them into analytical answer by another software
Look at **MBsums + SIGMA etc**
Not explored

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$



See also: Fleischer + Kotikov + Veretin 1998 [64]: integral N_3
– plus private commun. Kotikov (1 integral is wrong in [64])
– plus help in summing up by Ablinger + Bering + Blümlein + Schneider (priv. commun.)
The integral $I_{15}(\mathrm{0H0W0txZ})$ may serve as an instructive example. It is (up to some sign convention) the integral N_3 of [64]:

$$
\begin{aligned}
-N_3 &= I_{15}(\mathrm{0H0W0txZ}) \\
&= \frac{e^{2\gamma_E \epsilon}}{\pi^d} \int \frac{d^d k_1 d^d k_2}{D[k_1,0]D[k_1-k_2,0]D[k_2,0]D[k_2+p_2,0]D[k_1+k_2+p_2,0]D[k_1-k_2+p_1,M_Z]}
\end{aligned}
$$

The MB-representation is derived with calls to **PlanarityTest** and **AMBRE 3**.

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$ (ii)

The U- and F-polynomials are

```
Upoly1 = x[1]x[2] + x[1]x[3] + x[2]x[3] + x[1]x[4] + x[3]x[4] + x[1]x[5] + x[2]x[5]
               + x[4]x[5] + x[2]x[6] + x[3]x[6] + x[4]x[6] + x[5]x[6]

Fpoly2 = Upoly1 MZ^2 x[4] - s x[1]x[4]x[5] - s x[1]x[2]x[6] - s x[1]x[3]x[6]
                         - s x[2]x[3]x[6] - s x[1]x[4]x[6] - s x[1]x[5]x[6]
```

A naive MB-representation would become high-dimensional, and it would be plagued by the occurence of terms containing the ill-defined expression $\Gamma[0]$.
A dedicated introduction of Cheng-Wu variables leads to the following integrands for the x-integrations **See talk by J. Gluza at LL2016**:

```
Upoly2 = v[1] + v[2] v[3]

Fpoly2 = + MZ^2 Upoly2 C[2]v[3] - s A[1]A[2]v[1]^2 - s A[2]B[1]C[1]v[1]v[2]v[3]
                                                   - s A[1]B[2]C[2]v[1]v[2]v[3]
```

The x-integrations over `v[i]` can be easily performed, and we remain, from the four additive terms in `Fpoly2`, with a three-dimensional MB-integral.

```
N3 ~  (-s)^(-2-2eps)  Gamma[-eps]
      (-(s/MZ^2))^-z2 Gamma[-eps-z1] Gamma[-z1] Gamma[-eps-z2] Gamma[-z2]
       Gamma[-1-2eps-z1-z3] Gamma[-1-2eps-z2-z3] Gamma[-1-2eps-z1-z2-z3] Gamma[-z3]
       Gamma[1+z3]^2 Gamma[1+z1+z3] Gamma[2+2eps+z1+z2+z3]) / (Gamma[-2eps-z1]
       Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[-2eps-z1-z2])
```

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$ (iii)

When continuing in `eps` with **MB.m**, we derive a sum of two MB-representations, for vanishing, but finite ϵ:

```
N3 ~
{
MBint[ ((-s)^(-2-2eps) Gamma[-2eps] Gamma[-eps]
(-(s/MZ^2))^-z2 Gamma[-eps-z2] Gamma[-z2]^2 Gamma[1+z2] Gamma[-1-2eps-z2-z3]
Gamma[-z3] Gamma[1+z3] Gamma[1+eps+z3] Gamma[1+2eps+z3])
 / (Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[1-z2+z3]),
{{eps->0},{z2->-0.42644,z3->-0.826119}} ],

MBint[ ((-s)^(-2-2eps) Gamma[-eps]
 (-(s/MZ^2))^-z2 Gamma[-eps-z1] Gamma[-z1] Gamma[-eps-z2] Gamma[-z2]
 Gamma[-1-2eps-z1-z3] Gamma[-1-2eps-z2-z3] Gamma[-1-2eps-z1-z2-z3] Gamma[-z3]
 Gamma[1+z3]^2 Gamma[1+z1+z3]  Gamma[2+2eps+z1+z2+z3])
 / (Gamma[-2eps-z1] Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[-2 eps-z1-z2]),
{{eps->0},{z1->-0.268281,z2->-1.00065,z3->-0.171895}} ]
}
```

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$, cont'nd

Applying **MB + MBnumerics**, we get:

$$N_3 = \frac{1}{\epsilon^2}(1.23370055013617 - 6.20475892887384 \times 10^{-13} I) + \frac{1}{\epsilon}(2.8902545096591976 + 3.875784585038738 I) + (-0.778599608 3247692 - 4.1235126 00516016 I)$$

We have no number from **SecDec** or **Fiesta**, so we looked for an alternative for a comparison. In fact, this was the first promising result.

Non-planar one-scale integral $I_{15}(\mathrm{0H0W0txZ})$, cont'nd

Integral N3 according to equation (D.11) of Fleischer 1998 [64] in terms of harmonic sums, where $z = s/M_Z^2 = 1 + i\epsilon$:

$$
\begin{aligned}
N_3 \quad = \quad & \frac{1}{s^2} \sum_{n=1}^{\infty} (-z)^n \Bigg\{ \frac{1}{\epsilon^2} \left[-\frac{1}{2}\zeta_2 + K_2(n-1) \right] \\
& + \frac{1}{\epsilon} \Big[-\frac{1}{2}\zeta_3 - 2\zeta_2 S_1(n-1) + 2S_3(n-1) - 2K_3(n-1) + 4S_1(n-1)K_2(n-1) \\
& + (\zeta_2 - S_2(n-1))\ln(-z) \Big] \\
& + \Big[-\zeta_4 - 2\zeta_3 S_1(n-1) - 7\zeta_2 S_2(n-1) - 4\zeta_2 S_1(n-1)^2 + 7\zeta_2 K_2(n-1) - \frac{7}{2}S_4(n-1) \\
& + \frac{7}{2}S_2(n-1)^2 + 6S_1(n-1)S_3(n-1) + 2S_{13}(n-1) + 8K_4(n-1) \\
& - 8S_1(n-1)K_3(n-1) + 8S_1(n-1)^2 K_2(n-1) \\
& + \Big(\zeta_3 + 4\zeta_2 S_1(n-1) - S_{12}(n-1) - 3S_1(n-1)S_2(n-1) - 4K_3(n-1)\Big)\ln(-z) \\
& + \Big(-\zeta_2 + \frac{1}{2}S_2(n-1) + K_2(n-1)\Big)\ln^2(-z) \Big] \Bigg\}
\end{aligned}
$$

$S_a(n) = \sum_{j=1}^{n} 1/j^a$, $K_a(n) = -\sum_{j=1}^{n}(-1)^j/j^a$, $S_{ab}(n) = \sum_{j=1}^{n} S_b(j-1)/j^a$ [65, 66, 67]

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$, cont'd

The sum N_3 converges both in the Euklidean and the Minkowskian kinematics, but very slowly, so that it would need many terms in order to get our accuracy goal of eight digits.

N3 evaluated with 200 terms

```
time = 4.060519 sec        for 200 terms of the sum

N3 = (0.4 + 4 x I)

   + 1/eps x (2.8 + 3.87 x I) + 1 /eps^2 x (1.23 + 0 x I)
```

The agreement, with several 1000 terms (few hours running time), is suffiently good in order to see that the results from the numerical MB-approach appproach are reasonable.

Non-planar one-scale integral $I_{15}(0H0W0txZ)$, cont'd

One may improve this.
In fact, in appendix E of Fleischer 1998 [64], the necessary sums are performed.
We derive:[1]

$$\phi(z) = \frac{1}{2} S_{1,2}(z^2) - S_{1,2}(z) - S_{1,2}(-z) + \ln(1-z)\mathrm{Li}_2(-z)$$

and

$$-\frac{s^2 z}{1+z} N_3 = \frac{1}{\epsilon^2}\left[-\frac{\zeta_2}{2} - \mathrm{Li}_2(z)\right] + \frac{1}{\epsilon}\left[-\frac{1}{2}\zeta_3 - 2\zeta_2 \mathrm{Li}_2(-z) + 2\mathrm{Li}_3(-z) + 2\mathrm{Li}_3(z) + 4\left[\phi(-z) - S_{1,2}(z) - \mathrm{Li}_3(-z)\right] + (\zeta_2 - \mathrm{Li}_2(-z))\ln(-z)\right] - \frac{s^2 z}{1+z} N_3^{\mathrm{const}}$$

[1] In [64], the overall sign of (E.7) is wrong, and in the r.h.s. of (E.36) one has to replace $S_{1,3}$ under the integral by $S_{1,2}$ and to change the sign of $2\ln(1-z)$. We thank T. Kotikov for clarifying this.

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$, cont'd 2

Further,

$$
\begin{aligned}
\frac{s^2(-z)}{1+z} N_3^{\text{const}} \quad = \quad & -\zeta_4 + 2\zeta_3 \ln(1+z) - 7\zeta_2 \text{Li}_2(-z) - 4\,\zeta_2 \left[-\ln(1-r)^2 + \text{Li}_2(-z) \right] \\
& +7\,\zeta_2 \text{Li}_2(z) - \frac{7}{2}\text{Li}_4(-z) + \frac{7}{2} \left[\text{Li}_2(-z)^2 + \text{Li}_4(-z) - S_{2,2}(-z) \right] \\
& + 6\left[(\text{Li}_4(-z) - \ln(1+z)\text{Li}_3(-z) - \frac{1}{2}\text{Li}_2(-z)^2 + S_{2,2}(-z) \right] \\
& +2 \left[-\ln(1+z)\text{Li}_3(-z) - \frac{1}{2}\text{Li}_2(-z)^2 \right] + 8\text{Li}_4(z) \\
& - 8 \left[-\text{Li}_4(z) - S_{2,2}(z) - \int_0^{-z} \frac{dt}{1-t}\text{Li}_3(-t) \right] \\
& +8\Big[-\text{Li}_4(z) - \frac{1}{2}S_{2,2}(z) - 2S_{1,3}(z) + \phi(-z)[\ln(-z) - 2\ln(1+z)] \\
& - \int_0^{-z} \frac{dt}{1-t}[(2\ln(1-t) - \ln(t))\text{Li}_2(-t) + 2\text{Li}_3(-t) + 2S_{1,2}(-t)] \Big] \\
& +\Big[\zeta_3 - 4\zeta_2 \ln(1+z) + \left[\ln(1+z)\text{Li}_2(-z) + S_{1,2}(-z) \right] \\
& - 3\left(\text{Li}_3(-z) + (-\ln(1+z)\text{Li}_2(-z) - S_{1,2}(-z)) \right) - 4\text{Li}_3(z) \Big] \ln(-z) \\
& +\left[-\zeta_2 + \frac{1}{2}\text{Li}_2(-z) + \text{Li}_2(z) \right] \ln^2(-z)
\end{aligned}
$$

Non-planar one-scale integral $I_{15}(\mathbf{0H0W0txZ})$, cont'd 3

We calculated the sums for N_3 with two independent expressions, one derived with the aid of appendix of Fleischer:1998 [64], the other with the aid of Schneider et al.
The numerical calculation of the HPLs was done using the Matematica package **HPL4num.m** [61] based on appendix B of [5].
The results from Ablinger, Blümlein, Schneider (priv. commun.):

$$N_{3,-2}(1+i\,\epsilon) = \frac{3\zeta_2}{4}$$

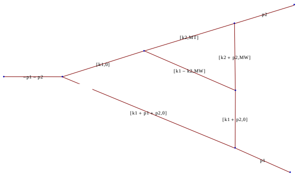
$$N_{3,-1}(1+i\,\epsilon) = -3\ln(2)\zeta_2 + \frac{21}{4}\zeta_3 + \frac{3}{4}\pi\zeta_2 I$$

We confirm with $N_3(1+i\,\epsilon)$ the 9 digits accuracy obtained with **MBnumerics**.

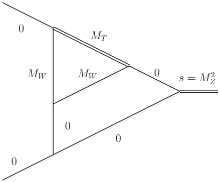
$$\begin{aligned} N_{3,const}(1+i\,\epsilon) &= 24\text{Li}_4(1/2) + \ln^4(2) - \frac{351}{80}\zeta_2^2 - 3\ln(2)2\pi\zeta_2 I - \frac{63}{16}\pi\zeta_3 I \\ &= 0.7785996089876216856345280569\,0 \\ &\quad - 4.1235125933364227264810336538\,3\,I \end{aligned}$$

One of the most difficult IR-divergent integrals with 2 scales

```
d = {PR[k1, 0, n1] PR[k1 - k2, MW, n2] PR[k2, MT, n3] PR[k1 + p2, 0, n4]
     PR[k2 + p2, MW, n5] PR[k1 + p1 + p2, 0, n6]}
```



=



$(-s)^{-2-2\epsilon}(-\frac{\mathbf{s}}{\mathbf{MT^2}})^{-z2}(-\frac{\mathbf{s}}{\mathbf{MW^2}})^{-z1}$
$\Gamma[-z1]\Gamma[-1-2\epsilon-z1-z2]\Gamma[-\epsilon-z1-z2]\Gamma[-z2]\Gamma[2+2\epsilon+z1+z2]\Gamma[-2\epsilon-z1-2z2-z3]\Gamma[-1-2\epsilon-z1-z2-z3]\Gamma[-\epsilon-z1-z2-z3]\Gamma[-z3]\Gamma[1+z2+z3]\Gamma[1+\epsilon+z1+z2+z3]$
/
$(\Gamma[-3\epsilon-z1-z2]\Gamma[1-2\epsilon-z1-z2]\Gamma[1-z3]\Gamma[-2\epsilon-2\mathbf{z1}-2\mathbf{z2}-\mathbf{z3}])$

One of the most difficult IR-divergent integrals with 2 scales, cont'd

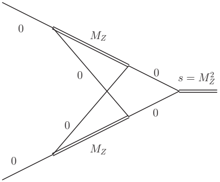
```
MBnumerics.m     2016-04-21     Johann Usovitsch
=
    1.541402128186602 + 0.248804198197504*I
+
   0.12361459942846659 - 1.0610332704387688 *I * eps^-1
+
  -0.33773737955057970 + 3.6*10^-17*I *eps^-2
```

Time needed **43 min.**

```
SecDec
=
  1.541 + 0.2487*I
+
  0.123615 - 1.06103*I *eps^-1
+
  -0.3377373796 - 5*10^-10*I*eps^-2
```

Time needed **24 hours**

Non-planar one-scale integral with 2 massive lines $I_{29}(\mathrm{0H0W0txZ})$



Scalar one-scale non-planar diagram with two massive lines I_{29} of the class of 100.
See also: Aglietti + Bonciani + Grassi + Remiddi 2007 [68]
As a check on the numerical reliability of our algorithms we calculated the scalar version of I^Z_{100}, let us call it $I^Z_{100,s}$. The MB-representation is four-fold:

```
MBint[((-(s/MZ^2))^-z2 Gamma[-z1]^2 Gamma[-z3]^2 Gamma[-z2 +
      z3]^2 Gamma[-1 - z2 - z4] Gamma[-1 - z1 - z2 - z4] Gamma[-1 -
      z1 - z2 + z3 - z4] Gamma[-z4] Gamma[1 + z4] Gamma[
     1 + z1 + z4] Gamma[2 + z1 + z2 + z4] Gamma[
     1 + z2 - z3 + z4])/(s^2 Gamma[-z2]^2 Gamma[-z1 - z3] Gamma[-z1 -
      z2 + z3]), eps -> 0,
    z1 -> -0.368723, z2 -> -0.859326, z3 -> -0.433081, z4 -> -0.35128],
```

where we define

```
MBint[A,{eps->0},{z1->z1_0, z2->z2_0,..., zn->zn_0}]
```

$$= \frac{const}{(2\pi i)^n} \lim_{\epsilon\to 0} \prod_{i=1,n} \int_{zi_0-i\infty}^{zi_0+i\infty} dzi\, A. \qquad (1)$$

Non-planar one-scale integral with 2 massive lines $I_{29}(\mathbf{0H0W0txZ})$, cont'd

Our numerical answer with use of MBnumerics is for $s = M_Z^2 + i\,\epsilon$:

```
I_{29}(0H0W0txZ)

  = -1.2116223301        + 4.9954503192        I   MBnumerics 2016-03-23
  = -1.211622330156316914 + 4.99545031920035447 I   Aglietti 2007, x16
```

Integral $I_{29}(\mathrm{0H0W0txZ})$ is $F_1(x;\epsilon)$, defined in equation (28), one of the three master integrals calculated in Aglietti+ Bonciani+ Grassi+ Remiddi 2007 [68] for the topology shown in Figure 1 there. In section 7.4, equation (280) its value is given numerically for $x = -1$,

$$a_0 = -0.07572639563476980715 + i\,0.3122156449500221544. \quad (2)$$

The normalizations differ by a factor of 16. The results agree with 11 digits, confirming the estimate of accuracy for $I_{29}(\mathrm{0H0W0txZ})$.

The dessert

The preliminary bosonic 2-loop contributions to κ_b

$$\Delta\kappa_{\mathrm{b}}^{(\alpha^2,\mathrm{bos})} = -1.0276 \times 10^{-4}$$

This amounts to about $\frac{1}{4}$ of the leptonic corrections to κ_b and $\sin^2\theta_{\mathrm{eff}}^{\mathrm{b}}$ and A_{b}

From experiment: $\Delta\sin^2\theta_{\mathrm{eff}}^{\mathrm{b}} \sim \pm 0.016$)
From experiment: $\Delta A_{\mathrm{b}} \sim \pm 0.016$).

In fact, the leptonic corrections are expected to be bigger than the bosonic ones.
It also amounts the relative contribution to the weak mixing angle of Γ_{Zbb}.

Number preliminary:
We have for several integrals only one calculation
We like to have two independent values for all of them.

Summary

- New automated tools: **AMBRE 3** + **MBnumerics** + **MBsums**
  See also talk by J. Gluza, this conference.
  Numerical approach to **planar + non-planar 2-loop vertices** in the **Minkowskian** kinematics:
  Mellin-Barnes integrals plus Sector Decomposition a la **SecDec** + **Fiesta**.
- Application to $Z \to \bar{b}b$.
- **No reduction** of integrals to masters.
- The preliminary result is: $\Delta\kappa_{\rm b}^{(\alpha^2,{\rm bos})} = -1.0276 \times 10^{-4}$
  This amounts to about $\frac{1}{4}$ of the leptonic corrections to $\sin^2\theta^{\rm b}_{\rm eff}$ and $A_{\rm b}$.
- The **715 integrals** have not all been duplicated so far.
  This is work in progress.
- Next – and last – steps: Γ_{Zbb}, $\Gamma_{Z_{tot}}$