Absence of Three-Loop Four-Point Ultraviolet Divergences in $N = 4$ Supergravity

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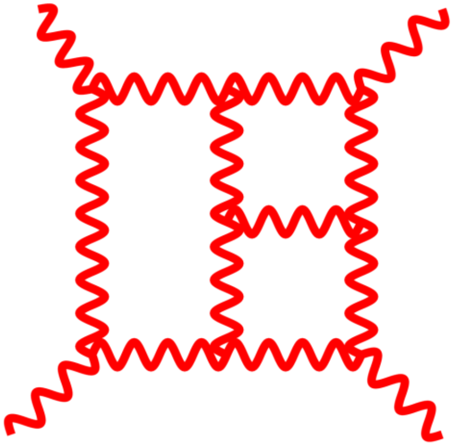
Zvi Bern, UCLA

With Tristan Dennen, Scott Davies and Yu-tin Huang

arXiv:1202.3423



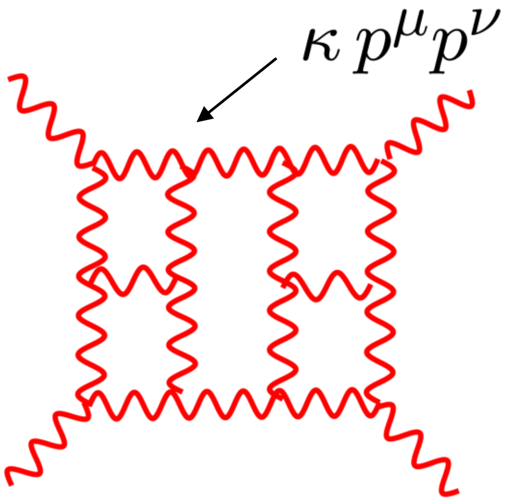
Outline

1. Overview of counterterms in pure supergravity theories.
2. Review of duality between color and kinematics.

   See also talks from Broedel, O'Connell, Dixon and Johansson
3. Gravity amplitudes as double copies of gauge-theory ones.
4. Generalized gauge invariance.
5. Very neat one-loop example in $N = 4$ supergravity.
6. Finiteness of UV $N = 4$ supergravity at three loops and four points.
7. Future directions and consequences.

Power Counting at High-Loop Orders

$$\kappa = \sqrt{32\pi G_N}$$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

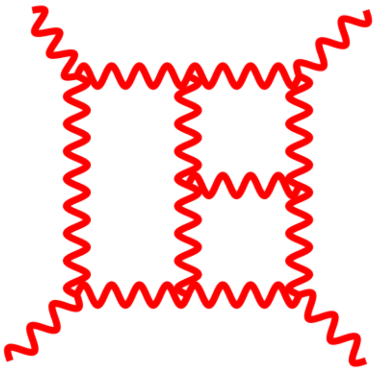
Non-renormalizable by power counting.

Recently, people have been looking mostly at N = 8 supergravity, mainly because that is where explicit calculations have been challenging the accepted status. **See Lance's talk**

R^4 Counterterm in Supergravity

Understanding from the mid 1980s was that R^4 was a valid counterterm and that *all* supergravity theories likely diverge at three loops.

Grisaru; Tomboulis; Deser, Kay, Stelle; Ferrara, Zumino; Green, Schwarz, Brink; Howe and Stelle; Marcus and Sagnotti, etc



$\frac{1}{\epsilon} R^4$ **was expected counterterm**

What we now *know*:

2007: $N = 8$ sugra, UV finite for $D < 6$ at 3 loops.

ZB, Carrasco, Dixon, Kosower, Johansson, Roiban

2010: $N =5, 6$ sugra R^4 not valid counterterm in $D = 4$

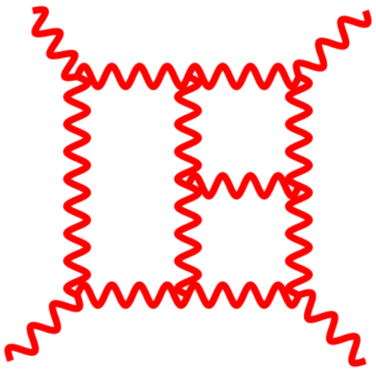
Bossard, Howe and Stelle

2012: $N = 4$ sugra, UV finite in $D = 4$ at 3 loops.

ZB, Davies, Dennen, Huang

$N = 4$ supergravity

$N = 4$ sugra at 3 loops ideal test case.



Recent consensus had it that a valid R^4 counterterm existed.

Bossard, Howe, Stelle (2010); Bossard, Howe, Stelle, Vanhove (2011)

- Need to maximize the susy for simplicity.
- Need to minimize the susy to lower the loop order where we might find potential divergences.
- BCJ duality gives us the power to do the calculation.

Recent Status of $N = 8$ counterterms

Two Recent Approaches:

- **Use susy identities and $E_{(7)7}$ soft scalar identities to rule out counterterms .**

  Bossard, Howe, Stelle; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger

- **Use Berkovits pure spinor string formalism and related first quantized formalism.** Green, Russo, Vanhove; Bjornsson and Green; Vanhove. Bjornsson

For $N = 8$ sugra in $D = 4$ *all* candidate counterterms ruled out until 7 loops.

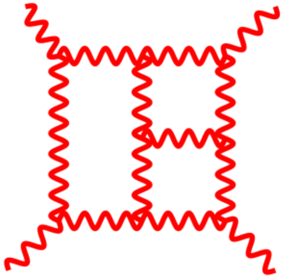
- **Candidate 7 loop superspace volume counterterm vanishes.**
- **But D^8R^4 apparently available at 7 loops (1/8 BPS) under all known symmetries.** Bossard, Howe, Stelle and Vanhove
- **At five loops expect divergence in 24/5 dimensions, worse than $N = 4$ sYM.**

Recent Status of *N* = 4 supergravity counterterms

Bossard, Howe and Stelle (2010)
Bossard, Howe, Stelle and Vanhove (2011)

Similar analysis for *N* = 4 supergravity in *D* = 4 shows:

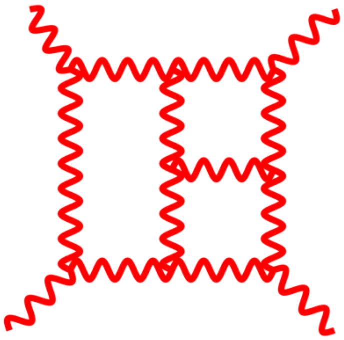
- **R^4 superspace volume counterterm vanishes.**
- **But R^4 apparently available at 3 loops (1/4 BPS) under all known symmetries.**
- **No nonrenormalization theorems known.**



$$\frac{c}{\epsilon} R^4$$

What is the coefficient of potential 3-loop counterterm?

N = 4 supergravity



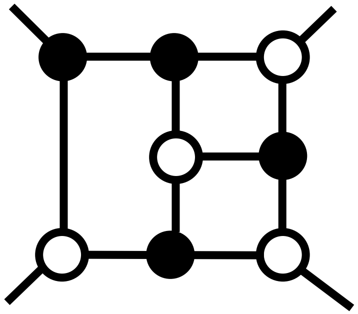
A no lose calculation:
Either we find first example of a divergence or once again we show an expected divergence is not present.

$\frac{c}{\epsilon} R^4$ **Is the constant vanishing or nonvanishing?**

One year everyone believed that supergravity was finite. The next year the fashion changed and everyone said that supergravity was bound to have divergences even though none had actually been found. — ***Stephen Hawking, 1994***

To this day, no one has ever proven that *any* pure supergravity diverges in $D = 4$.

Hidden UV cancellations in $N = 4$ supergravity?



- **Maximal cuts suggest we have a UV divergence**
- **Similar to prediction of 7 loop divergence in $N = 8$ supergravity**

Bjornsson and Green; Vanhove

See discussion in Johansson's talk

$$\int (d^4 l)^3 \frac{l^8}{(l^2)^{10}} \sim \log(\Lambda)$$

At least log divergent (actually linear divergence)

But this is too quick:

- **In $D = 4$ no one loop subamplitude can have a divergence. These cancellations unaccounted in above.**
- **There exists a web of cancellations in all 1 and 2 loop subamplitudes. Nontrivial cancellations between cuts!**
- **If loop momenta are cancelling in subdiagrams you can lower the overall power count!**

Need the full calculation to find out what really happens!

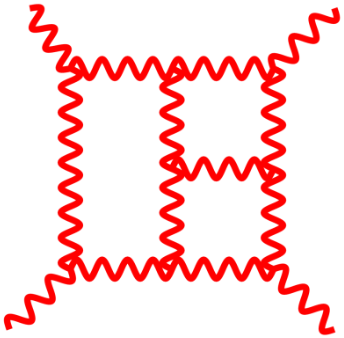
Heavy Artillery for Multiloop Gravity

Need ironclad method and also need complete amplitudes.

- **Duality between color and kinematics and gravity double copy.**

  ZB, Carrasco, Johansson
- **Unitarity method to guarantee constructions OK.**

  ZB, Dixon, Dunbar, Kosower



Duality Between Color and Kinematics

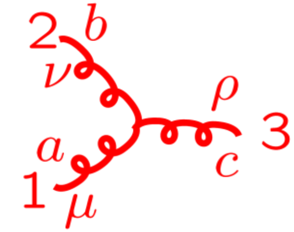
ZB, Carrasco, Johansson (BCJ)

coupling constant

color factor

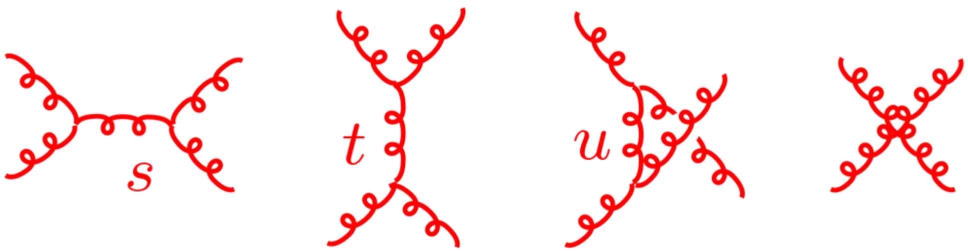
momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1a_2b}f^{ba_4a_3} + f^{a_4a_2b}f^{ba_3a_1} + f^{a_4a_1b}f^{ba_2a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1+k_2)^2$ $\quad u = (k_1+k_3)^2$

$t = (k_1+k_4)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

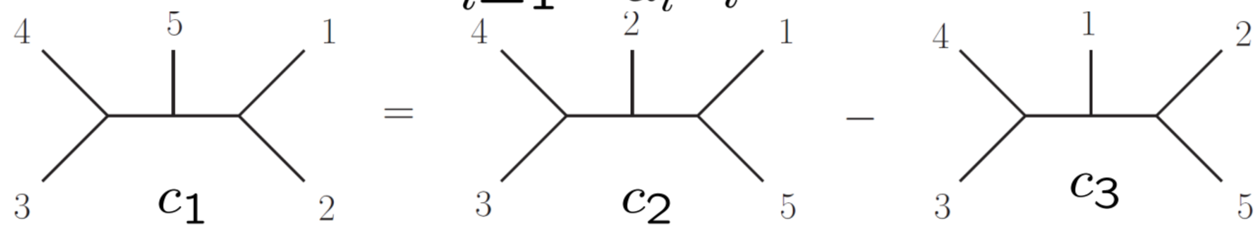
Consider five-point tree amplitude: BCJ

color factor

kinematic numerator factor

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i \, n_i}{\prod_{\alpha_i} p_i^2}$$

Feynman propagators



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 - c_3 = 0 \Leftrightarrow n_1 - n_2 - n_3 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du ,Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

kinematic numerator

color factor

gauge theory:

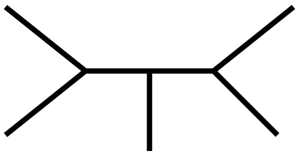
$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n)=\sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

sum over diagrams with only 3 vertices

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity:

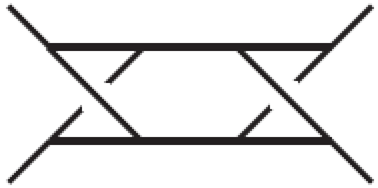
$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n)=\sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

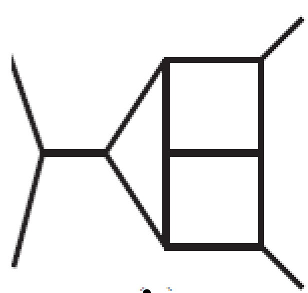
Gravity numerators are a double copy of gauge-theory ones!

This works for ordinary Einstein gravity and susy versions!

Loop-Level BCJ Conjecture

ZB, Carrasco, Johansson





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

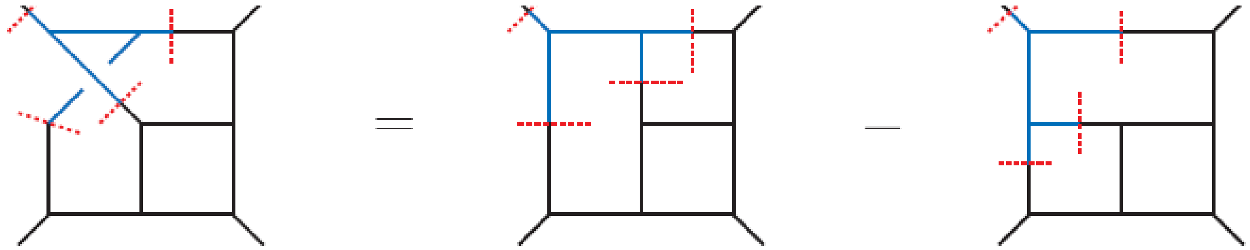
gravity

symmetry factor

- Loop-level conjecture is identical to tree-level one except for symmetry factors and loop integration.
- Double copy works if numerator satisfies duality.

Nonplanar from Planar

BCJ



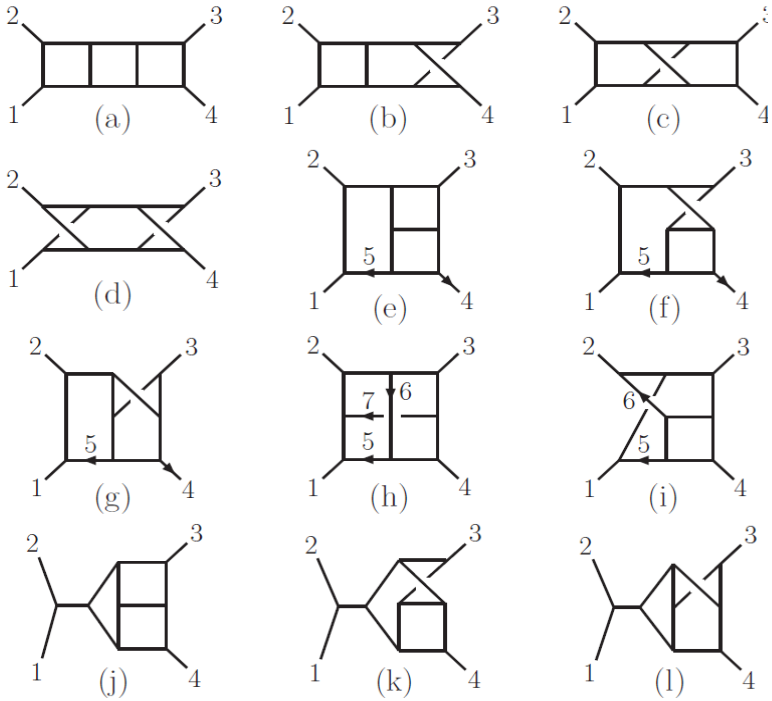
$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

Planar determines nonplanar

- **We can carry advances from planar sector to the nonplanar sector.**
- **Yangian symmetry must have consequences at in nonplanar sector.**
- **Only at level of the integrands, so far, but bodes well for the future.**

BCJ Representation

ZB, Carrasco, Johansson (2010)



Three-loop four-point $\mathcal{N} = 4$ super-Yang-Mills integrand.

This representation of amplitude has manifest duality between color and kinematics.

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2 \big)/3$
(h)	$\big(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u)$ $+ t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2 \big)/3$
(i)	$\big(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t)$ $+ t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2 \big)/3$
(j)-(l)	$s(t-u)/3$

One diagram to rule them all

ZB, Carrasco, Johansson (2010)

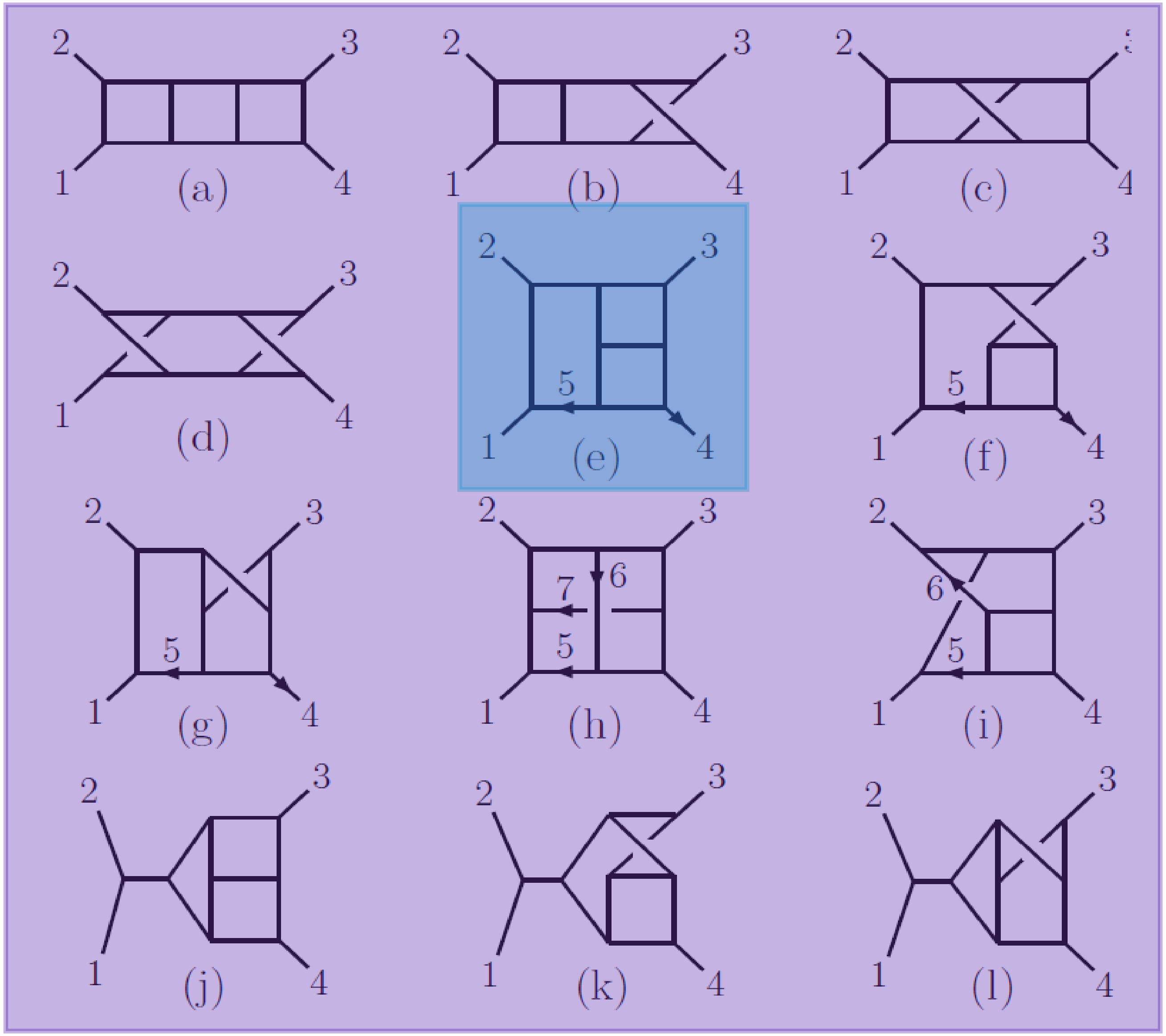


Diagram (e) is the master diagram.

Determine the master integrand in proper form and duality gives all others.

One diagram to rule them all

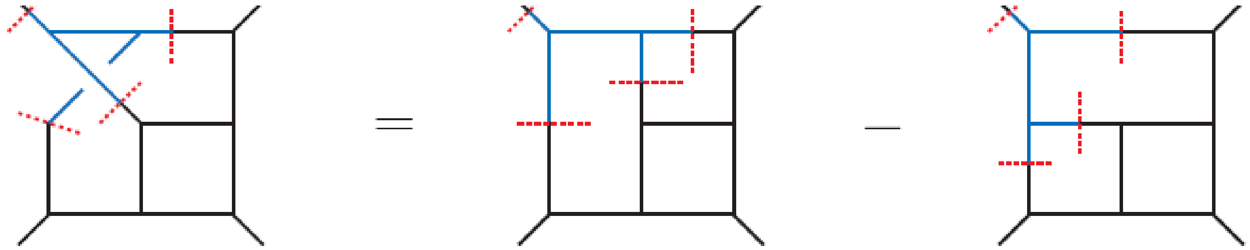
$$
\begin{aligned}
N^{(a)} &= N^{(b)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(b)} &= N^{(d)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(c)} &= N^{(a)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,\\
N^{(d)} &= N^{(h)}(k_3,k_1,k_2,l_7,l_6,k_{1,3}-l_5+l_6-l_7)+N^{(h)}(k_3,k_2,k_1,l_7,l_6,k_{2,3}+l_5-l_7)\,,\\
N^{(f)} &= \boxed{N^{(e)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,}\\
N^{(g)} &= \boxed{N^{(e)}(k_1,k_2,k_3,l_5,l_6,l_7)\,,}\\
N^{(h)} &= -N^{(g)}(k_1,k_2,k_3,l_5,l_6,k_{1,2}-l_5-l_7)-N^{(i)}(k_4,k_3,k_2,l_6-l_5,l_5-l_6+l_7-k_{1,2},l_6)\,,\\
N^{(i)} &= N^{(e)}(k_1,k_2,k_3,l_5,l_7,l_6)-N^{(e)}(k_3,k_2,k_1,-k_4-l_5-l_6,-l_6-l_7,l_6)\,,\\
N^{(j)} &= N^{(e)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(e)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(k)} &= N^{(f)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(f)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,\\
N^{(l)} &= N^{(g)}(k_1,k_2,k_3,l_5,l_6,l_7)-N^{(g)}(k_2,k_1,k_3,l_5,l_6,l_7)\,,
\end{aligned}
$$

triangle subdiagrams vanish in *N* = 4 sYM

All numerators solved in terms of numerator (e)

Gravity integrands are trivial!

BCJ



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

If you have a set of duality satisfying numerators.
To get:

gauge theory ⟶ gravity theory

simply take

color factor ⟶ kinematic numerator

Gravity integrands are free!

Gravity From Gauge Theory

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\rm loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

	n		$\tilde{n}$
N **= 8 sugra:**	**(*N* = 4 sYM)**	$\times$	**(*N* = 4 sYM)**
N **= 6 sugra:**	**(*N* = 4 sYM)**	$\times$	**(*N* = 2 sYM)**
N **= 4 sugra:**	**(*N* = 4 sYM)**	$\times$	**(*N* = 0 sYM)**
N **= 0 sugra:**	**(*N* = 0 sYM)**	$\times$	**(*N* = 0 sYM)**

N **= 0 sugra: graviton + antisym tensor + dilaton**

In this talk we discuss *N* = 4 supergravity

Generalized Gauge Invariance

BCJ
Bern, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{m-2+2L}}\mathcal{A}_m^{\text{loop}} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

$$(c_\alpha + c_\beta + c_\gamma) f(p_i) = 0$$

Above is just a definition of generalized gauge invariance

gravity

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory.**
- **Double copy works even if only one of the two copies has duality manifest.**

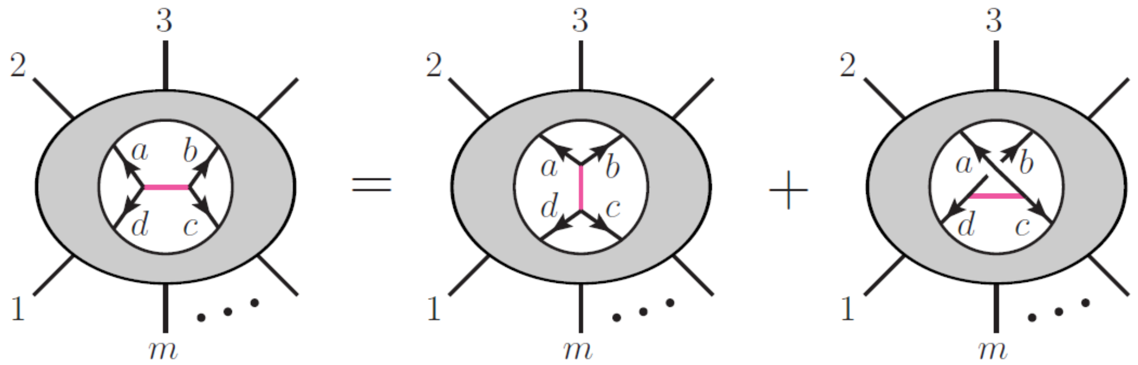
Gravity Generalized Gauge Invariance

Key point: **Only one copy needs to satisfy BCJ duality. Second copy can be *any* valid representation.**

Key trick: **Choose second copy representation to make calculation as simple as possible.**

Choose representations so that many diagrams vanish!

Generalized Gauge Invariance



think of these
as color diagrams

Replace left color factor with other two.

$$\frac{(-i)^L}{g^{m-2+2L}} \mathcal{A}_m^{\rm loop} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

Color factor eliminated → Numerator factor vanishes

Any single diagram can be set to zero this way.

In general, all but a small fraction of diagrams can be set to zero.

Implication for Gravity Double Copy

if this numerator vanishes

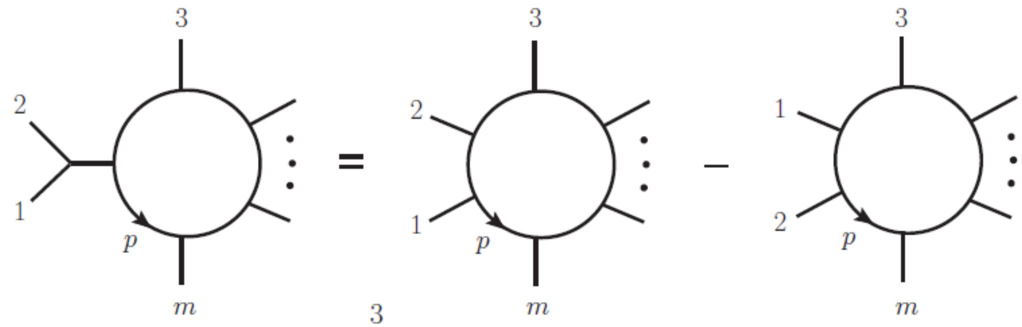
this numerator is irrelevant

$$\sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

- **A trivial but very helpful observation.**
- **Enhanced by using color Jacobi to generate zeros.**

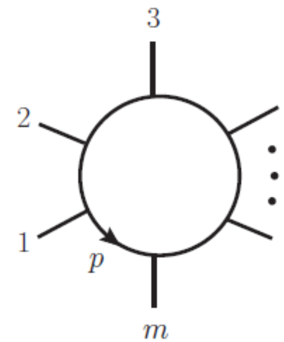
Color Jacobi to Eliminate Diagrams

color
Jacobi identity



m legs

All color factors
expressed in terms of
m-gon color factors



This is color
basis of Del Duca,
Dixon and Maltoni

integrand

$$\mathcal{A}^{1\text{-loop}}(1,2,\ldots,m) = g^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^Dp}{(2\pi)^D}\, c_{123\ldots m}\, \mathscr{A}(1,2,\ldots,m;p)$$

only *m*-gon color factors

- All other diagrams effectively set to zero: coefficient of color factor vanishes.
- All terms pushed into *m*-gons.

ZB, Boucher-Veronneau, Johansson

Gravity *m*-point Consequences

General one-loop gravity formula

Replace color factor with numerator factor

integrand

$$\mathcal{M}^{\text{1-loop}}(1,2,\ldots,m) = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^D p}{(2\pi)^D} \tilde{n}_{123\ldots m}(p)\, \mathscr{A}(1,2,\ldots,m;p)$$

Let's suppose that you had a case where $\tilde{n}_{123\ldots m}$ **independent of loop momentum**

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}+4\ \text{susy}}(1,2,\ldots,m) = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \tilde{n}_{123\ldots m}\, A^{\text{1-loop}}_{\mathcal{N}\ \text{susy}}(1,2,\ldots,m)$$

integrated amplitude

Same considerations work at any loop order

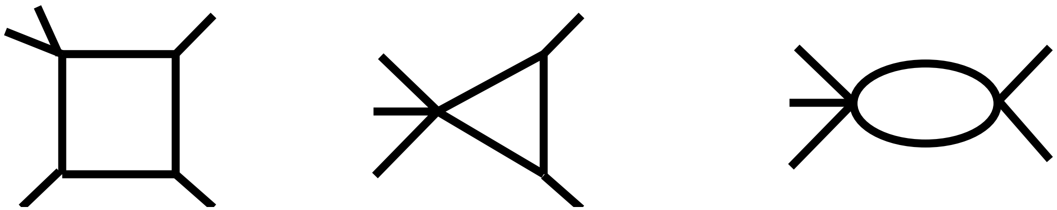
Do we have any such cases with where numerators independent of loop momentum?

Yes, *N* = 4 sYM 4,5 points at one-loop and 4 points at 2 loops !

Five-Point Lower Susy Confirmation

ZB, Boucher-Veronneau, Johansson

Integrated expression in terms of basis of scalar integrals:



rational

Carrasco and Johansson

$$n_{12345}^{N=4} = -\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}{4i\epsilon(1,2,3,4)}$$

color factor replaced by *N* = 4 numerator

$$M_{\mathcal{N}=6,\mathrm{mat.}}(1^-,2^-,3^+,4^+,5^+) = \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^4 \sum_{S_5/(Z_5\times Z_2)} n_{12345}\, A^{\text{1-loop}}_{\mathcal{N}=1,\mathrm{mat.}}(1^-,2^-,3^+,4^+,5^+)$$

$$M_{\mathcal{N}=4,\mathrm{mat.}}(1^-,2^-,3^+,4^+,5^+) = \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^4 \sum_{S_5/(Z_5\times Z_2)} n_{12345}\, A^{\text{1-loop}}_{\text{scalar}}(1^-,2^-,3^+,4^+,5^+)\,,$$

known from Dunbar, Ettle and Perkins (2011)

known from ZB, Dixon and Kosower (1993)

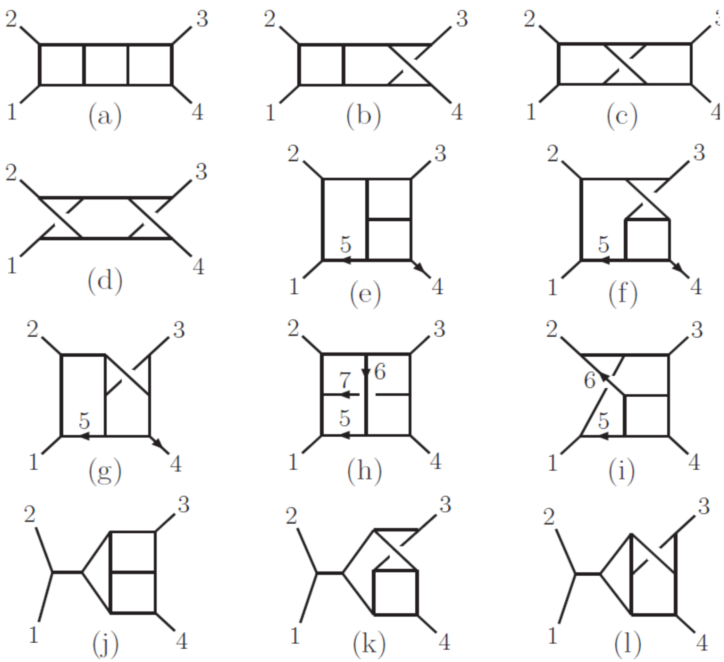
It works!

- **Two-loop four-point example from Boucher-Veronneau and Dixon.**
- **Naculich, Nastase and Schnitzer have recent paper exploring amplitude consequences: relations between *N*≥ 4 sugra and subleading color.**

Three-loop $N = 4$ Supergravity Construction

ZB, Davies, Dennen, Huang

$N = 4$ sugra : ($N = 4$ sYM) x ($N = 0$ YM)



- **No integrated YM expressions available.**
- **For $N = 4$ sYM copy use known BCJ representation.**
- **What representation should we use for pure YM side?**

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u) \right.$ $\left. +t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t) \right.$ $\left. +t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

$N = 4$ sYM integrand

Not a conjecture!

Three-loop $N = 4$ supergravity

What is a convenient representation for pure YM copy?

Answer: Feynman diagrams.

Yes, I did say Feynman diagrams!

This case is very special

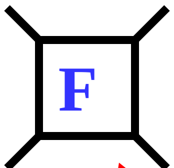
- **We can drop all Feynman diagrams where corresponding $N = 4$ numerators vanish.**
- **The Feynman diagrams are in a far simpler theory!**
- **We need only the leading UV parts.**
- **Completely straightforward. Faster to just do it than to argue about which way might be better.**

Multiloop $N = 4$ supergravity

Does it work? Test at 1, 2 loops

All pure supergravities finite at 1,2 loops

One-loop: keep only box Feynman diagrams

$st A_4^{\text{tree}} \times$ 

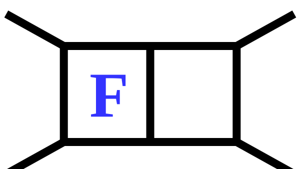
$+ \text{perms} = \frac{0}{\epsilon} + \text{finite}$

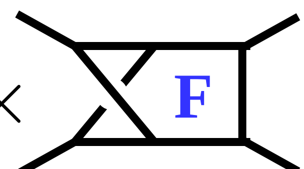
N = 4 sYM box numerator

N = 0 Feynman diagram, including ghosts

Becomes gauge invariant after permutation sum.

Two-loop: keep only double box Feynman diagrams

$s^2 t A_4^{\text{tree}} \times$ 

$+ s^2 t A_4^{\text{tree}} \times$ 

$+ \text{perms} = \frac{0}{\epsilon} + \text{finite}$

N = 4 sYM numerator

Feynman diagram including ghosts

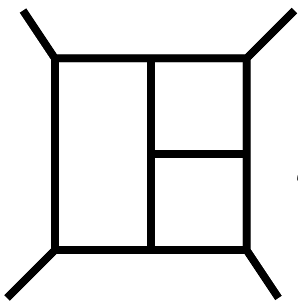
Get correct results. Who would have imagined it is this simple to get gravity results?

Three-Loop Construction

Now apply the construction to three loops.

***N* = 4 sugra : (*N* = 4 sYM) x (*N* = 0 YM)**

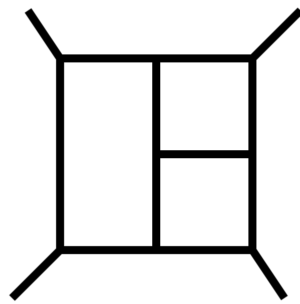
***N* = 4 sYM**



$\sim l \cdot k \, s^2 t A_4^{\text{tree}}$

Use BCJ representation

pure YM



$\sim (\varepsilon_i \cdot l)^4 \; l^4$

Feynman representation

***N* = 4 sugra linear divergent**

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

simple to see finite for *N*=5,6 sugra

Pure YM 4 point amplitude has never been done at three loops.

Numerator: $k^7 l^9 + k^8 l^8$ **+ finite**

log divergent

Will series expand in external momenta *k*

Extracting UV divergences from the integrals

We get many integrals, with up to 9 powers of loop momenta in the numerator.

- **Linear and log three-loop divergences.**
- **Individual integrals have subdivergence.**
- **Not technically feasible to evaluate integrals directly (non-planar 3 loop integrals never done).**

Well understood how to extract UV divergences:

1. **Expand the integrals in external momenta.** Marcus and Sagnotti (1984)
2. **Introduce uniform mass regulator to deal with physical and unphysical IR singularities.**
3. **Systematically subtract all subdivergences.**
4. **Integrate.**

See discussion in Lance's talk on 4-loop *N* = 8 supergravity

Dealing With Subdivergences

Marcus, Sagnotti (1984)

The problem was solve nearly 30 years ago.

Recursively subtract all subdivergences.

regulator dependent

reparametrize subintegral

$$\mathcal{S}\left[\int \prod_{i=1}^{L} dp_i\, I\right] \equiv \mathrm{Div}\left[\int \prod_{i=1}^{L} dp_i\, I\right] - \sum_{l=1}^{L-1} \sum_{\substack{l-\mathrm{loop} \\ \mathrm{subloops}}} \mathrm{Div}\left[\int \prod_{j=l+1}^{L} dp'_j\, \mathcal{S}\left[\int \prod_{i=1}^{l} dp'_i\, I\right]\right]$$

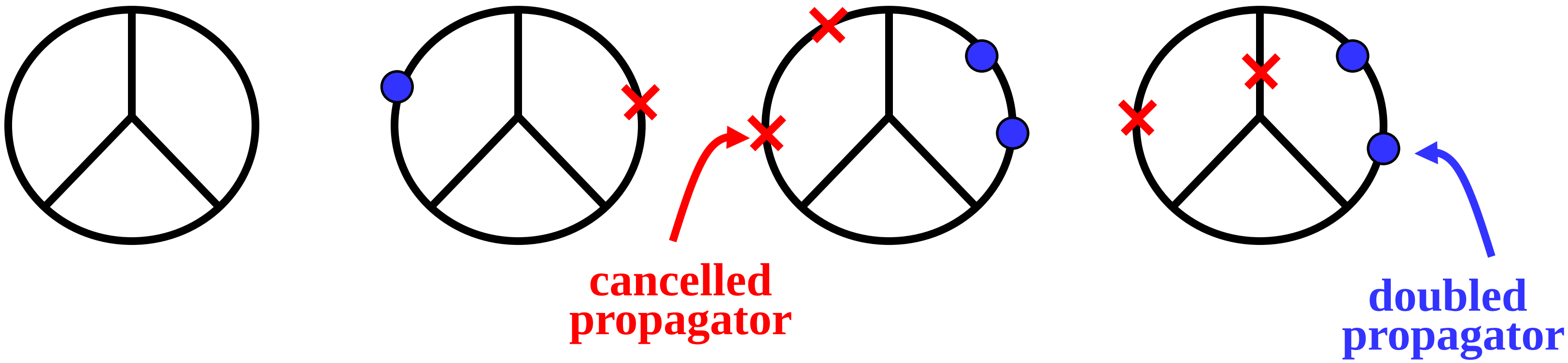
Regulator Independent

Nice consistency check: all log(*m*) terms must cancel

Extracting UV divergence in the presence of UV subdivergences and IR divergences is a well understood problem.

Vacuum Integrals

After series expansion in external momenta, and after tensor reduction, get about 600 vacuum integrals containing the UV information, e.g.



Evaluation:

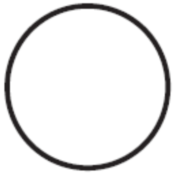
MB: Mellin Barnes integration Czakon

FIESTA: Sector decomposition A.V. Simirnov and Tentyukov

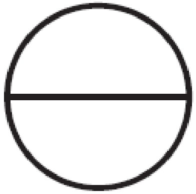
FIRE: Integral reduction using integration by parts identities. A.V. Simirnov

Integral Basis

Using FIRE we obtain a basis of integrals:

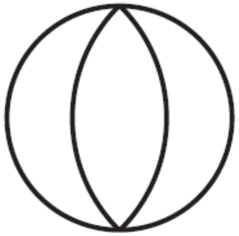


$$(m^2)^{1-\epsilon}\left(\frac{1}{\epsilon}+1+\left(1+\frac{1}{2}\zeta_2\right)\epsilon\right)$$



$$(m^2)^{1-2\epsilon}\left(\frac{3}{2\epsilon^2}+\frac{9}{2\epsilon}+\frac{21}{2}+\frac{3}{2}\zeta_2-2c\right)$$

$$(m^2)^{1-3\epsilon}\left(\frac{1}{\epsilon^3}+\frac{17}{3\epsilon^2}+\left(\frac{67}{3}+\frac{3}{2}\zeta_2-4c\right)\frac{1}{\epsilon}\right)$$



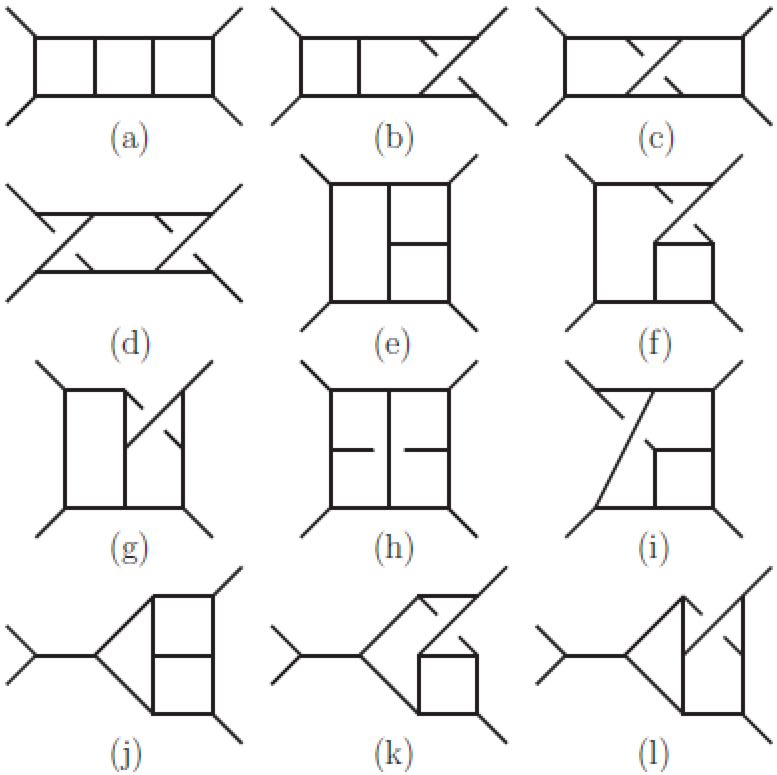
$$(m^2)^{2-3\epsilon}\left(\frac{2}{\epsilon^3}+\frac{23}{3\epsilon^2}+\left(\frac{35}{2}+3\zeta_2\right)\frac{1}{\epsilon}\right)$$

$$(m^2)^{-3\epsilon}\left(\frac{2\zeta_3}{\epsilon}\right)$$

$$c=\sqrt{3}\,\mathrm{Im}\left(\mathrm{Li}_2(e^{i\pi/3})\right)$$

Use Mellin-Barnes resummation of residues method of Davydychev and Kalmykov on all but last integral. Last one doable by staring at paper from Grozin or Smirnov's book (easy because no subdivergences).

The N = 4 Supergravity UV Cancellation



Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity with reference momenta

$q_1 = q_2 = k_3, \quad q_3 = q_4 = k_1$

Sum over diagrams is gauge invariant and independent of spinor-helicity reference momentum choices.

All divergences cancel completely!

Explanations?

Key Question: Is there an ordinary symmetry explanation for this? Or is something extraordinary happening?

Quantum corrected duality current nonconservation. Kallosh (2012)

Non-renormalization understanding from heterotic string. Tourkine and Vanhove (2012)

- **Is there a simple way to understand our result?**
- **What role does the global U(1) anomaly play?** Marcus (1985)
- **Why don't similar cancellations happen for $N = 8$ supergravity, killing potential 7 loop counterterm?**

Questions upon questions …

Answering questions

- Is the cancellation special to $N = 4$ supergravity?
  - We need to know result of $N = 8$ supergravity 4 pt 5 loop calculation. See Henrik Johansson's talk for progress.
- Four loops: D^2R^4 counterterm available.
  - Full superspace integral – analogous to 8-loop counterterm in $N = 8$ sugra.
  - Doable because $N = 4$ sYM BCJ representation exists.

    See Lance's talk
- Three- loop five points: ϕR^4 counterterm might be available.
  - Doable because BCJ representation exists.

    (Carrasco & Johansson, unpublished)

A critical problem for gravity is to find BCJ representations for as many gauge-theory amplitudes as possible…

See O'Connell's talk for tree-level progress

Summary

- **When duality between color and kinematics manifest, gravity integrands follow immediately from gauge-theory ones.**
- **In special cases, we can immediately obtain *integrated* gravity amplitudes from *integrated* gauge theory ones.**
- **Generalized gauge invariance a powerful tool.**
- **Duality between color and kinematics gives us a powerful way to explore the UV properties of gravity theories.**
- **$N = 4$ sugra has no three-loop four-point divergence, contrary to expectations from symmetry considerations. A surprise!**
- **Power counting using known symmetries and understood consequences can be misleading.**

The new tools are going to greatly clarify the UV structure of gravity.