Renormalization freedom in de Sitter spacetime

Simone Dresti

Georg-Augustus Universität Göttingen · Institute for Theoretical Physics

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Outline of the talk

Motivations

- Out-of-equilibrium phenomena played an important role during the evolution of the universe
  - At the inflationary epoch
  - ...
- In these phenomena one is typically interested in the time evolution of expectation values of observables
- The appropriate framework is the CTP formalism, first developed by Schwinger and Keldysh that allows to choose an arbitrary initial state and to follow its causal evolution including quantum effects

Motivations...

- By considering QFTs with time dependent backgrounds (e.g. quasi de Sitter), the time translation symmetry of the Lagrangian is broken

▶ Time-dependent freedom in the renormalization
  - In the Energy-Momentum Tensor [Baacke, Covi et al, '10]
  - In composite operators [Dresti, Riotto, '13]

- The arbitrariness of the counterterms can have physical implications in out-of equilibrium phenomena (e.g. end of inflation)
- Better understanding of the formalism
  - Renormalization freedom in the context of distributions
  - Non covariance in the CTP formalism

Renormalization in spacetime

Renormalization: Extension of distributions on manifolds

- Tadpoles: vanish after a proper choice of normal ordering
- Fish diagram: $\Delta_F^2 \sim \left(\frac{1}{x^2}\right)^2$ in $\mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$
  It is ill-defined in the language of distributions. The renormalized product $\tilde{\Delta}_F^2$ exists (in the sense of [Epstein, Glaser, '73]) but is not unique. The renormalization freedom is related to the scaling of the product about the singularity and is given by $c\delta$

Remarks:

- The renormalization freedom should depend only on the mass and on geometrical quantities [Hollands, Wald, '01]
- Since the construction is local, an extension to manifolds is possible

The tadpole diagram in Minkowski spacetime

$$\propto \int_{t_{in}}^{t} dt_1 \left[\left(\frac{-i\lambda}{2} \right) \int \frac{dp^3}{(2\pi)^3} F(p, t_1, t_1) \right] \times$$

$$\times \left[\left(-iG^R(k, t, t_1) \right) F(k, t_1, t) + F(k, t, t_1) \left(-iG^A(k, t_1, t) \right) \right]$$

The divergent part is $\int \frac{dp^3}{(2\pi)^3} F(p, t_1, t_1) \sim \left(\frac{-i\lambda}{16\pi^2} \right) \Lambda^2 + C$.

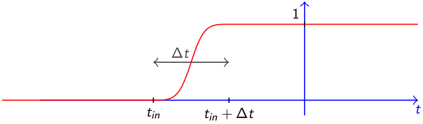
$$\left(\right. $$  $$\left. \right)_{\mathrm{CTP}} = \left(\right. $$  $$\left. \right)_{\mathrm{STD}} + \frac{\lambda m^2 C}{128\pi^2 w_k^3} \cos\left(2w_k(t - t_{in})\right)$$

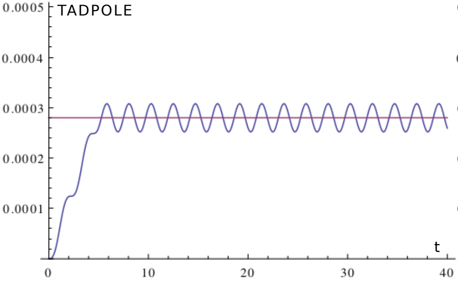
Since the spacetime is time-translational invariant we expect to recover the Poincaré symmetry for $t - t_{in} \gg 0$. The extra contribution is an artifact of the fact that the interaction profile contains a Heaviside distribution, i.e. $\lambda(t) = \lambda\, \theta(t - t_{in})$

Coupling constant profile...

Example:

$$g(t) = \frac{f(t - t_{in})}{f(t - t_{in}) + f(\Delta t - t + t_{in})}$$





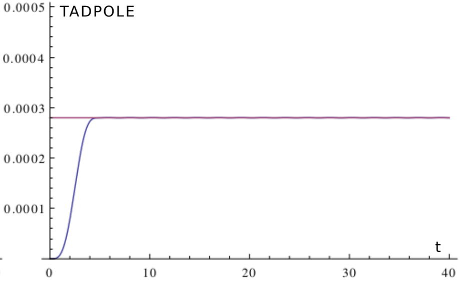


Figure: linear and quadratic transient extension for $\Delta t = 5$. The amplitude tends to a constant value (for large Δt) which is profile-independent

The adiabatic limit

The limit $t_{in} \to -\infty$ is not defined $\Rightarrow$ Distribution theory

$$\propto \int_{\tau_{in}}^{\tau} d\tau_1 \left(-iG^R(k,\tau,\tau_1)\right) F(k,\tau_1,\tau) f_\lambda(\tau_1).$$

Adiabatic limit: $f_\lambda(t) \to 1$

Example:

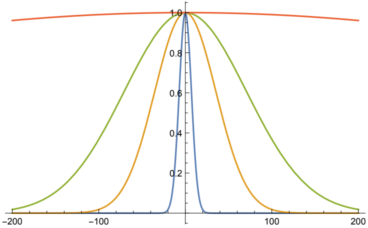


Figure: $f_\lambda(t) = e^{\frac{-\tau^2}{\lambda^2}}$ for $\lambda = 10, 50, 100, 1000$.

The adiabatic limit - example

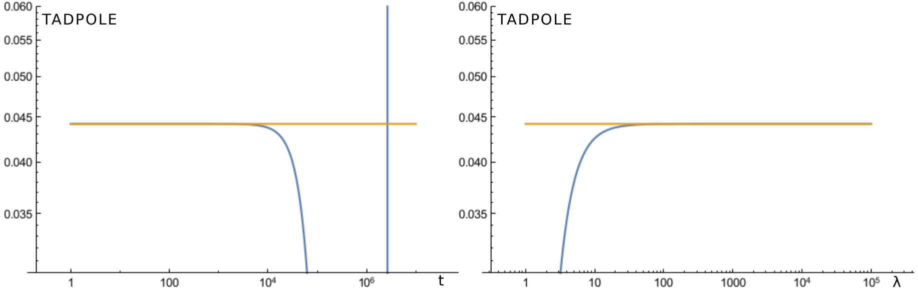


Figure: Tadpole as a function of time(left) and λ(right).

Adiabatic limit:

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\tau} d\tau_1 \left(-iG^R(k,\tau,\tau_1)\right) F(k,\tau_1,\tau) f_\lambda(\tau_1) \to \mathrm{Const}$$

The tadpole in de Sitter spacetime

Tadpole in de Sitter space for external legs taken at lowest order in k

$$= \frac{\lambda H^2}{48\pi^2 k^3}\frac{1}{2}\left[\left(\frac{1}{\epsilon}-\frac{C}{H^2}+\log\left(\frac{\mu^2}{H^2}\right)\right)\left(\log\left(\frac{\tau}{\tau_{in}}\right)+\frac{1}{3}-\frac{\tau^3}{3\tau_{in}^3}\right)\right]$$

General result

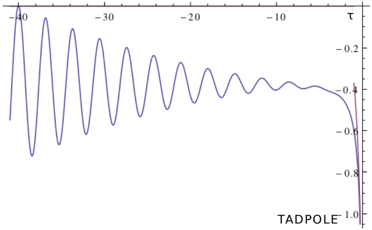
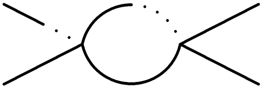
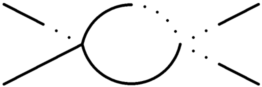
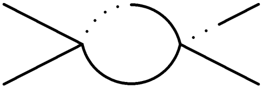
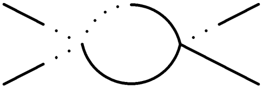
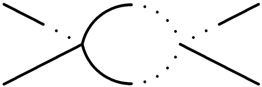
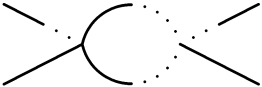
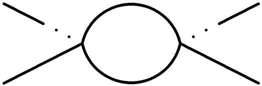


Figure: Time-dependence of the tadpole assuming $\tau_{in} = -40$ and $k = 1$

- Using different profiles it is possible to get rid of the rational dependence in τ_{in}
- The result has in any case a logarithmic dependence in time $\log(\tau)$

Fish diagram in de Sitter spacetime

Inequivalent Feynman diagrams

A1		**A2**	
A3		**A4**	
C1		**C2**	
B			

Fish diagram - contributions

Contribution $B + C_1 + C_2$:

- Each diagram diverges linearly but the sum is finite
- Infrared divergent (ϵ regulator)

$$\frac{H^4\lambda^2\left(k_1^3+k_2^3\right)\left(k_3^3+k_4^3\right)\left(3\log\left(\frac{\tau}{\tau_{in}}\right)+1\right)}{15552\pi^2k^3k_1^3k_2^3k_3^3k_4^3\epsilon}\Bigg(9\log\left(\frac{\tau}{\tau_{in}}\right)\times$$
$$\times\left(2\epsilon\log\left(k^2\tau\,\tau_{in}\right)+1\right)+12\epsilon\log(-k\tau)+4\epsilon+3\Bigg)$$

Contribution $A_1 + A_2$:

- Diagrams A_i diverge logarithmically

$$-\frac{H^4\lambda^2\tau_{in}^6\left(k_1^3\left(k_2^3\left(k_3^3+k_4^3\right)+k_3^3k_4^3\right)+k_2^3k_3^3k_4^3\right)}{186624\pi^2k_1^3k_2^3k_3^3k_4^3\epsilon}\Bigg(9\epsilon\log\left(\frac{\mu}{H}\right)+\log\left(\frac{1}{\tau^6}\right)+$$
$$2\epsilon\left(3\log\left(\frac{4\tau_{in}}{\tau}\right)+6\gamma-17\right)\log\left(\frac{\tau}{\tau_{in}}\right)+\log\left(\tau_{in}^6\right)$$
$$+\epsilon\left(2\pi^2+19\gamma-48+19\log(2)\right)-5\Bigg)$$

Conclusions

- Time-dependent contributions appear in the CTP formalism in Minkowski spacetime because the non-covariant interaction profile $\lambda(t) = \lambda\,\theta(t - t_{in})$

$\Rightarrow$ The covariance is restored in the adiabatic limit or with a different choice of the interaction profile

- In de Sitter spacetime one has in addition that the background depends on time (through the scale factor)

$\Rightarrow$ There are terms that don't depend on the interaction profile and might be physical

For the future...

- Understand the phenomenology of the time-dependence
- More diagrams and more loops...
- Better understanding of the formalism in the context of AQFT

$\Rightarrow$ Comparison with the scheme *analytic renormalization in spacetime* [Géré, Hack, Pinamonti, '15]

Backup

Closed-Time-Path Formalism

Description of the system

- Hamiltonian: $H(t) = H_0(t) + H_I(t)$
- Density matrix $\rho(t)$

Expectation value: $\langle \mathcal{O}(t) \rangle = \mathrm{Tr}(\rho(t)\mathcal{O}(t))$
Dynamics :

$$\begin{cases} i\dot{\rho}(t) = [H_I(t), \rho(t)] \\ \rho(t_{in}) = \rho_{in} \end{cases}$$

The solution is $\rho(t) = U_I(t, t_{in})\rho_{in} U_I^\dagger(t, t_{in})$, where U_I is the time-evolution operator, i.e.

$$U_I(t, t_{in}) = \mathrm{T} e^{-i \int_{t_{in}}^{t} d\tau \, H_I(\tau)}$$

Closed-Time-Path Formalism - contour

Expectation value

$$\langle \mathcal{O}(t) \rangle = \mathrm{Tr}\Bigg\{ \rho_{\text{in}} \left(\bar{\mathrm{T}} e^{+i \int_{t_{\text{in}}}^{\infty} d\tau H_I(\tau)} \right) \left(\mathrm{T} e^{-i \int_{t}^{\infty} d\tau H_I(\tau)} \right) \times$$
$$\times\, \mathcal{O}(t) \left(\mathrm{T} e^{-i \int_{t_{\text{in}}}^{t} d\tau H_I(\tau)} \right) \Bigg\}$$

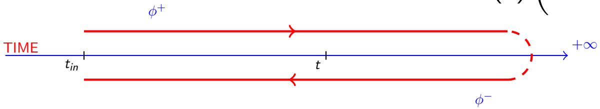


Figure: Closed time contour $\mathcal{C}$

After a proper definition of time-ordering...

$$\langle \mathcal{O}(t) \rangle = \mathrm{Tr}\left\{ \rho_{\text{in}} \mathrm{T}_{\mathcal{C}} \left(\mathcal{O}^+ e^{-i \int_{t_{\text{in}}}^{\infty} d\tau \left[H_I^+(\tau) - H_I^-(\tau) \right]} \right) \right\}$$

In-in action

$$S[\phi] \to S[\phi^+, \phi^-] = \int_{t_{in}}^{\infty} d\tau \int d^3x (\mathcal{L}[\phi^+] - \mathcal{L}[\phi^-])$$

Scaling of a function/distribution

Let f be a function in $\mathbb{R}^n$.

Definitions

- Scaling Degree of f: $\mathrm{sd}(f) := \inf_\omega \{\omega\colon \lim_{\lambda\to 0} \lambda^\omega f(\lambda x) = 0\}$
- Distribution $u_f : g \mapsto \langle f, g\rangle$

Because duality, the definition of the scaling at the level of the test functions is

$$\begin{aligned}\langle x \mapsto \lambda^\omega f(\lambda x), g\rangle &= \int d^n x\, \lambda^\omega f(\lambda x) g(x) \\ &= \int d^n x\, f(x) \lambda^{\omega-n} g\left(\frac{x}{\lambda}\right) = \left\langle f, x \mapsto \lambda^{\omega-n} g\left(\frac{x}{\lambda}\right)\right\rangle\end{aligned}$$

Example: the function (and then the associated distribution) $x \mapsto x^{-2}$ has scaling degree 2, because $\lambda^\omega \frac{1}{(\lambda x)^2} \to 0$ for $\omega > 2$

Extension of a distribution

Let $u \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and $\mathrm{div}(u) < 0$, the extension of u to $\mathcal{D}(\mathbb{R}^n)$ is constructed through a sequence of smooth functions

$$\theta_n = \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad x \in U^c \end{cases},$$

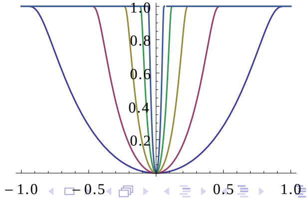
where U is a neighborhood of the origin such that

$$\theta_n \overset{n\to\infty}{\rightarrow} \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad \text{otherwise} \end{cases}$$

The extended distribution is given by $\tilde{u} = \lim_{n\to\infty} \theta_n u$ and is independent of the choice of the sequence $\{\theta_n\}_{n\in\mathbb{N}}$.

Example:

$$\theta_n(x) = 1 - e^{\frac{(2^n x)^2}{(2^n x)^2 - 1}} \chi_{[-2^{-n}, 2^{-n}]}$$



Extension for positive singular order $\omega > 0$

The extension theorem guarantees a unique extension on test function that vanish at the origin up to order ω

Definition (W-operation)

Let $\mathcal{D}^\omega(\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}^n)$ the subspace of function vanishing up to order ω at 0. The function W is a projection into that subspace

$$W_\omega \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}^\omega(\mathbb{R}^n), \quad \varphi \mapsto W_\omega \varphi$$

where

$$(W_\omega \varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w} \right)(0),$$

with $w \in \mathcal{D}(\mathbb{R}^n), w(0) \neq 0$.

Remark: The following relation holds $W_\omega^w(w\varphi) = w\, W_\omega^{x \mapsto 1}(\varphi)$

Extension for positive singular order - theorem

Theorem [Brunetti, Fredenhagen, '97]

Let $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ be a distribution with singular order ω. Given a W_ω-operation and constants $C^\alpha \in \mathbb{C}$, then there is one distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ with singular order ω such that

- $\langle u, \varphi \rangle = \langle u_0, \varphi \rangle \,, \; \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$
- $\langle u, wx^\alpha \rangle = C^\alpha, \; \alpha \leq \omega.$

u is then given by

$$\langle u, \varphi \rangle = \left\langle u_0^{\mathrm{ext}}, W_\omega \varphi \right\rangle + \sum_{|\alpha| \leq \omega} \frac{C^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w} \right)(0)$$

Remark: w is a function used to make quantities like $\langle u, wx^\alpha \rangle$ meaningful

Renormalization of composite operators

Statement

Let A be a bare local composite operator, consider all local operators B such that

- B has the same symmetry property of A
- $\mathrm{dim} B \leq \mathrm{dim} A$

then it is possible to define a divergent coefficient Z_{AB} for each B such that

$$A_R(x) = \sum_B Z_{AB} \left\{ B(x) - \langle B \rangle \right\}$$

is finite[a].

[a] i.e. all the n-point functions $\langle A_R(x)\phi(x_1)\ldots\phi(x_n)\rangle$ are finite

Renormalization

Definitions

- Renormalization: technique used to treat infinities arising in calculated quantities
- Regularization scheme: scheme used to absorb the infinities that arise in the perturbative approach
- Renormalization prescription: set of rules that describe what finite part one has to consider in the counterterms
- Renormalization conditions: conditions used to fix the the freedom of the counterterms.