Quantization of Moduli Spaces of Flat Connections
Applications to Supersymmetric Gauge Theories

Ioana Coman-Lohi

(based on joint work with Jörg Teschner and Maxime Gabella arXiv:1505.05898)

DESY Theory Workshop, 30 September 2015

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

MARIE CURIE ACTIONS

DESY

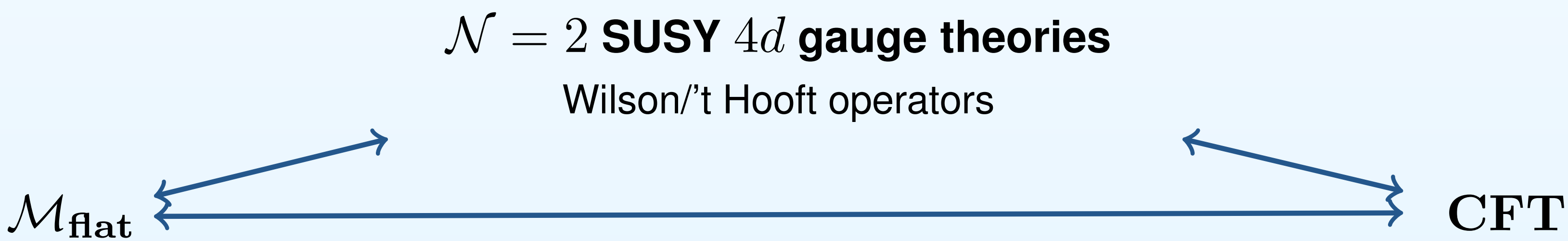
Context

Moduli spaces of flat connections on a Riemann surface $\mathcal{C}_{g,n}$ are relevant for 4d class $\mathcal{S}$ gauge theories from compactification of 6d $\mathcal{N} = (2,0)$ SCFT's on $\mathcal{C}_{g,n}$

[Gaiotto, Moore, Neitzke '09]

Context and motivation:

- AGT correspondence [Alday, Gaiotto, Tachikawa '10]



- Study moduli spaces of flat $SL_N(\mathbb{C})$-connections on $\mathcal{C}_{g,n}$

$$\mathcal{M}_{\text{flat}}(\mathcal{C}_{g,n}) \simeq \text{Hom}(\pi_1(\mathcal{C}_{g,n}), SL_N(\mathbb{C}))/SL_N(\mathbb{C})$$

dim $\mathcal{M} = (2g-2+n)(N^2-1)$

Outline

- Study the algebra of functions $\mathcal{A}_{g,n}$ on $\mathcal{M}_{\text{flat}}(\mathcal{C}_{g,n})$.
- Find a preferred set of generators w.r.t. a pair of pants decomposition.
  - Algebraic relations between functions on $\mathcal{M}_{\text{flat}}$
- Describe $\mathcal{A}^{q}_{g,n} \equiv$ a quantization of the algebra of functions on $\mathcal{M}_{\text{flat}}$.
- Investigate the relation to the algebra of Verlinde operators in Toda CFT.

I. $\mathcal{A}_{g,n}$ and tinkertoys

Basis of algebraic generators:

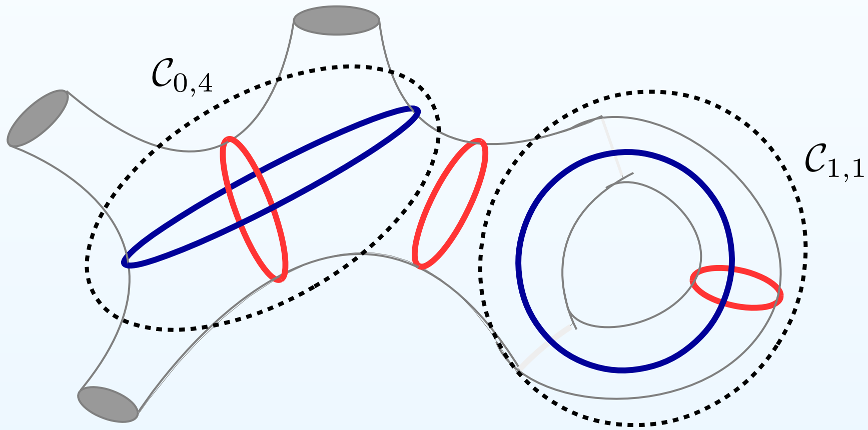


Fig. 1: Loops and networks on $\mathcal{C}_{1,3}$.

- Construct generators of $\mathcal{A}_{g,n}$ from $SL_N(\mathbb{C})$ holonomy matrices
  - Trace functions for simple loops on $\mathcal{C}_{g,n}$, from characteristic polynomial
  - Networks – contractions of $\prod$holonomies by $SL_N(\mathbb{C})$-invariant tensors

I. $\mathcal{A}_{g,n}$ and tinkertoys

Basis of algebraic generators:

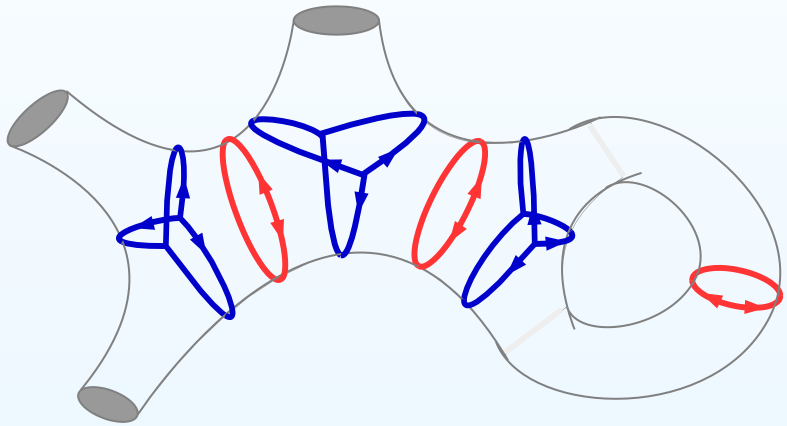
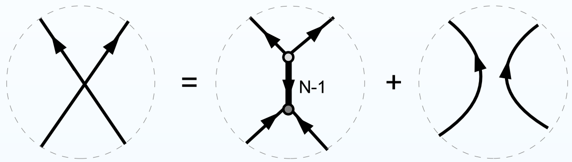


Fig. 1: Loops and networks on $\mathcal{C}_{1,3}$.

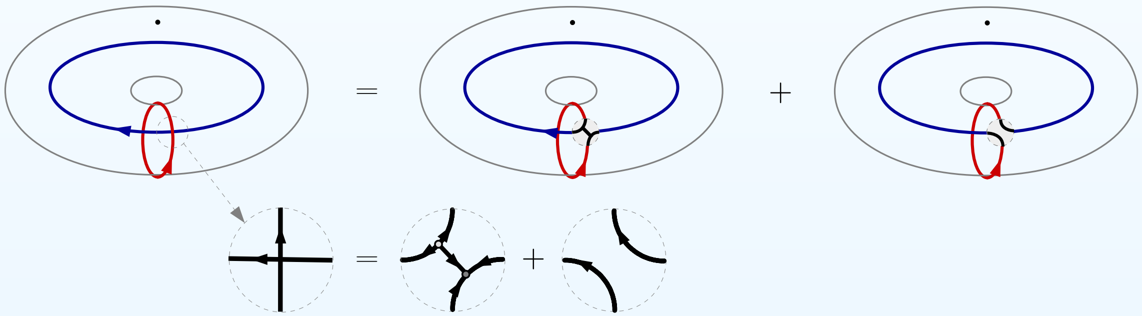
- Construct generators of $\mathcal{A}_{g,n}$ from $SL_N(\mathbb{C})$ holonomy matrices
  - Trace functions for simple loops on $\mathcal{C}_{g,n}$, from characteristic polynomial
  - Networks – contractions of $\prod$holonomies by $SL_N(\mathbb{C})$-invariant tensors
- Nicely localized w.r.t. a pair of pants decomposition of $\mathcal{C}_{g,n}$, as in CFT.

Relations I

- Skein relations express the product of two functions as a sum over generators.

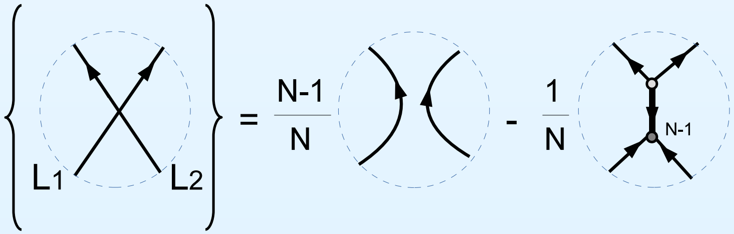


- Punctured torus:



Representations of crossing relations.

- Poisson relations - determined by applying the Goldman bracket: $\{L_{\gamma_1}, L_{\gamma_2}\} = L_{\{\gamma_1,\gamma_2\}}$ [Goldman '86].



Generators and relations

- Coordinates on a triangulation of $\mathcal{C}_{g,n}$
  - Fock-Goncharov coordinates x_i: attach to triangulation. [Fock, Goncharov '06]
  - Construct the holonomy matrices from elementary matrices: edge - crossing or moving through a face.

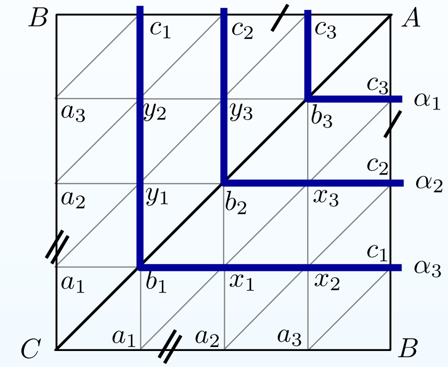


Fig. 2: $\mathcal{C}_{0,3}$ coordinates.

- Example: $SL(4,\mathbb{C})$ holonomy around puncture A on $\mathcal{C}_{0,3}$

$$A_1 = \mathrm{tr}\mathbf{A} = \prod_i \alpha_i^{-\kappa_{1i}^{-1}} \left(1+\alpha_1+\alpha_1\alpha_2+\alpha_1\alpha_2\alpha_3\right) \ .$$

- Networks expansion $N_i = \sum_{\underline{\mathbf{a}}} c_{\underline{\mathbf{a}}} x_{\underline{\mathbf{a}}}$ for $x_{\underline{\mathbf{a}}}$ monomials of x_i coordinates.

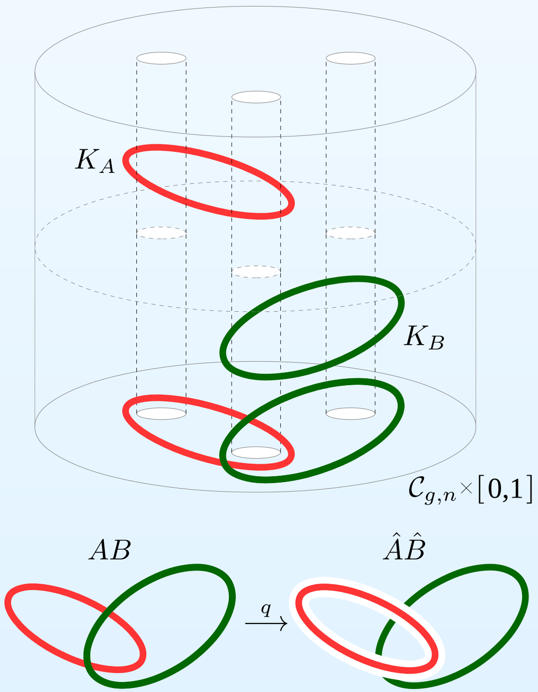
II. Quantization of $\mathcal{A}_{g,n}$

- Algebra: q-deformed algebraic relations between the generators of $\mathcal{A}^q_{g,n}$ using q-skein relations derived from quantum groups, with $q = e^{i\hbar}$.
- Representation: canonically quantize Fock-Goncharov coordinates. Construct the quantized generators in terms of $\hat{x}_i$ coordinates. $\hat{x}\hat{y} = q^{\epsilon_{xy}}\hat{y}\hat{x}$

The quantized $\mathcal{A}^q_{g,n}$ is provided by a 1-parameter family of skein non - commutative algebras of links in an oriented 3-manifold. [Turaev '91]

One can define networks in terms of $\mathcal{U}_q(sl(N))$ invariant tensors. [Sikora '05]

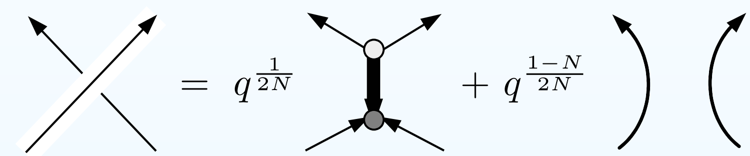
$$[\hat{A}, \hat{B}] \xrightarrow{q\to 1} \hbar\{A, B\}$$



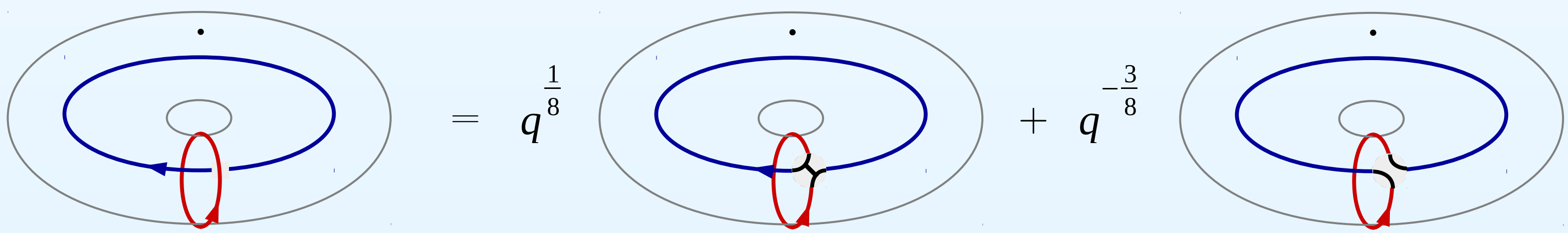
q-deformed relations

Examples :

- $\mathcal{U}_q(sl(N))$ quantum crossing relation: by *rescaling* networks constructed from tensor contractions.



- Representation: $\mathcal{U}_q(sl(4))$ quantum skein relation



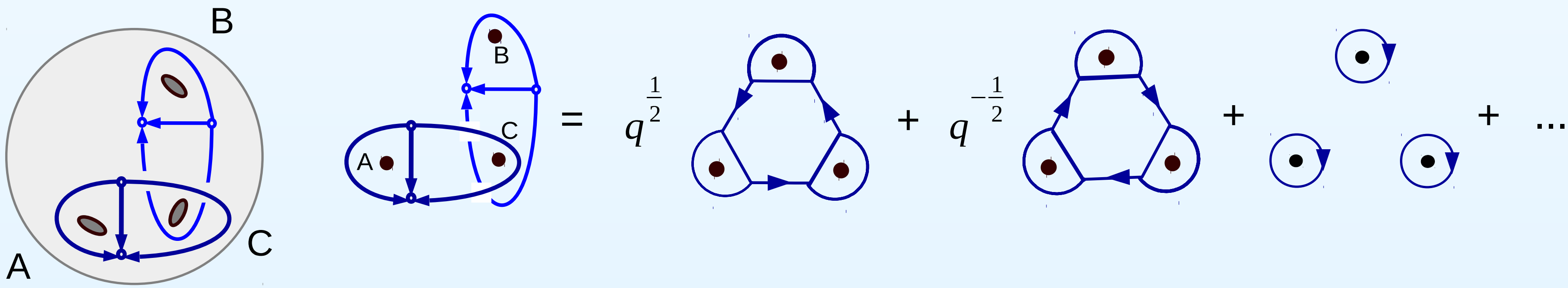
Quantized generators

Fock-Goncharov coordinates:

- Quantized Fock-Goncharov coordinates satisfy $\hat{x}_\alpha \hat{x}_\beta = q^{\epsilon_{\alpha\beta}} \hat{x}_\beta \hat{x}_\alpha$.
- Monomials: $x_{\underline{\mathbf{a}}} = \exp \sum_\alpha \mathbf{a}_\alpha X_\alpha \quad \rightarrow \quad \hat{x}_{\underline{\mathbf{a}}} = \exp \sum_\alpha \mathbf{a}_\alpha \hat{X}_\alpha$

  We construct quantized networks $\hat{N}_i = \sum_{\underline{\mathbf{a}}} c^q_{\underline{\mathbf{a}}} \hat{x}_{\underline{\mathbf{a}}}$ and trace functions.

Example: quantum skein relation for $\mathcal{U}_q(sl(3))$



$$\hat{N}_{AC}\hat{N}_{BC} = q^{1/2}\hat{\hat{W}}_1 + q^{-1/2}\hat{W}_1 + A_1B_1C_1 + A_2B_2C_2 + A_1A_2 + B_1B_2 + C_1C_2 + [3]$$

General expanded form of quantized functions $\hat{F} = \sum_{\underline{\mathbf{a}}} c^q_{\underline{\mathbf{a}}} \hat{x}_{\underline{\mathbf{a}}}$

So far...

- Classically: studied the algebra of functions $\mathcal{A}_{g,n}$ on $\mathcal{M}_{\rm flat}(\mathcal{C}_{g,n})$.

  Tinkertoys – preferred set of generators w.r.t. pair of pants decomposition.

- Quantization:

  Described a quantization $\mathcal{A}^q_{g,n}$ of the algebra of functions on $\mathcal{M}_{\rm flat}$.

  $\mathcal{A}^q_{g,n}$ in FG-coordinates $\leftrightarrow$ representation of quantum skein relations.

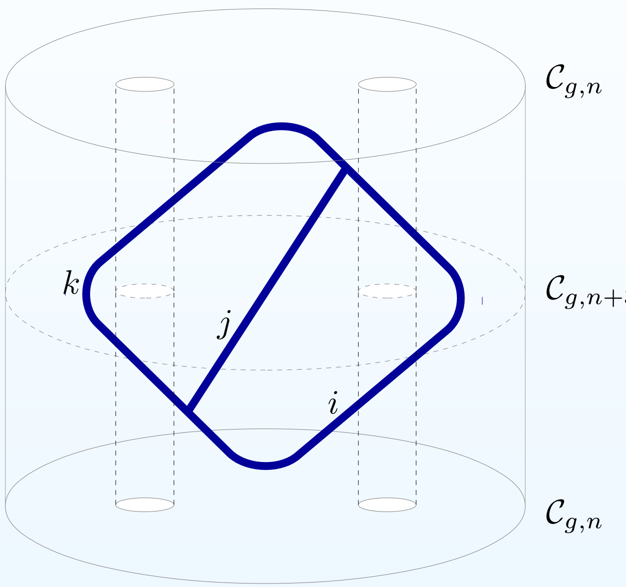
  Examples of quantization of BPS indices for higher rank.

- Claim:

  The algebra of Verlinde line and network operators in Toda CFT on $\mathcal{C}_{g,n}$ provides a representation of $\mathcal{A}^q_{g,n}$.

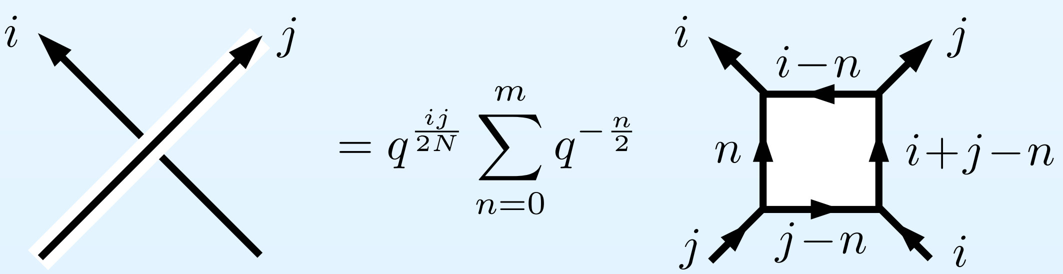
III. Toda field theory and Verlinde operators

- Verlinde loop and network operators
  describe the monodromy acquired by a vertex operator as it moves along a path.



- Fusion/braiding on conformal blocks.
- Braiding matrix $B(\alpha) \to \widetilde{R} \in \mathcal{U}_q(sl_N)^{\otimes 2}$
- Drinfeld twist: standard $R \;\to\; J^{-1}\widetilde{R}J$

- $VO_m \;\simeq\; M_{\lambda_m} \otimes \ldots \otimes M_{\lambda_1}$



-

Thank you for your attention!

GATIS

Gauge Theories as Integrable Systems