$R^*$ and five loop calculations

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Sep 25, 2017
Physics of the pole parts

Interesting physics just from the pole parts of diagrams:

- Anomalous dimensions (beta function, etc.)
- Splitting functions
- Decay rates (Higgs decay, etc.)

Pole parts are easier to compute than the finite parts

Goal

Compute poles of five loop diagrams
**$R^*$ operation**

- The poles parts come from the divergent momentum configurations.
- The recursive $R^*$ operation takes care of combinatorics of subdivergences \[\text{[Chetyrkin, Smirnov '83]}\]
- We have extended $R^*$ to Feynman integrals with arbitrary numerator structure \[\text{[Herzog, Ruijl '17]}\]

**UV counterterm operation**

$\Delta(G) = \text{poles of } G \text{ when all momenta go to } \infty \text{ with all contributions from subdivergences subtracted}$
Identifying divergent (sub)diagrams

\[ \mu \nu \]

\[ \int d^D p_1 \int d^D p_2 \frac{Q \cdot p_2 p_1^\nu}{p_1^2 p_2^2 p_3^2 p_4^4} \]

Get degree of divergence through power counting

Each loop contributes +4 due to the measure

All momenta \( \to \infty \): 8 + 1 + 1

- 2 - 2 - 4 = 0 (log)

\( p_2, p_3 \to \infty \): 4 + 1

- 2 = 1 (linear)

\( p_4 \to 0 \): 4

- 4 = 0 (log IR)
Identifying divergent (sub)diagrams

\[ \int d^D p_1 \int d^D p_2 \frac{Q \cdot p_2 p_1^\nu}{p_1^2 p_2^2 p_3^2 p_4^4} \]

- Get degree of divergence through power counting
- Each loop contributes +4 due to the measure
- All momenta \( \to \infty \): \( 8 + 1 + 1 - 2 - 2 - 2 - 4 = 0 \) (log)
- \( p_2, p_3 \to \infty \): \( 4 + 1 - 2 - 2 = 1 \) (linear)
- \( p_4 \to 0 \): \( 4 - 4 = 0 \) (log IR)
$R^*$-operation by example

$$K\left(\frac{2}{\epsilon^2} + 4 + 2\epsilon\right) \equiv \frac{2}{\epsilon^2}$$

$$K\begin{array}{c}
 1 \\
 2 
\end{array} = \Delta\begin{array}{c}
 1 \\
 2 
\end{array}$$

$$K\begin{array}{c}
 1 \\
 2 \\
 3 \\
 4 
\end{array} = \Delta\begin{array}{c}
 1 \\
 2 \\
 3 \\
 4 
\end{array} + \Delta\begin{array}{c}
 2 \\
 3 \\
 4 \\
 1 
\end{array} \cdot \begin{array}{c}
 1 \\
 4 
\end{array}$$

Consider all sets of non-overlapping divergent subdiagrams
$R^*$-operation by example

\[ K(\frac{2}{\epsilon^2} + 4 + 2\epsilon) \equiv \frac{2}{\epsilon^2} \]

\[ K \begin{array}{c}
1 \\
2
\end{array} = \Delta \left( \begin{array}{c}
1 \\
2
\end{array} \right) \]

\[ K \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} = \Delta \left( \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \right) + \Delta \left( \begin{array}{c}
2 \\
3
\end{array} \right) \cdot \begin{array}{c}
1 \\
4
\end{array} \]

Consider all sets of non-overlapping divergent subdiagrams

\[ K \begin{array}{c}
1 \\
2 \\
3 \\
5 \\
6 \\
4
\end{array} = \Delta \left( \begin{array}{c}
1 \\
2 \\
3 \\
5 \\
6 \\
4
\end{array} \right) + \Delta \left( \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array} \right) \cdot \begin{array}{c}
4 \\
5 \\
6
\end{array} \]

\[ -\Delta \left( \begin{array}{c}
5 \\
6
\end{array} \right) \Delta \left( \begin{array}{c}
2 \\
3
\end{array} \right) \cdot \begin{array}{c}
1 \\
4
\end{array} \]

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Counterterm operation $\Delta$

For log diagrams, $\Delta$ does not depend on external momenta or masses!

$$\Delta \left( \begin{array}{c}
\end{array} \right) = \Delta \left( \begin{array}{c}
\end{array} \right) = \Delta \left( \begin{array}{c}
\end{array} \right)$$

Infrared rearrangement (IRR)

Rearrange diagrams to simpler ones we can compute (always possible)
How to compute $\Delta$?

- We use its definition:

$$
\Delta(G)_{\text{IRR}} = \Delta(G') = K(G') - \text{subdivergences}(G')
$$

- Simpler than $G$
- Lower-loop diagrams
How to compute $\Delta$?

- We use its definition:

$$\Delta(G)^{\text{IRR}} = \Delta(G') = K(G') - \text{subdivergences}(G')$$

Simpler than $G$  ~ Lower-loop diagrams

- Recursive application:

$$\Delta\left(\begin{array}{c}
  \quad
  \end{array}\right) = K\left(\begin{array}{c}
  \quad
  \end{array}\right)$$

$$\Delta\left(\begin{array}{c}
  \quad
  \end{array}\right) = K\left(\begin{array}{c}
  \quad
  \end{array}\right) - \Delta\left(\begin{array}{c}
  \quad
  \end{array}\right)$$

- **Forcerer** can compute all rearranged diagrams [Ruijl, Ueda, Vermaseren ’17]
Five-loop example

\[ K_{\mu\nu} = \Delta_{\mu\nu} + \text{subdiv} \]

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Five-loop example

\[ K_{\mu \nu} - K_{\nu \mu} = \Delta_{\mu \nu} + \text{subdiv} \]

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Five-loop example

\[ K^{\mu\nu}_\nu^\mu = \Delta^{\mu\nu}_\nu^\mu + \text{subdiv}^{\mu\nu}_\nu^\mu \]

\[ \Delta^{\mu\nu}_\nu^\mu = K^{\mu\nu}_\nu^\mu - \text{subdiv}^{\mu\nu}_\nu^\mu \]

**computable!**

**four loops or fewer**

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\( R^* \) and five loop calculations
$R^*$-operation overview

1. Make input diagram logarithmic [creates a lot of terms]
2. Infrared rearrange diagrams
3. Identify counterterms
4. Make each counterterm log [millions of counterterms]
5. Tensor reduce the diagrams
6. Repeat process for new counterterms
Computational blow-up

- Troublesome five loop diagrams:

- Represents 12,029,521 scalar integrals!
- Computation may require a terabyte of disk space
- Time-consuming integral reductions
- $R^*$ is slow (subgraph finding, tensor reduction)
Results at five loops

- Computed five loop beta function for general colour group
  - Verified QCD result of [Baikov, Chetyrkin, Kühn '16]
  - Took 6 days on a pc with 32 cores
- Recomputed $H \rightarrow b\bar{b}$, $R$-ratio
  - Easy: took a few hours on one pc
- Computed $H \rightarrow gg$ [HRUVV '17]
  - Quartically divergent diagrams...
  - Hard: took two months
- Five-loop splitting functions (new!)
Five-loop beta function

- Behaviour of series in $\overline{\text{MS}}$ for QCD:

  $n_f = 3 : 1 + 0.56588 \alpha_S + 0.45301 \alpha_S^2 + 0.67697 \alpha_S^3 + 0.58093 \alpha_S^4$

  $n_f = 4 : 1 + 0.49020 \alpha_S + 0.30879 \alpha_S^2 + 0.48590 \alpha_S^3 + 0.28060 \alpha_S^4$

  $n_f = 5 : 1 + 0.40135 \alpha_S + 0.14943 \alpha_S^2 + 0.31722 \alpha_S^3 + 0.08092 \alpha_S^4$

- In MiniMOM (with $\alpha_S \equiv \alpha_S^{\text{MM}}$) [Ruijl, Ueda, Vermaseren ‘17]:

  $n_f = 3 : 1 + 0.56588 \alpha_S + 0.94199 \alpha_S^2 + 2.30450 \alpha_S^3 + 6.64749 \alpha_S^4$

  $n_f = 4 : 1 + 0.49020 \alpha_S + 0.64521 \alpha_S^2 + 1.63846 \alpha_S^3 + 3.46687 \alpha_S^4$

  $n_f = 5 : 1 + 0.40135 \alpha_S + 0.32886 \alpha_S^2 + 1.02690 \alpha_S^3 + 0.84177 \alpha_S^4$

- No sign of divergence of the series
Five-loop beta function

Figure: Left: the corrections of the QCD beta function. Right: the scale dependence of $\alpha_S$ for a value of 0.2 at 40 GeV$^2$
Five-loop beta function

- Adding the $N^4\text{LO}$ contributions changes the beta function by less than 1% at $\alpha_S = 0.47$ for $n_f = 4$ and at $\alpha_S = 0.39$ for $n_f = 3$
- The $N^4\text{LO}$ effect on the values of $\alpha_S$ are as small as 0.08% at $\mu^2 = 3 \text{ GeV}^2$ and 0.4% at $\mu^2 = 1 \text{ GeV}^2$
- The perturbative running of $\alpha_S$ is now fully under control for all practical purposes
Higgs decay to gluons

- Heavy top limit: effective coupling of Higgs to gluons

\[ \Gamma_{H \rightarrow gg} = \frac{\sqrt{2} G_F}{M_H} |C_1|^2 \text{Im} \Pi^{GG}(-M_H^2 - i\delta) \]

- \( C_1 \) is the N\(^4\)LO Wilson coefficient, determined by [Chetyrkin, Baikov, Kühn '16]

- \( \text{Im} \Pi^{GG}(-q^2 - i\delta) = \sin(L\pi\varepsilon)\Pi^{GG}(q^2) = \Pi^{GG}(q^2)L\pi\varepsilon + \ldots \)

- Only the pole parts of \( \Pi^{GG}(q^2) \) are needed
Higgs decay to gluons

- \( G(q^2) \equiv \frac{4\pi}{N_A q^4} \text{Im} \Pi^{GG}(q^2) \)
- At \( \mu^2 = q^2 \), \( G(q^2) \) is given by

\[
\begin{align*}
    n_f = 1 & : 1 + 5.437794 \alpha_S + 20.72031 \alpha_S^2 + 58.9218 \alpha_S^3 + 118.008 \alpha_S^4 \\
    n_f = 3 & : 1 + 4.695071 \alpha_S + 13.47244 \alpha_S^2 + 20.6640 \alpha_S^3 - 15.9624 \alpha_S^4 \\
    n_f = 5 & : 1 + 3.952348 \alpha_S + 6.955514 \alpha_S^2 - 6.85175 \alpha_S^3 - 75.2591 \alpha_S^4 \\
    n_f = 7 & : 1 + 3.209625 \alpha_S + 1.169536 \alpha_S^2 - 24.4579 \alpha_S^3 - 76.9977 \alpha_S^4 \\
    n_f = 9 & : 1 + 2.466902 \alpha_S - 3.885496 \alpha_S^2 - 32.9870 \alpha_S^3 - 37.3025 \alpha_S^4
\end{align*}
\]

- Large \( \alpha_S^4 \) coefficient due to coincidence and not a sign of diverging series
Renormalization scale dependence $\Gamma_{H \rightarrow gg}$

**Figure:** Left: the renormalization-scale dependence of $\tilde{G} = (\beta(a_S)/a_S)^2 G(M_H^2)$ at $n_f = 5$ in $\overline{\text{MS}}$. Right: decay width for $\alpha_S(M_Z^2) = 0.118$, $M_H = 125$ GeV and $\mu_t = 164$ GeV.
The effect of the $N^4$LO correction is $-0.6\%$ at $\mu = M_H$, $-0.8\%$ at $\mu = M_H/2$, and $+0.9\%$ at $\mu = 2M_H$.

The $N^4$LO scale variation between $\mu = 1/3 M_H$ and $\mu = 3M_H$ is as small as $0.9\%$ (full width), a reduction of almost a factor of four with respect to the corresponding $N^3$LO result.

Uncertainty of perturbation series now much smaller than that due to uncertainty of $\alpha_S(M_Z)$.

Our results are beyond LHC accuracy, but may be in reach of future colliders.
Five loop splitting functions

Non-singlet splitting function:

- Calculated $N = 1$ with one power of $\xi$ and verified with $Z_q$
- $N = 2$ computed
- $N = 3$ is almost done
- Computing higher moments is hard due to the number of terms and their complexity
Five loop $N = 2$ non-singlet splitting function

$$\gamma^{(4)+}_{\text{ns}} (N=2) =$$

$$C_F^5 \left[ \frac{9306376}{19683} - \frac{802784}{729} \zeta_3 - \frac{557440}{81} \zeta_5 + \frac{12544}{9} \zeta_3^2 + 8512 \zeta_7 \right]$$

$$- C_A C_F^4 \left[ \frac{81862744}{19683} - \frac{1600592}{243} \zeta_3 + \frac{59840}{81} \zeta_4 - \frac{142240}{27} \zeta_5 + 3072 \zeta_3^2 - \frac{35200}{9} \zeta_6 + 19936 \zeta_7 \right]$$

$$+ C_A^2 C_F^3 \left[ \frac{63340406}{6561} - \frac{1003192}{243} \zeta_3 - \frac{229472}{81} \zeta_4 + \frac{61696}{27} \zeta_5 + \frac{30976}{9} \zeta_3^2 - \frac{35200}{9} \zeta_6 + 15680 \zeta_7 \right]$$

$$- C_A^3 C_F^2 \left[ \frac{220224724}{19683} + \frac{4115536}{729} \zeta_3 - \frac{170968}{27} \zeta_4 - \frac{3640624}{243} \zeta_5 + \frac{70400}{27} \zeta_3^2 + \frac{123200}{27} \zeta_6 + \frac{331856}{27} \zeta_7 \right]$$

$$+ C_A^4 C_F \left[ \frac{266532611}{39366} + \frac{2588144}{729} \zeta_3 - \frac{221920}{81} \zeta_4 - \frac{3102208}{243} \zeta_5 + \frac{74912}{81} \zeta_3^2 + \frac{334400}{81} \zeta_6 + \frac{178976}{27} \zeta_7 \right]$$

$$- \frac{d_A^{abcd} d_A^{abcd}}{n_a} C_F \left[ \frac{15344}{81} - \frac{12064}{27} \zeta_3 - \frac{704}{3} \zeta_4 + \frac{58400}{27} \zeta_5 - \frac{6016}{3} \zeta_3^2 - \frac{19040}{9} \zeta_7 \right]$$

$$+ \frac{d_R^{abcd} d_A^{abcd}}{N_R} C_F \left[ \frac{23968}{81} - \frac{733504}{81} \zeta_3 + \frac{176320}{81} \zeta_5 + \frac{6400}{3} \zeta_3^2 + \frac{77056}{9} \zeta_7 \right]$$

$$- \frac{d_R^{abcd} d_A^{abcd}}{N_R} C_A \left[ \frac{82768}{81} - \frac{555520}{81} \zeta_3 + \frac{10912}{9} \zeta_4 - \frac{1292960}{81} \zeta_5 + \frac{84352}{27} \zeta_3^2 + \frac{140800}{27} \zeta_6 + 12768 \zeta_7 \right]$$

$$+ \text{terms with } n_f$$
Conclusions

- Local $R^*$ + Forcer very successful for five loop computations
- $\Gamma_{H\rightarrow gg}$ now known to within 1% uncertainty
- Splitting functions remain a challenge, but improvements are being made
Acknowledgements

This work is supported by the ERC Advanced Grant no. 320651, “HEPGAME”
Renormalization scale dependence of $\Gamma_{H \to gg}$ (II)

Figure: On-shell top mass of 173 GeV, $\alpha_s(M_Z^2) = 0.118$, $M_H = 125$ GeV