The Analytical Calculation Of The Four-Loop Cusp Anomalous Dimensions Of QCD

Robert M. Schabinger


and work in progress

Trinity College Dublin
1 Overview
   - Form Factors And Cusp Anomalous Dimensions
   - The Dipole Conjecture
   - Calculational Method

2 Bases Of Finite Master Integrals
   - The General Idea
   - Computational Complexity

3 Linear Reducibility
   - The Compatibility Graph Algorithm
   - “Universality Classes” Of Variable Changes
   - New Results For Master Integrals

4 Outlook
What We’re Doing

Our goal is to compute $\gamma^* \rightarrow q \bar{q}$ and $h \rightarrow gg$ at four loops in QCD.
Our goal is to compute $\gamma^* \rightarrow q\bar{q}$ and $h \rightarrow gg$ at four loops in QCD.
What We’re Doing

Our goal is to compute $\gamma^* \rightarrow q\bar{q}$ and $h \rightarrow gg$ at four loops in QCD.
Why We’re Doing It

The four loop cusp anomalous dimensions in QCD!

The QCD form factors in dimensional regularization satisfy a renormalization group equation which was understood long ago

\[ \frac{q^2}{\partial q^2} \ln \left( \frac{F(q^2/\mu^2, \alpha_s, \epsilon)}{\mu^2 \partial \mu^2 + \beta(\alpha_s) \partial \alpha_s} \right) = \frac{1}{2} K(\alpha_s) + \frac{1}{2} G(q^2/\mu^2, \alpha_s, \epsilon) \]

At \( L \) loops, \( \Gamma_L \) characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

\( \Rightarrow \Gamma_4 \) is the last unknown ingredient needed for \( N^3 \)LL resummation!
Why We’re Doing It

The four loop cusp anomalous dimensions in QCD!
Why We’re Doing It

The four loop cusp anomalous dimensions in QCD!
The QCD form factors in dimensional regularization satisfy a
renormalization group equation which was understood long ago


\[
q^2 \frac{\partial}{\partial q^2} \ln \left( \mathcal{F} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) \right) = \frac{1}{2} \mathcal{K}(\alpha_s) + \frac{1}{2} \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right)
\]

\[
\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) = \Gamma(\alpha_s)
\]

\[
\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{K}(\alpha_s) = -\Gamma(\alpha_s)
\]
Why We’re Doing It

The four loop cusp anomalous dimensions in QCD!

The QCD form factors in dimensional regularization satisfy a renormalization group equation which was understood long ago


\[ q^2 \frac{\partial}{\partial q^2} \ln \left( \mathcal{F} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) \right) = \frac{1}{2} \mathcal{K}(\alpha_s) + \frac{1}{2} \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) \]

\[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) = \Gamma(\alpha_s) \]

\[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{K}(\alpha_s) = -\Gamma(\alpha_s) \]

At $L$ loops, $\Gamma_L$ characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.
Why We’re Doing It

The four loop cusp anomalous dimensions in QCD!
The QCD form factors in dimensional regularization satisfy a renormalization group equation which was understood long ago


\[ q^2 \frac{\partial}{\partial q^2} \ln \left( \mathcal{F} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) \right) = \frac{1}{2} \mathcal{K}(\alpha_s) + \frac{1}{2} \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) \]

\[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{G} \left( \frac{q^2}{\mu^2}, \alpha_s, \epsilon \right) = \Gamma(\alpha_s) \]

\[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) \mathcal{K}(\alpha_s) = -\Gamma(\alpha_s) \]

At \( L \) loops, \( \Gamma_L \) characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

\[ \implies \Gamma_4 \text{ is the last unknown ingredient needed for N}^3\text{LL resummation!} \]
A Dipole Formula For Gauge Theory IR Divergences?


The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c) \, \mathcal{N} = 4$ super Yang-Mills, are believed to be of the form:

$$A_1^{\mathcal{N}=4}(p_1, \ldots, p_n) = \exp \left\{ -\frac{1}{2} \sum_{L=1}^{\infty} \left( \frac{\alpha_s}{4\pi} \right)^L \mu_\epsilon^{2L\epsilon} \int_0^{\mu_\epsilon^2} d\mu^2 \left( \mu^2 \right)^{-1-L\epsilon} \right. $$

$$\sum_{i,j=1}^{n} \sum_{i<j}^{\Gamma^{\mathcal{N}=4}_{1;L}} \ln \left( \frac{\mu^2}{-s_{ij}} \right) + \mathcal{G}^{\mathcal{N}=4}_{1;L} \frac{T_i \cdot T_j}{\mathcal{N}_c} \right. $$

$$\left. \left\{ \sum_{L=0}^{\infty} \mathcal{H}^{\mathcal{N}=4}_{1;L} (\epsilon; p_1, \ldots, p_n) \right. \right\}$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, JHEP 0706 (2007) 064). In principle, the above structure could hold for more general gauge theories like QCD.
Although some three-loop evidence was collected by Dixon \cite{Dixon:2009ri} for the $n_f$ terms, it is now clear that the dipole conjecture fails in general.


Although some three-loop evidence was collected by Dixon (Phys. Rev. D79 (2009) 091501) for the $n_f$ terms, it is now clear that the dipole conjecture fails in general.


The Casimir scaling part of the conjecture

$$\Gamma^g_L \equiv \frac{C_A}{C_F} \Gamma^q_L$$

has received a lot of attention in the last few years.
Although some three-loop evidence was collected by Dixon (Phys. Rev. D79 (2009) 091501) for the $n_f$ terms, it is now clear that the dipole conjecture fails in general.


The Casimir scaling part of the conjecture

$$\Gamma^g_L \equiv \frac{C_A}{C_F} \Gamma^q_L$$

has received a lot of attention in the last few years.

Although some three-loop evidence was collected by Dixon (\textit{Phys. Rev. D79} (2009) 091501) for the $n_f$ terms, it is now clear that the dipole conjecture fails in general. 


The Casimir scaling part of the conjecture

\[ \Gamma^g_L \equiv \frac{C_A}{C_F} \Gamma^q_L \]

has received a lot of attention in the last few years.


A. Grozin \textit{et. al.}, \textit{JHEP} 1601 (2016) 140
Although some three-loop evidence was collected by Dixon (Phys. Rev. D79 (2009) 091501) for the $n_f$ terms, it is now clear that the dipole conjecture fails in general.


The Casimir scaling part of the conjecture

\[
\Gamma^q_L \equiv C_A/C_F \Gamma^q_L
\]

has received a lot of attention in the last few years.


A. Grozin et. al., JHEP 1601 (2016) 140

Although some three-loop evidence was collected by Dixon (Phys. Rev. D79 (2009) 091501) for the $n_f$ terms, it is now clear that the dipole conjecture fails in general.


The Casimir scaling part of the conjecture

$$\Gamma^g_L = C_A/C_F \Gamma^q_L$$

has received a lot of attention in the last few years.


A. Grozin et. al., JHEP 1601 (2016) 140


How To Survive The Calculation

Use a decent-sized cluster to do numerator algebra.\(^\sim\) 50,000 diagrams with QGraf + FORM/Mathematica.


Find integral reductions for up to twelve-line integrals with as many as six inverse propagators (with Finred).


Construct an alternative basis of finite integrals and rewrite everything in terms of it using a set of auxiliary reductions.


Evaluate all finite master integrals analytically using HyperInt.


How To Survive The Calculation

- Use a decent-sized cluster to do numerator algebra.
  ($\sim 50,000$ diagrams with QGraf + FORM/Mathematica)

How To Survive The Calculation

- Use a decent-sized cluster to do numerator algebra.
  (∼ 50,000 diagrams with QGraf + FORM/Mathematica)
  

- Find integral reductions for up to twelve-line integrals with as many as six inverse propagators (with Finred).
  
  Phys. Rev. D95 (2017) no.3, 034030
How To Survive The Calculation

- Use a decent-sized cluster to do numerator algebra.
  (~ 50,000 diagrams with QGraf + FORM/Mathematica)
  

- Find integral reductions for up to twelve-line integrals
  with as many as six inverse propagators (with Finred).
  
  Phys. Rev. **D95** (2017) no.3, 034030

- Construct an alternative basis of finite integrals and rewrite
  everything in terms of it using a set of auxiliary reductions.
  
  A. von Manteuffel et. al., **JHEP** 1502 (2015) 120;
  Phys. Rev. **D93** (2016) no.12, 125014
How To Survive The Calculation

- Use a decent-sized cluster to do numerator algebra. 
  ($\sim 50,000$ diagrams with QGraf + FORM/Mathematica)
  

- Find integral reductions for up to twelve-line integrals with as many as six inverse propagators (with Finred).
  
  Phys. Rev. D95 (2017) no.3, 034030

- Construct an alternative basis of finite integrals and rewrite everything in terms of it using a set of auxiliary reductions.
  

- Evaluate all finite master integrals **analytically** using HyperInt.
  
From Conventional To Finite Integral Bases

For each irreducible topology, test progressively more complicated integrals for convergence.


For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of "extra" powers of the propagators or "dots"), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$.

Rotate from the old basis to the new basis using auxiliary IBPs. The computationally expensive part at this stage is to perform a Tarasov shift (Phys. Rev. D54 (1996) 6479) on the old basis and then IBP reduce the resulting linear combination of integrals in $d + 2$ with a number of additional dots equal to the loop order. This connects the "conventional" integral bases in $d$ and $d + 2$; it can be used iteratively if multiple dimension shifts are required.
From Conventional To Finite Integral Bases

- For each irreducible topology, test progressively more complicated integrals for convergence.

For each irreducible topology, test progressively more complicated integrals for convergence.


For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of “extra” powers of the propagators or “dots”), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$. 
For each irreducible topology, test progressively more complicated integrals for convergence.


For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of “extra” powers of the propagators or “dots”), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$.

Rotate from the old basis to the new basis using auxiliary IBPs.

The computationally expensive part at this stage is to perform a Tarasov shift (Phys. Rev. D54 (1996) 6479) on the old basis and then IBP reduce the resulting linear combination of integrals in $d + 2$ with a number of additional dots equal to the loop order. This connects the “conventional” integral bases in $d$ and $d + 2$; it can be used iteratively if multiple dimension shifts are required.
What About The Auxiliary Reductions Needed For The Basis Rotation?

Consider the three-loop gluon form factor, where $s_{\text{max}} = 5$: 
What About The Auxiliary Reductions Needed For The Basis Rotation?

Consider the three-loop gluon form factor, where $s_{\text{max}} = 5$:
What About The Auxiliary Reductions Needed For The Basis Rotation?

Consider the three-loop gluon form factor, where $s_{\text{max}} = 5$:

$$\Rightarrow \text{Auxiliary reductions are a subleading problem!}$$
An Illustrative Comparison

J. M. Henn et. al., JHEP 1605 (2016) 066

\[
\epsilon^8 \left[ (k_4^2)^2 \right] = \frac{1}{576} + \frac{1}{36} \zeta_2 \epsilon^2 + \frac{151}{864} \zeta_3 \epsilon^3 + \frac{173}{288} \zeta_2^2 \epsilon^4
\]

\[+ \left( \frac{505}{216} \zeta_2 \zeta_3 + \frac{5503}{1440} \zeta_5 \right) \epsilon^5 + \left( \frac{6317}{720} \zeta_2^3 + \frac{9895}{2592} \zeta_3^2 \right) \epsilon^6 + \mathcal{O}(\epsilon^7)\]
An Illustrative Comparison

J. M. Henn et. al., JHEP 1605 (2016) 066

\[ \epsilon^8 \frac{(4-2\epsilon)}{k_1 + p_1} \]

\[ \left[ (k_4^2)^2 \right] = \frac{1}{576} + \frac{1}{36} \zeta_2 \epsilon^2 + \frac{151}{864} \zeta_3 \epsilon^3 + \frac{173}{288} \zeta_2^2 \epsilon^4 \]

\[ + \left( \frac{505}{216} \zeta_2 \zeta_3 + \frac{5503}{1440} \zeta_5 \right) \epsilon^5 + \left( \frac{6317}{720} \zeta_2^3 + \frac{9895}{2592} \zeta_3^2 \right) \epsilon^6 + \mathcal{O}(\epsilon^7) \]
An Illustrative Comparison

J. M. Henn et. al., JHEP 1605 (2016) 066

$\epsilon^8_{k_4+p_1} \left[(k_4^2)^2\right] = \frac{1}{576} + \frac{1}{36} \zeta_2 \epsilon^2 + \frac{151}{864} \zeta_3 \epsilon^3 + \frac{173}{288} \zeta_2^2 \epsilon^4$

$+ \left( \frac{505}{216} \zeta_2 \zeta_3 + \frac{5503}{1440} \zeta_5 \right) \epsilon^5 + \left( \frac{6317}{720} \zeta_2^3 + \frac{9895}{2592} \zeta_3^2 \right) \epsilon^6 + \mathcal{O}(\epsilon^7)$

$\epsilon^8_{p_1} = -\frac{3}{5} \zeta_2^2 + 5 \zeta_2 \zeta_3 + \frac{25}{2} \zeta_5 - \frac{7}{10} \zeta_3^3 - \frac{3}{10} \zeta_2^2 \zeta_3 - \frac{5}{2} \zeta_2 \zeta_5 - \frac{147}{16} \zeta_7 + \mathcal{O}(\epsilon)$
The Correlation Of The Maximal Weight At Leading Order With The Number Of Edges Of The Graph


Data for the 22 three-loop form factor master sectors:

<table>
<thead>
<tr>
<th># Edges</th>
<th># Wt. 0</th>
<th># Wt. 2</th>
<th># Wt. 3</th>
<th># Wt. 4</th>
<th># Wt. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
The Correlation Of The Maximal Weight At Leading Order With The Number Of Edges Of The Graph

R. N. Lee et. al., Nucl. Phys. B856 (2012) 95

Data for the 197 genuine four-loop form factor master sectors:

<table>
<thead>
<tr>
<th># Edges</th>
<th># Wt. 2</th>
<th># Wt. 3</th>
<th># Wt. 4</th>
<th># Wt. 5</th>
<th># Wt. 6</th>
<th># Wt. 7</th>
<th># Wt. 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>17</td>
<td>2</td>
<td>19</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>18</td>
<td>8</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>23</td>
<td>6</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>11 + (0-5)</td>
<td>4 + (0-5)</td>
</tr>
</tbody>
</table>

Based on our experience, we expect that just 117 of the above are relevant to the calculation of the cusp anomalous dimensions!
The Compatibility Graph Algorithm


For each integration order, the algorithm associates a cascade of polynomials and their compatibilities to the integral topology under consideration, starting with the “compatibility graph”

\[ \mathcal{U} \leftrightarrow \mathcal{F} \]
The Compatibility Graph Algorithm


For each integration order, the algorithm associates a cascade of polynomials and their compatibilities to the integral topology under consideration, starting with the “compatibility graph”

\[ \mathcal{U} \quad \longleftrightarrow \quad \mathcal{F} \]

The algorithm assumes that, at each step, all factors are linear with respect to at least one of the remaining Feynman parameters:

\[
f_j(x_{k_1}, \ldots, x_{k_m}) = q_j^{(i)}(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m}) \cdot x_i + r_j^{(i)}(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m})
\]
The Compatibility Graph Algorithm


For each integration order, the algorithm associates a cascade of polynomials and their compatibilities to the integral topology under consideration, starting with the “compatibility graph”

\[
(U) \quad \longleftrightarrow \quad (F)
\]

The algorithm assumes that, at each step, all factors are linear with respect to at least one of the remaining Feynman parameters:

\[
f_j(x_{k_1}, \ldots, x_{k_m}) = q_j^{(i)}(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m}) x_i \\
\quad + r_j^{(i)}(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m})
\]

One obtains a tight upper bound on the factors which are relevant to the integration! Non-trivial mathematics, but the intuition is clear.
The Compatibility Graph Algorithm

All $q_j^{(i)}$ and $r_j^{(i)}$ may appear as letters after $x_i$ is integrated out, but that is not the end of the story. We have compatibility resultants

$$\{f_\ell, f_n\} x_i = \det \begin{pmatrix} q_\ell^{(i)} & r_\ell^{(i)} \\ q_n^{(i)} & r_n^{(i)} \end{pmatrix} \quad \{f_j, f_\infty\} x_i = q_j^{(i)} \quad \{f_j, f_0\} x_i = r_j^{(i)}$$

Any set of compatibility resultants with indices in common, including 0 and $\infty$, generate polynomial factors which are then considered to be compatible at the next iteration of the algorithm.
The Compatibility Graph Algorithm

All $q_j^{(i)}$ and $r_j^{(i)}$ may appear as letters after $x_i$ is integrated out, but that is not the end of the story. We have compatibility resultants

\[
\{f_{\ell}, f_n\}_{x_i} = \det \begin{pmatrix} q_{\ell}^{(i)} & r_{\ell}^{(i)} \\ q_n^{(i)} & r_n^{(i)} \end{pmatrix} \quad \{f_j, f_\infty\}_{x_i} = q_j^{(i)} \quad \{f_j, f_0\}_{x_i} = r_j^{(i)}
\]

Any set of compatibility resultants with indices in common, including 0 and $\infty$, generate polynomial factors which are then considered to be compatible at the next iteration of the algorithm.

\[
\frac{1}{f_1^{\nu_1} \cdots f_N^{\nu_N}} = \sum_{k=1}^{N} \sum_{j=1}^{\nu_k} \frac{(-1)^{\nu_k-j} \{f_k, \infty\}_{x_i}^{\nu_k}}{(\nu_k-j)! f_k^j} \sum_{\substack{\ell_1 \cdots \ell_N \ \ell_k-1, \ell_k+1, \cdots \ell_N \ \ell_1 \cdots \ell_k-1, \ell_k+1, \cdots \ell_N \ \ell_k \neq \ell_i \neq k}}^{N} \prod_{r=1}^{N} \frac{\nu_r + \ell_r - 1}{(\{f_r, f_r\}_{x_i})^{\nu_r + \ell_r}} \frac{\nu_k - j}{\ell_1 \cdots \ell_{k-1} \ell_{k+1} \cdots \ell_N}
\]

Robert M. Schabinger

Analytical Four-Loop QCD Cusp Anomalous Dimensions
“Universality Classes” Of Variable Changes

Remarkably, making certain simple variable changes in $\mathcal{U}$ and $\mathcal{F}$ can dramatically improve the linear reducibility of most tough sectors:

\[
x_i = x'_i x_j x_k \implies \quad \ldots \quad \implies \quad \ldots
\]

\[
x_i = x'_i x_j \quad x_k = x'_k x'_i \implies \quad \ldots
\]

\[
x_i = x'_i x_j \implies \quad \ldots
\]
Remarkably, making certain simple variable changes in $\mathcal{U}$ and $\mathcal{F}$ can dramatically improve the linear reducibility of most tough sectors:

\[ x_i = x'_i x_j x_k \implies \]

\[ x_i = x'_i x_j \quad x_k = x'_k x'_i \implies \]

Only two top-level sectors left which we cannot access analytically!
Selected Results

\[(4-2\epsilon)\]

\[= \frac{612}{5} \zeta_2^2 \zeta_3 - 300 \zeta_2 \zeta_5 - \frac{147}{2} \zeta_7 + \mathcal{O}(\epsilon)\]

\[(6-2\epsilon)\]

\[= -12 \zeta_2 \zeta_3 + 30 \zeta_5 + \frac{418}{105} \zeta_2^3 + 12 \zeta_3^2 - \frac{204}{5} \zeta_2 \zeta_3 + 100 \zeta_2 \zeta_5\]

\[+ \frac{49}{2} \zeta_7 - \frac{49151}{5250} \zeta_2^4 - 3 \zeta_2 \zeta_3^2 - 15 \zeta_3 \zeta_5 + \frac{72}{5} \zeta_5,3 + \mathcal{O}(\epsilon)\]

\[(6-2\epsilon)\]

\[= -\frac{128}{15} \zeta_2^3 - 48 \zeta_3^2 - 4 \zeta_2 \zeta_3 - 76 \zeta_2 \zeta_5 + \frac{343}{2} \zeta_7 + \frac{50503}{2625} \zeta_2^4\]

\[+ 18 \zeta_2 \zeta_3^2 - 80 \zeta_3 \zeta_5 - \frac{222}{5} \zeta_5,3 + \mathcal{O}(\epsilon)\]
Outlook

Our to-do list looks as follows:

- Buy new computers to more effectively run Laporta’s algorithm.
- Keep thinking about algorithmic improvements.
- Obtain analytical results for the cusp anomalous dimensions.
- Obtain analytical results for the finite parts of the form factors.