RECENT PROGRESS ON INFRARED SINGULARITIES

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Recent progress on Infrared singularities

Infrared singularities: motivation

- Gauge-theory scattering amplitudes feature infrared singularities. The singularities cancel order by order in cross sections, between REAL and VIRTUAL corrections.

- Knowing the singularity structure analytically(!) is essential for cross section calculations.

- The singularity structure is also an essential element in resumming leading radiative corrections.

- The singularities are **universal** and significantly simpler than finite parts. **They provide insight into the structure of gauge-theory amplitudes for any number of loops and legs, and beyond the planar limit.**

Recent progress on infrared singularities

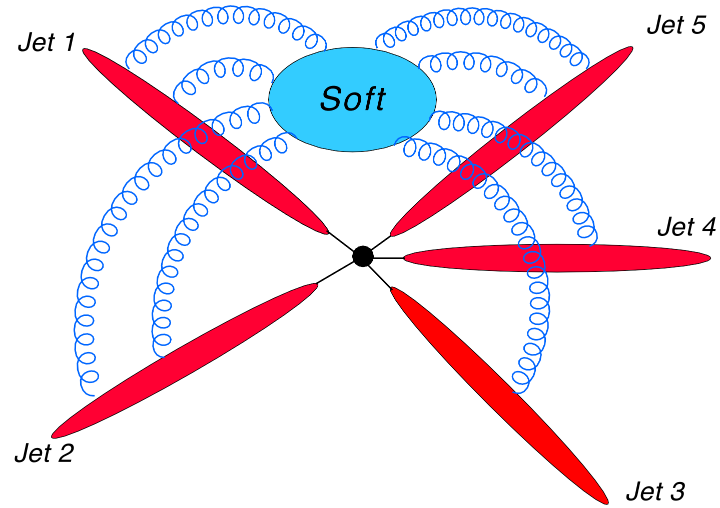
Plan of the talk

- Brief review of the structure of IR singularities (any number of massless legs, general kinematics, general colour)
- 3-loop correction to the dipole formula: kinematic variables and the space of functions
- IR singularities in the high-energy limit of 2 to 2 scattering
- IR singularities of collinear splitting amplitudes
- The bootstrap approach
- 3-loop result for the Soft Anomalous Dimension

Factorization of amplitudes with massless legs

Fixed angle scattering
with **lightlike partons** $p_i^2 = 0$

$$s_{ij} \equiv 2p_i \cdot p_j = 2\beta_i \cdot \beta_j Q^2 \gg \Lambda^2$$



IR singularities can be factorised
— all originate in soft and collinear region

Soft (matrix in colour flow space)

Jets (colour singlet)

$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right) H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right) \prod_{i=1}^{n} \frac{J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)}$$

The soft function: lightlike Wilson lines $\quad \mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

IR Singularities for Amplitudes with Massless Legs

Exponentiation:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

$$\Gamma_{\mathrm{Dip.}}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

Lightlike Cusp anomalous dimension

Catani (1998)
Dixon, Mert-Aybat and Sterman (2006)
Becher & Neubert, EG & Magnea (2009)

Soft/jet factorisation and **rescaling symmetry** with regards to the Wilson line velocities completely fix the anomalous dimension Γ to two loops as a *sum of dipoles*,
and constrain the form of higher-order corrections.

Corrections to the dipole formula

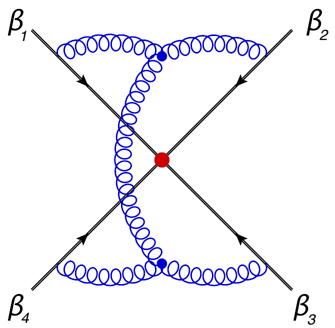
There are two types of **corrections to the dipole formula**:

Becher & Neubert, EG & Magnea (2009)

1. Corrections induced by higher Casimir contributions
   to the cusp anomalous dimension — starting at 4 loops.

Boels, Huber and Yang, 1705.03444
Moch et al., 1707.08315
Grozin, Henn, Stahlhofen, 1708.01221
see also Schabinger's talk

2. Functions of **conformally-invariant cross ratios** — starting at 3-loops:



$$\Gamma = \Gamma_{\text{Dip.}} + \Delta(\rho_{ijkl}) \qquad \rho_{ijkl} = \frac{(p_i \cdot p_j)(p_k \cdot p_l)}{(p_i \cdot p_k)(p_j \cdot p_l)}$$

$\Delta_n^{(3)}$ was computed using Feynman diagrams in

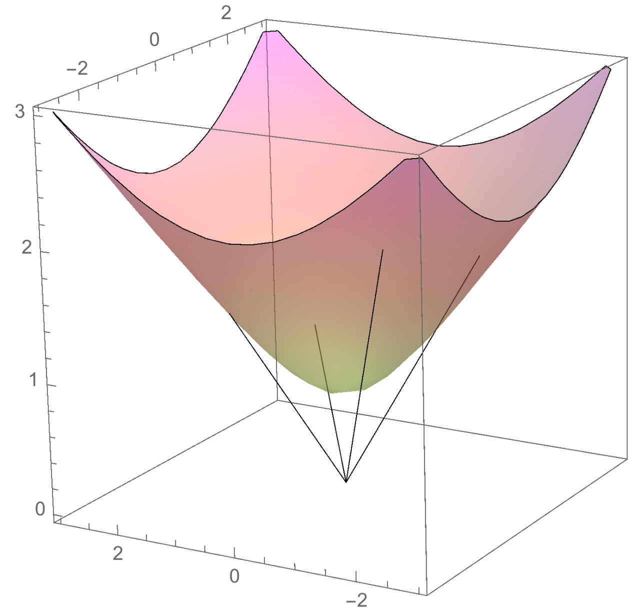
Ø. Almelid, C. Duhr, EG, Phys. Rev. Lett. **117**, 172002

Kinematic variables

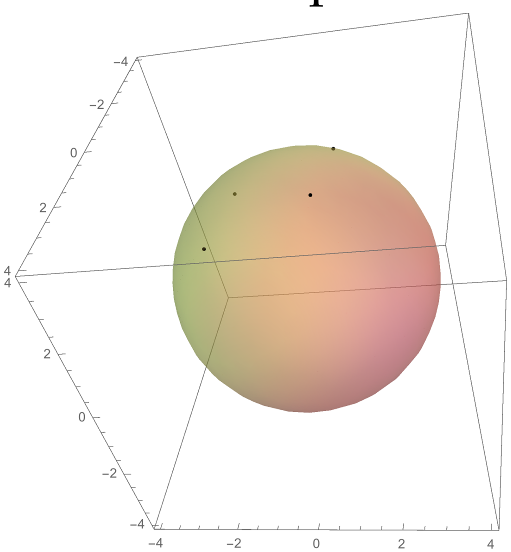
Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White, "Bootstrapping the QCD soft anomalous dimension" JHEP 09 (2017) 073

For $\beta_i^2 \neq 0$: using **rescaling symmetry** the velocities β_i map to a hyperbolic 3D space:

$$(\beta_i^0)^2 - (\beta_i^1)^2 - (\beta_i^2)^2 - (\beta_i^3)^2 = R^2, \quad \beta_i^0 > 0$$



The **lightlike limit** corresponds to the boundary of this space: β_i map to points on a Riemann sphere.



Parametrising: $\beta_i = \left(1 + \frac{z_i \bar{z}_i}{4}, \frac{z_i + \bar{z}_i}{2}, + \frac{z_i - \bar{z}_i}{2i}, 1 - \frac{z_i \bar{z}_i}{4}\right)$

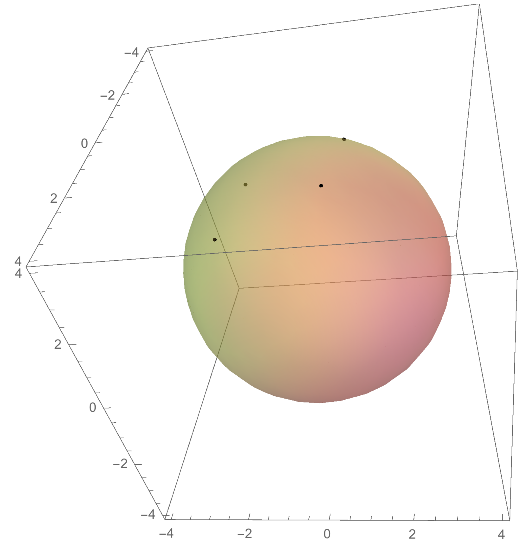
angles map to distances: $2\beta_i \cdot \beta_j = |z_i - z_j|^2$

The **rescaling-invariant** kinematic variables are:

$$\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left|\frac{(z_i - z_j)\,(z_k - z_l)}{(z_i - z_k)\,(z_j - z_l)}\right|^2$$

THE SPACE OF FUNCTIONS

n lightlike velocities β_i are described by
n points of the Riemann sphere



For a given set of 4 lines $\{i, j, k, l\}$ there are two independent cross ratios $\{\rho_{ijkl}, \rho_{ilkj}\}$

$$\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left| \frac{(z_i - z_j)\,(z_k - z_l)}{(z_i - z_k)\,(z_j - z_l)} \right|^2$$

Since only distances matter we can use $\mathrm{SL}(2, \mathbb{C})$ invariance to fix three of the points: $z_i = z,\ z_j = 0,\ z_k = \infty,\ z_l = 1$ so $\{\rho_{ijkl},\ \rho_{ilkj}\} = \{z\bar{z},\ (1-z)(1-\bar{z})\}$

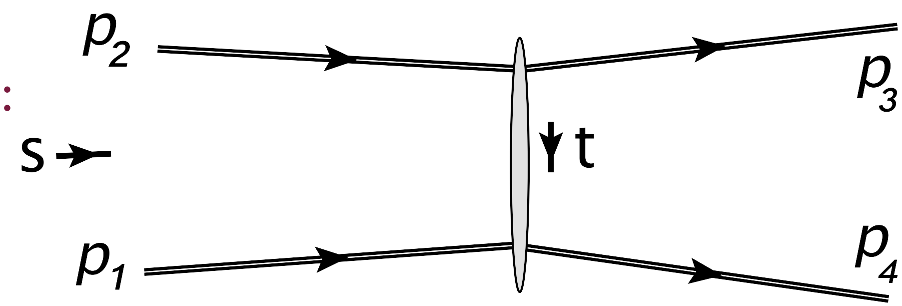
Iterated integrals on a Riemann sphere with n marked points are combinations of multiple polylogarithms (MPLs) with rational functions as coefficients.

F.C.S. Brown (2009)

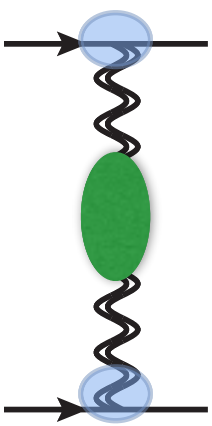
- Absence of branch points in Euclidean kinematics implies: **Single-Valued** MPLs.
- Singularities only appear when points coincide: must be Single-Valued **HPLs**.
  Symbol alphabet: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$
- At 3-loops we expect **pure functions** of **uniform weight** — a property of $\mathcal{N} = 4$ SYM

THE HIGH-ENERGY LIMIT: A REGGEIZED GLUON

The high-energy limit is dominated by t-channel exchange:



Leading logs of (-t/s) are summed through gluon Reggeization:



$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

$$\boxed{\alpha(t) = \frac{1}{4}\mathbf{T}_t^2 \int_0^{-t} \frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))}$$

Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

which is **fully consistent with the dipole formula for IR singularities**. This "Regge pole" factorization can be improved to NLL by introducing impact factors and corrections to the trajectory.

This holds to **NLL** for the **Real part** of the amplitude — **but** beyond that the exchange of multiple Reggeized gluons kick in!

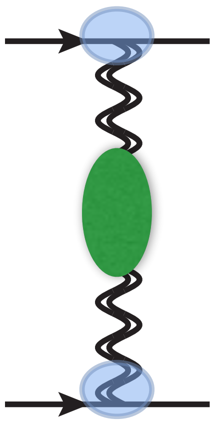
Del Duca, Glover (2001), Del Duca, Falcioni, Magnea, Vernazza (2013)

HIGH-ENERGY LIMIT: EXCHANGE OF MULTIPLE REGGEIZED GLUONS

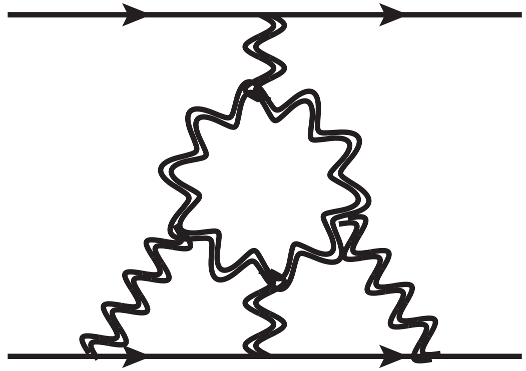
Recent progress: we now know to use JIMWLK/BFKL rapidity evolution to **compute** multiple Reggeized gluon contributions to 2-to-2 amplitudes.

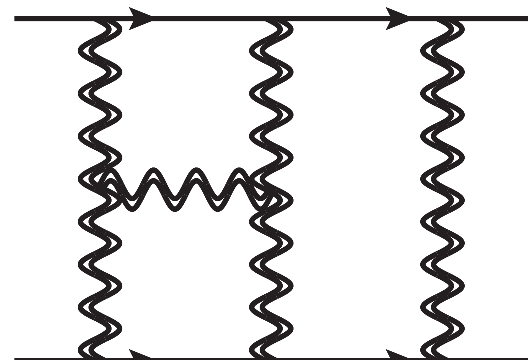
Signature odd (Real) part of the amplitude

LL, NLL



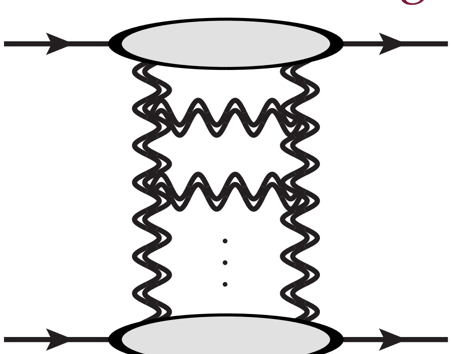
NNLL





Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016
checked against Henn & Mistlberger (2017)

Signature even (Imaginary) amplitude



NLL

Caron-Huot JHEP 05 (2015) 093
Caron-Huot, EG, Reichel, Vernazza - to appear

The Soft Anomalous Dimension in the High-Energy Limit (NLL)

Results (based on rapidity evolution):

$$\mathbf{\Gamma}\left(\alpha_s\right)=\frac{\alpha_s}{\pi}L\,\mathbf{T}_t^2+\mathbf{\Gamma}_{\rm NLL}\left(\alpha_s,L\right)+\mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s,L\right)+\dots.$$

In the High-Energy Limit the Soft Anomalous Dimension for 2 to 2 scattering is now known **to all orders** at NLL accuracy:

Odd Amplitude (Real part)

$$\mathbf{\Gamma}_{\rm NLL}^{(+)}=\left(\frac{\alpha_s(\lambda)}{\pi}\right)^2\frac{\gamma_K^{(2)}}{2}L\,\mathbf{T}_t^2+\left(\frac{\alpha_s(\lambda)}{\pi}\right)\sum_{i=1}^{2}\left(\frac{\gamma_K^{(1)}}{2}C_i\log\frac{-t}{\lambda^2}+2\gamma_i^{(1)}\right)$$

Even Amplitude (Imaginary part)

$$\mathbf{\Gamma}_{\rm NLL}^{(-)}=i\pi\frac{\alpha_s}{\pi}\,G\left(\frac{\alpha_s}{\pi}L\right)\frac{1}{2}(\mathbf{T}_s^2-\mathbf{T}_u^2)$$

Caron-Huot, EG, Reichel, Vernazza - to appear

$$G^{(l)}=\frac{1}{(l-1)!}\left[\frac{(C_A-\mathbf{T}_t^2)}{2}\right]^{l-1}\left(1-\frac{C_A}{C_A-\mathbf{T}_t^2}R(\epsilon)\right)^{-1}\Bigg|_{\epsilon^{l-1}}$$

$$R(\epsilon)=\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}-1=-2\zeta_3\,\epsilon^3-3\zeta_4\,\epsilon^4-6\zeta_5\epsilon^5-\left(10\zeta_6-2\zeta_3^2\right)\epsilon^6+\mathcal{O}(\epsilon^7)$$

THE SOFT ANOMALOUS DIMENSION IN THE HIGH-ENERGY LIMIT (BEYOND NLL)

Results beyond NLL accuracy:

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\mathrm{NLL}}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\mathrm{NNLL}}\left(\alpha_s, L\right) + \ldots\,.$$

Based on rapidity evolution $\boxed{\mathbf{\Gamma}^{(+)}_{\mathrm{NNLL}} = \mathcal{O}(\alpha_s^4)}$ Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016

— consistent with the Soft Anomalous Dimension 3-loop result.

The absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the Real part and for $k \geq 2$ in the Imaginary part, is a non-trivial prediction from rapidity evolution, which underpins the structure of corrections to the dipole formula.

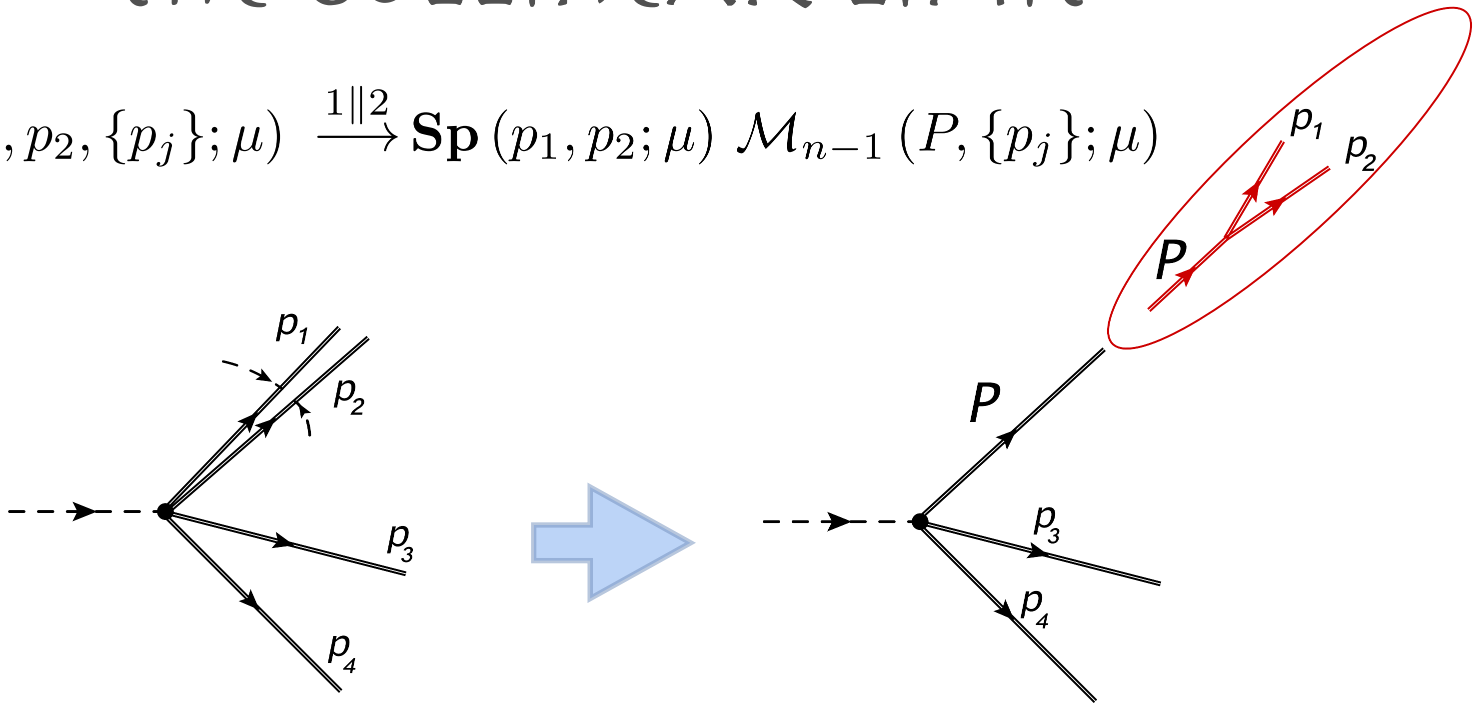
Based on the Soft Anomalous Dimension 3-loop result we also know:

$$\mathbf{\Gamma}^{(-)}_{\mathrm{NNLL}} = i\pi \left[\frac{\zeta_3}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 L + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}^2_{s-u}$$

$$\mathbf{\Gamma}^{(-)}_{\mathrm{N^3LL}} = i\pi \left[\frac{11\zeta_4}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}^2_{s-u} \qquad \mathbf{\Gamma}^{(+)}_{\mathrm{N^3LL}} = \mathcal{O}(\alpha_s^3)$$

THE COLLINEAR LIMIT

$$\mathcal{M}_n\left(p_1, p_2, \{p_j\}; \mu\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu\right) \mathcal{M}_{n-1}\left(P, \{p_j\}; \mu\right)$$



In particular, IR singularities of the splitting amplitude are those present in n-parton scattering (with 1 ‖ 2) and not in (n-1)-parton scattering:

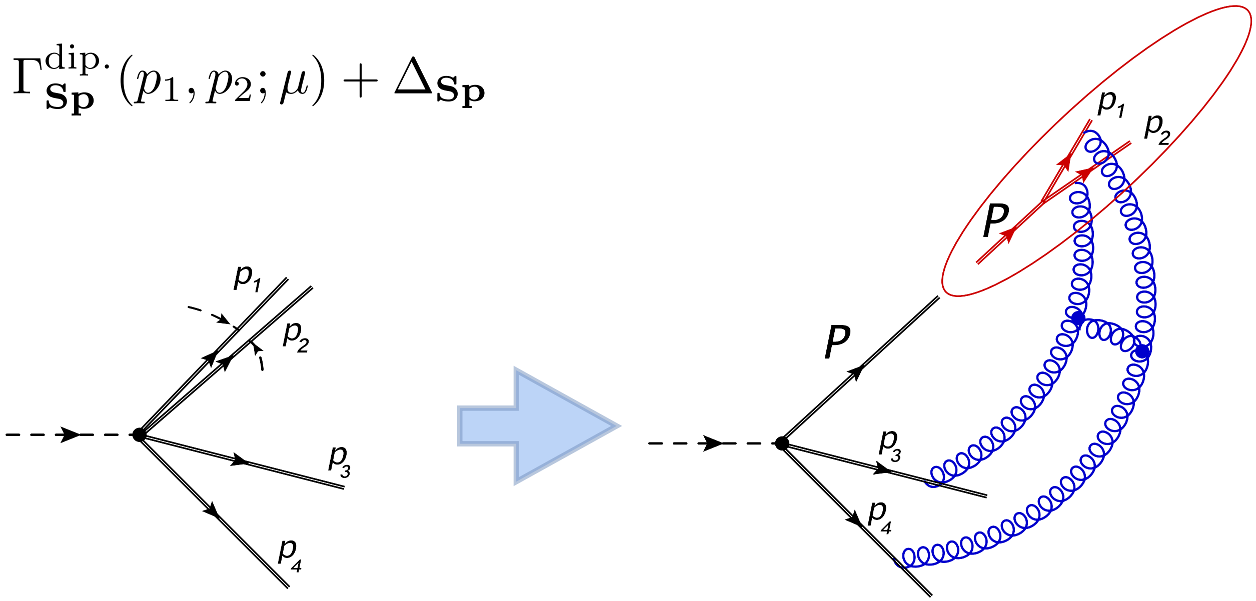
$$\Gamma_{\mathbf{Sp}} = \Gamma_n - \Gamma_{n-1}$$

The expectation (see e.g. [Catani, de Florian, Rodrigo 1112.4405, Feige & Schwartz 1403.6472]) is that the final-state splitting amplitude depends exclusively on the variables of the collinear pair.
This is *automatically realised* by the dipole formula for the singularities.

The collinear limit at 3 loops

At three loops there are diagrams that could introduce correlation between collinear partons and the rest of the process:

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu) = \Gamma_{\mathbf{Sp}}^{\text{dip.}}(p_1, p_2; \mu) + \Delta_{\mathbf{Sp}}$$



Requiring that the splitting amplitude singularities are independent of the rest of the process amounts to a strong constraint on the structure of the correction.

Becher & Neubert (2009)
Dixon, EG & Magnea (2010)

Bootstrapping the 3-loop soft anomalous dimension

Non-Abelian exponentiation theorem [EG, Smillie, White (2013)] **implies that has *fully connected* colour factors, such as** $f^{abe}f^{cde}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d$

Knowing that tripoles are excluded, applying **colour conservation** one finds that the answer can be expressed in terms of colour structures involving four generators correlating 3 and 4 lines.

Bose symmetry then implies the structure:

$$\begin{aligned}\Delta_n^{(3)}(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\leq i<j<k<l\leq n}\Big[\;&\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)\\ &+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)\\ &+\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]\\ &-C\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\Bigg\}\end{aligned}$$

where we regard the function $F(z)$ and the constant C as unknown.

Bootstrapping the 3-loop soft anomalous dimension

We expect a **pure function** involving **weight 5** SVHPLs.
Symmetry under $z \longleftrightarrow \bar{z}$ implies **palindromic** words.
The general ansatz has 13 parameters (rational numbers):

$$
\begin{aligned}
F(z) = {} & a_1 \mathcal{L}_{00000} + a_2 \mathcal{L}_{00100} + a_3 \mathcal{L}_{10001} + a_4 \mathcal{L}_{10101} + a_5 \left(\mathcal{L}_{01001} + \mathcal{L}_{10010}\right) \\
& + a_6 \left[\mathcal{L}_{00101} + \mathcal{L}_{10100} + 2(\mathcal{L}_{00011} + \mathcal{L}_{11000})\right] + a_7 \left[\mathcal{L}_{11010} + \mathcal{L}_{01011} + 3(\mathcal{L}_{00011} + \mathcal{L}_{11000})\right] \\
& + a_8 \, \zeta_2 \mathcal{L}_{000} + a_9 \, \zeta_2 \left(\mathcal{L}_{001} + \mathcal{L}_{100}\right) + a_{10} \, \zeta_3 \, \mathcal{L}_{00} + a_{11} \, \zeta_2^2 \, \mathcal{L}_0 , \\
C = {} & a_{12} \, \zeta_5 + a_{13} \, \zeta_2 \, \zeta_3 \, ,
\end{aligned}
$$

Collinear limits: independence of $\Delta_{\mathbf{Sp}} = (\Delta_n - \Delta_{n-1})|_{1\|2}$ of the rest of the process provides **6 constraints**.

Regge Limits: absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the Real part and for $k \geq 2$ in the Imaginary part, provides **8 constraints**.

Together these constraints **completely fix the coefficients above,** up to an overall normalization. We thus recover the 3-loop result computed from Feynman diagrams.

Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White,
"Bootstrapping the QCD soft anomalous dimension" JHEP 09 (2017) 073

The 3-loop correction to soft anomalous dimension

Ø. Almelid, C. Duhr, EG
Phys. Rev. Lett. **117**, 172002

$$\Delta_n^{(3)}(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\le i<j<k<l\le n}\Big[\ \mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)$$
$$+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)$$
$$+\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]$$
$$-\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$

$$\boxed{F(z)=\mathcal{L}_{10101}(z)+2\zeta_2\Big(\mathcal{L}_{100}(z)+\mathcal{L}_{001}(z)\Big)}$$

$$\rho_{1234}=z\bar{z}$$
$$\rho_{1432}=(1-z)(1-\bar{z})$$

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms (SVHPLs) introduced by Francis Brown in 2009. They are single-valued in the region where $\bar{z}=z^*$

Conclusions

- IR singularities of massless scattering amplitudes are now known to **3-loops**.

- As expected, the first correction to the dipole formula occurs at three loops.
  For three partons it is a constant, while for four or more, a quadrupole interaction correlating simultaneously colour and kinematics of 4 patrons.

- High-energy limit: recent calculations based on rapidity evolution fix the Soft Anomalous Dimension in the high energy limit at NLL to all orders, and NNLL at three loops.

- Collinear limits works as expected: final-state splitting amplitudes remain independent of the rest of the process.

$$\Delta_{\mathbf{Sp}} = \left(\Delta_n - \Delta_{n-1}\right)\big|_{1\|2} = -24\left(\frac{\alpha_s}{4\pi}\right)^3 (\zeta_5 + 2\zeta_2\zeta_3)\left[f^{abe}f^{cde}\left\{\mathbf{T}_1^a, \mathbf{T}_1^c\right\}\left\{\mathbf{T}_2^b, \mathbf{T}_2^d\right\} + \frac{1}{2}C_A^2\,\mathbf{T}_1\cdot\mathbf{T}_2\right]$$

- Understanding the space of functions and constraints based on non-Abelian exponentiation, symmetries, weight and information from collinear and Regge limits allow us to recover the form of the soft anomalous dimension by bootstrap.
  With these methods, four loops may be within reach.