Solving differential equations for Feynman integrals by expansions near singular points

Vladimir A. Smirnov

Skobeltsyn Institute of Nuclear Physics of Moscow State University

St. Gilgen, September 25, 2017
Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]
Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]

The method of differential equations
[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]
Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]

The method of differential equations
[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]

Gehrmann & Remiddi: a method to evaluate master integrals.
Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]

The method of differential equations
[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]

Gehrmann & Remiddi: a method to evaluate master integrals.
Let us consider Feynman integrals with two scales and let \( x \) be the ratio of these scales.
Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]

The method of differential equations
[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann &
E. Remiddi’00, J. Henn’13]

Gehrmann & Remiddi: a method to evaluate master integrals.
Let us consider Feynman integrals with two scales and let $x$ be the ratio of these scales.

DE

$$\partial_x J = M(x, \epsilon) J,$$

where $J$ is a column-vector of $N$ primary master integrals, and $M$ is an $N \times N$ matrix with elements which are rational functions of $x$ and $\epsilon = (4 - D)/2$. 
[J.H. Henn’13]: turn to a new basis where DE take the form

\[ \partial_x J = \epsilon M(x) J. \]
[J.H. Henn’13]: turn to a new basis where DE take the form

\[ \partial_x J = \epsilon M(x) J. \]

Then solving DE is straightforward.
[J.H. Henn’13]: turn to a new basis where DE take the form

\[ \partial_x J = \varepsilon M(x) J. \]

Then solving DE is straightforward.

The \( \varepsilon \)-form is not always possible. The simplest counter example is the two-loop sunset diagram with three equal non-zero masses.
[J.H. Henn’13]: turn to a new basis where DE take the form

\[ \partial_x J = \epsilon M(x) J. \]

Then solving DE is straightforward.

The \( \epsilon \)-form is not always possible. The simplest counter example is the two-loop sunset diagram with three equal non-zero masses.

However this form is possible:

\[ \partial_x J = (M_0(x) + \epsilon M_1(x)) J. \]
[J.H. Henn’13]: turn to a new basis where DE take the form

\[ \partial_x J = \epsilon M(x) J. \]

Then solving DE is straightforward.

The \( \varepsilon \)-form is not always possible. The simplest counter example is the two-loop sunset diagram with three equal non-zero masses.

However this form is possible:

\[ \partial_x J = (M_0(x) + \epsilon M_1(x)) J. \]

However, 'integrating out' the constant term appears to be an essentially more complicated problem.
Elliptic generalization of multiple polylogarithms motivated by two-loop examples, where the ε-form is impossible
[L. Adams, C. Bogner, A. Schweitzer & S. Weinzierl’16; E. Remiddi & L. Tancredi’17; talk by E. Remiddi]
Elliptic generalization of multiple polylogarithms motivated by two-loop examples, where the $\varepsilon$-form is impossible
[L. Adams, C. Bogner, A. Schweitzer & S. Weinzierl’16; E. Remiddi & L. Tancredi’17; talk by E. Remiddi]

An example of a calculation of a full set of the master integrals with ‘elliptic sectors’
[R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello & V.S. ’16]
Elliptic generalization of multiple polylogarithms motivated by two-loop examples, where the $\varepsilon$-form is impossible
[L. Adams, C. Bogner, A. Schweitzer & S. Weinzierl’16; E. Remiddi & L. Tancredi’17; talk by E. Remiddi]

An example of a calculation of a full set of the master integrals with ‘elliptic sectors’
[R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello & V.S. ’16]

Elliptic functions appear only in two sectors and final results are expressed either in terms of multiple polylogarithms or, for the elliptic sectors, in terms of two and three-fold iterated integrals suitable for numerical evaluation.
We are very far, even in lower loops orders, from answering the following question: ‘What is the class of functions which can appear in results for Feynman integrals in situations where $\epsilon$-form is impossible’?
We are very far, even in lower loops orders, from answering the following question:
‘What is the class of functions which can appear in results for Feynman integrals in situations where $\epsilon$-form is impossible’?

Knowing a differential system and the corresponding boundary conditions gives almost as much information about Feynman integrals as knowing their explicit expressions in terms of some class of functions.
We are very far, even in lower loops orders, from answering the following question:
‘What is the class of functions which can appear in results for Feynman integrals in situations where $\epsilon$-form is impossible’?

Knowing a differential system and the corresponding boundary conditions gives almost as much information about Feynman integrals as knowing their explicit expressions in terms of some class of functions.

Some properties of the integrals are more accessible via DE.
We are very far, even in lower loops orders, from answering the following question:
‘What is the class of functions which can appear in results for Feynman integrals in situations where $\epsilon$-form is impossible’?

Knowing a differential system and the corresponding boundary conditions gives almost as much information about Feynman integrals as knowing their explicit expressions in terms of some class of functions.

Some properties of the integrals are more accessible via DE. Singularities of DE provide a way to examine the branching properties of integrals.
We are very far, even in lower loops orders, from answering the following question:
‘What is the class of functions which can appear in results for Feynman integrals in situations where $\epsilon$-form is impossible’?

Knowing a differential system and the corresponding boundary conditions gives almost as much information about Feynman integrals as knowing their explicit expressions in terms of some class of functions.

Some properties of the integrals are more accessible via DE. Singularities of DE provide a way to examine the branching properties of integrals. Numerical values of the integrals can be obtained from a numerical solution of DE.
The goal: to describe an algorithm which enables one to find a solution of a given differential system in the form of an $\epsilon$-expansion series with numerical coefficients.
The goal: to describe an algorithm which enables one to find a solution of a given differential system in the form of an $\epsilon$-expansion series with numerical coefficients.

The idea: to use generalized power series expansions near the singular points of the differential system and solve difference equations for the corresponding coefficients in these expansions.
The goal: to describe an algorithm which enables one to find a solution of a given differential system in the form of an $\epsilon$-expansion series with numerical coefficients.

The idea: to use generalized power series expansions near the singular points of the differential system and solve difference equations for the corresponding coefficients in these expansions.

The idea is very well known in mathematics.
The goal: to describe an algorithm which enables one to find a solution of a given differential system in the form of an $\epsilon$-expansion series with numerical coefficients.

The idea: to use generalized power series expansions near the singular points of the differential system and solve difference equations for the corresponding coefficients in these expansions.

The idea is very well known in mathematics.
[B. A. Kniehl, A. F. Pikelner & O. L. Veretin’17]
An algorithm to solve difference equations for coefficients of the series expansions at a given singular point.
An algorithm to solve difference equations for coefficients of the series expansions at a given singular point.

A matching procedure which enables one to connect series expansions at two neighboring points.
- An algorithm to solve difference equations for coefficients of the series expansions at a given singular point.
- A matching procedure which enables one to connect series expansions at two neighboring points.
- As a proof of concept: a computer code where this algorithm is implemented for a simple example of a family of four-loop Feynman integrals where the $\epsilon$-form is impossible.
\[ \partial_x J = M(x) J. \]
\[ \partial_x J = M(x) J. \]

We imply that all the singular points of DE are regular, i.e. we can reduce the DE to a local Fuchsian form in any singular point.
\[ \partial_x J = M(x) J. \]

We imply that all the singular points of DE are regular, i.e. we can reduce the DE to a local Fuchsian form in any singular point.

General solution

\[ J(x) = U(x) C, \]
\[ \partial_x J = M(x) J. \]

We imply that all the singular points of DE are regular, i.e. we can reduce the DE to a local Fuchsian form in any singular point.

General solution

\[ J(x) = U(x) \mathcal{C}, \]

where \( \mathcal{C} \) is a column of constants, and \( U \) is an evolution operator

\[ U(x) = P \exp \left[ \int M(x) \, dx \right]. \]
Expanding in a vicinity of each singular point.
Expanding in a vicinity of each singular point.
Take $x = 0$. 
Expanding in a vicinity of each singular point.
Take \( x = 0 \).
The expansion is

\[
U(x) = \sum_{\lambda \in S} x^\lambda \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(n + \lambda, k) x^n \ln^k x,
\]

where \( S \) is a finite set of powers of the form \( \lambda = r\epsilon \) with integer \( r \), \( K_\lambda \geq 0 \) is an integer number corresponding to the maximal power of the logarithm.
Expanding in a vicinity of each singular point.
Take \( x = 0 \).
The expansion is

\[
U(x) = \sum_{\lambda \in S} x^\lambda \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(n + \lambda, k) x^n \ln^k x,
\]

where \( S \) is a finite set of powers of the form \( \lambda = r \epsilon \) with integer \( r \), \( K_\lambda \geq 0 \) is an integer number corresponding to the maximal power of the logarithm.
The goal is to determine \( S \), \( K_\lambda \), and the matrix coefficients \( C(n + \lambda, k) \).
Suppose that DE are in a global normalized Fuchsian form

\[ M(x) = \frac{A_0}{x} + \sum_{k=1}^{s} \frac{A_k}{x - x_k} \]
Suppose that DE are in a global normalized Fuchsian form

\[ M(x) = \frac{A_0}{x} + \sum_{k=1}^{s} \frac{A_k}{x - x_k} \]

and for any \( k = 0, \ldots, s \) the matrix \( A_k \) is free of resonances, i.e. the difference of any two of its distinct eigenvalues is not integer.
Suppose that DE are in a global normalized Fuchsian form

\[ M(x) = \frac{A_0}{x} + \sum_{k=1}^{s} \frac{A_k}{x - x_k} \]

and for any \( k = 0, \ldots, s \) the matrix \( A_k \) is free of resonances, i.e. the difference of any two of its distinct eigenvalues is not integer.

In particular, the ‘elliptic’ cases, as a rule, can algorithmically be reduced to a global normalized Fuchsian form using, e.g., the algorithm of Lee [R.N. Lee’14].
Multiply both sides by the common denominator \( xQ(x) \), where

\[
Q(x) = \prod_{k} (x - x_k) = \sum_{m=0}^{s} q_m x^m.
\]

with \( q_0 \neq 0 \).
Multiply both sides by the common denominator \( xQ(x) \), where

\[
Q(x) = \prod_{k} (x - x_k) = \sum_{m=0}^{s} q_m x^m.
\]

with \( q_0 \neq 0 \).
Define the polynomial matrix \( B(x, \alpha) \) and its coefficients \( B_m(\alpha) \) by

\[
B(x, \alpha) = Q(x)(xM(x) - \alpha) = \sum_{m=0}^{N} B_m(\alpha) x^m.
\]

with \( B_0(\alpha) = q_0(A_0 - \alpha) \).
Then the recurrence relations read

\[- \text{BJF}(B_0(\lambda + n), -q_0, K_\lambda)C(\lambda + n, 0..K_\lambda)\]

\[= \sum_{m=1}^{s} \text{BJF}(B_m(\lambda + n - m), -q_m, K_\lambda)C(\lambda + n - m, 0..K_\lambda)\]
Then the recurrence relations read

\[- \text{BJF}(B_0(\lambda + n), -q_0, K_\lambda)C(\lambda + n, 0..K_\lambda)\]

\[= \sum_{m=1}^{s} \text{BJF}(B_m(\lambda + n - m), -q_m, K_\lambda)C(\lambda + n - m, 0..K_\lambda).\]

(BJF means ‘Block Jordan Form’.)
Then the recurrence relations read

\[ - \text{BJF}(B_0(\lambda + n), -q_0, K_\lambda) \mathcal{C}(\lambda + n, 0..K_\lambda) \]

\[ = \sum_{m=1}^{s} \text{BJF}(B_m(\lambda + n - m), -q_m, K_\lambda) \mathcal{C}(\lambda + n - m, 0..K_\lambda) \, . \]

(BJF means ‘Block Jordan Form’.)

\[ \mathcal{C}(\alpha, 0..K) = \begin{bmatrix} C(\alpha, 0) \\ \vdots \\ C(\alpha, K) \end{bmatrix} \]

denotes a \((K + 1)N \times N\) matrix built from blocks \(C(\alpha, k)\).
Then the recurrence relations read

\[- \text{BJF}(B_0(\lambda + n), -q_0, K_\lambda)C (\lambda + n, 0..K_\lambda)\]

\[= \sum_{m=1}^{s} \text{BJF}(B_m(\lambda + n - m), -q_m, K_\lambda)C (\lambda + n - m, 0..K_\lambda).\]

(BJF means ‘Block Jordan Form’.)

\[C (\alpha, 0..K) = \begin{bmatrix}
C (\alpha, 0) \\
\vdots \\
C (\alpha, K)
\end{bmatrix}\]

denotes a \((K + 1)N \times N\) matrix built from blocks \(C (\alpha, k),\)

\[\text{BJF}(A, B, K) = \begin{bmatrix}
A & B & 0 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ddots & B \\
0 & 0 & 0 & A
\end{bmatrix}_{K+1}\]
The evolution operator $U$ is determined up to a multiplication by a constant matrix from the right.
The evolution operator $U$ is determined up to a multiplication by a constant matrix from the right. We fix it by the condition

$$U(x) \xrightarrow{x \to 0} x^{A_0}$$
The evolution operator $U$ is determined up to a multiplication by a constant matrix from the right. We fix it by the condition

$$U(x) \stackrel{x \to 0}{\sim} x^{A_0}$$

We determine $S$, i.e. the set of distinct eigenvalues of $A_0$, and $K$, i.e. the highest power of the logarithm, and the leading coefficients $C(\lambda, k)$, representing

$$x^{A_0} = \sum_{\lambda \in S} x^\lambda \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(\lambda, k) \ln^k x.$$
The matrix $-\mathrm{BJF}(B_0(\lambda + n), -q_0, K_{\lambda})$ on the lhs of the difference equation is invertible for $\lambda \in S$ and $n > 0$ because

$$\det \mathrm{BJF}(B_0(\lambda + n), -q_0, K_{\lambda}) = (\det B_0(\lambda + n))^{K_{\lambda}+1}$$

$$= q_0^{(K_{\lambda}+1)n} \left[\det(A_0 - \lambda - n)\right]^{K_{\lambda}+1}$$

with $q_0 \neq 0$ and (due to the absence of resonances in $A_0$)

$\det(A_0 - \lambda - n) \neq 0$, 
The matrix $-\mathbf{BJF}(B_0(\lambda + n), -q_0, K_\lambda)$ on the lhs of the difference equation is invertible for $\lambda \in S$ and $n > 0$ because

$$\det \mathbf{BJF}(B_0(\lambda + n), -q_0, K_\lambda) = (\det B_0(\lambda + n))^{K_\lambda + 1} = q_0^{(K_\lambda + 1)n} [\det(A_0 - \lambda - n)]^{K_\lambda + 1}$$

with $q_0 \neq 0$ and (due to the absence of resonances in $A_0$) $\det(A_0 - \lambda - n) \neq 0$.

The recurrence relation takes the form

$$C(\lambda + n, 0..K_\lambda) = \sum_{m=1}^{s} T(\lambda, n, m)C(\lambda + n - m, 0..K_\lambda),$$
The matrix $-\text{BJF}(B_0(\lambda + n), -q_0, K_\lambda)$ on the lhs of the difference equation is invertible for $\lambda \in S$ and $n > 0$ because
\[
\det \text{BJF}(B_0(\lambda + n), -q_0, K_\lambda) = (\det B_0(\lambda + n))^{K_\lambda + 1} = q_0^{(K_\lambda + 1)n} [\det(A_0 - \lambda - n)]^{K_\lambda + 1}
\]
with $q_0 \neq 0$ and (due to the absence of resonances in $A_0$)
\[
\det(A_0 - \lambda - n) \neq 0,
\]
The recurrence relation takes the form
\[
C(\lambda + n, 0..K_\lambda) = \sum_{m=1}^{s} T(\lambda, n, m)C(\lambda + n - m, 0..K_\lambda),
\]
with
\[
T(\lambda, n, m) = - [\text{BJF}(B_0(\lambda + n), -q_0, K_\lambda)]^{-1}
\]
\[
\times \text{BJF}(B_m(\lambda + n - m), -q_m, K_\lambda).
\]
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms.
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms. The data necessary to obtain the expansion at $x = 0$: 
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms. The data necessary to obtain the expansion at $x = 0$:

- The set $S = \{\lambda_1, \lambda_2, \ldots\}$ of the eigenvalues of the matrix residue $A_0$. 
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms. The data necessary to obtain the expansion at $x = 0$:

- The set $S = \{\lambda_1, \lambda_2, \ldots \}$ of the eigenvalues of the matrix residue $A_0$.
- For each $\lambda \in S$:
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms. The data necessary to obtain the expansion at $x = 0$:

- The set $S = \{\lambda_1, \lambda_2, \ldots\}$ of the eigenvalues of the matrix residue $A_0$.
- For each $\lambda \in S$:
  - the maximal power of the logarithm $K_\lambda$ and the leading coefficients $C(\lambda, 0..K_\lambda)$ determined by the boundary conditions.
This finite-order recurrence relation, together with the initial conditions, is solved with a linear growth of the computational complexity wrt the number of expansion terms. The data necessary to obtain the expansion at $x = 0$:

- The set $S = \{\lambda_1, \lambda_2, \ldots\}$ of the eigenvalues of the matrix residue $A_0$.
- For each $\lambda \in S$:
  - the maximal power of the logarithm $K_\lambda$ and the leading coefficients $C(\lambda, 0..K_\lambda)$ determined by the boundary conditions.
  - the matrix coefficients $T(\lambda, n, 1), \ldots, T(\lambda, n, s)$ which are $(K_\lambda + 1) N \times (K_\lambda + 1) N$ matrices, where the dependence on $n$ is explicit.
There are three possible scenarios of evaluating $C(\lambda + n, k)$:

- with analytic numbers exactly in $\epsilon$, 

There are three possible scenarios of evaluating \( C (\lambda + n, k) \):

- with analytic numbers exactly in \( \epsilon \),
- with analytic numbers in an expansion in \( \epsilon \),
There are three possible scenarios of evaluating $C (\lambda + n, k)$:

- with analytic numbers exactly in $\epsilon$,
- with analytic numbers in an expansion in $\epsilon$,
- with approximate numerical numbers in an expansion in $\epsilon$. 
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series.
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series. In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points.
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series.

In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points. Suppose that the next singular point closest to the origin is $x = 1$. 

---

Solving differential equations for Feynman integrals by expansions near singular points

Matching
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series.

In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points. Suppose that the next singular point closest to the origin is \( x = 1 \).

We can construct the evolution operator also in an expansion near this point.
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series.
In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points.
Suppose that the next singular point closest to the origin is \( x = 1 \).
We can construct the evolution operator also in an expansion near this point. Let it be \( \tilde{U}(x) \).
After solving the recurrence relations, the evolution operator can be evaluated within the convergence region of the power series.

In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points. Suppose that the next singular point closest to the origin is \( x = 1 \).

We can construct the evolution operator also in an expansion near this point. Let it be \( \tilde{U}(x) \). Due to the freedom in definition of the evolution operator, we have

\[
U(x) = \tilde{U}(x) L.
\]

where \( L \) is a constant matrix.
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. 
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. We obtain $L = \tilde{U}^{-1} (1/2) U (1/2)$,
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. We obtain $L = \tilde{U}^{-1} (1/2) U (1/2)$, so that in the whole convergence region of $\tilde{U}$ we have

$$U(x) = \tilde{U}(x) \tilde{U}^{-1} (1/2) U (1/2).$$
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. We obtain $L = \tilde{U}^{-1} (1/2) U (1/2)$, so that in the whole convergence region of $\tilde{U}$ we have

$$U (x) = \tilde{U} (x) \tilde{U}^{-1} (1/2) U (1/2).$$

Analytic continuation to the whole complex plane of $x$. 
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. We obtain $L = \tilde{U}^{-1}(1/2) U(1/2)$, so that in the whole convergence region of $\tilde{U}$ we have

$$U(x) = \tilde{U}(x) \tilde{U}^{-1}(1/2) U(1/2).$$

Analytic continuation to the whole complex plane of $x$. In the case where the singularities lie on the real axis and if we are interested in the evaluation for real $x$, we can avoid expansions near regular points.
To fix $L$, choose a point which belongs to both regions of convergence, e.g. $x = 1/2$. We obtain $L = \tilde{U}^{-1} (1/2) U (1/2)$, so that in the whole convergence region of $\tilde{U}$ we have

$$U(x) = \tilde{U}(x) \tilde{U}^{-1} (1/2) U (1/2) .$$

Analytic continuation to the whole complex plane of $x$. In the case where the singularities lie on the real axis and if we are interested in the evaluation for real $x$, we can avoid expansions near regular points. A sequence of the singular points

$$x_0 < x_1 < \ldots x_s < \infty = x_{s+1} = x_{-1}$$
To fix \( L \), choose a point which belongs to both regions of convergence, e.g. \( x = 1/2 \). We obtain \( L = \tilde{U}^{-1} (1/2) U (1/2) \), so that in the whole convergence region of \( \tilde{U} \) we have

\[
U (x) = \tilde{U} (x) \tilde{U}^{-1} (1/2) U (1/2) .
\]

Analytic continuation to the whole complex plane of \( x \). In the case where the singularities lie on the real axis and if we are interested in the evaluation for real \( x \), we can avoid expansions near regular points. A sequence of the singular points

\[
x_0 < x_1 < \ldots x_s < \infty = x_{s+1} = x_{-1}
\]

then for each \( 0 \leq k \leq s \) we make the (Moebius) transformation

\[
y_k(x) = \frac{ax + b}{cx + d}
\]
which maps the points $x_{k-1}$, $x_k$, $x_{k+1}$ to $\mp 1$, 0, $\pm 1$, respectively.
which maps the points $x_{k-1}$, $x_k$, $x_{k+1}$ to $\mp 1$, 0, $\pm 1$, respectively.

Explicitly,

$$y_k(x) = \pm \frac{(x - x_k)(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$
which maps the points $x_{k-1}$, $x_k$, $x_{k+1}$ to $\pm 1$, 0, $\pm 1$, respectively.

Explicitly,

$$y_k(x) = \pm \frac{(x - x_k)(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$

The boundary conditions are included at one of the points, e.g. $x = 0$ and then series expansions at other points can be obtained by matching, step by step, pairs of expansions at neighboring points.
Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines
Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines

\[ F_{a_1, \ldots, a_{14}} = \]

\[ \int \cdots \int \frac{d^D k_1 \ldots d^D k_4 (k_1 \cdot p)^{a_6} (k_2 \cdot p)^{a_7} (k_3 \cdot p)^{a_8} (k_4 \cdot p)^{a_9}}{(-k_1^2)^{a_1} (-k_2^2)^{a_2} (m^2 - k_3^2)^{a_3} (m^2 - k_4^2)^{a_4} (m^2 - (\sum k_i + p)^2)^{a_5}} \times (k_1 \cdot k_2)^{a_{10}} (k_1 \cdot k_3)^{a_{11}} (k_1 \cdot k_4)^{a_{12}} (k_2 \cdot k_3)^{a_{13}} (k_2 \cdot k_4)^{a_{14}}, \]

with \( x = p^2 / m^2 \).
Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines

\[
F_{a_1, \ldots, a_{14}} = \\
\int \cdots \int \frac{d^D k_1 \cdots d^D k_4 (k_1 \cdot p)^{a_6} (k_2 \cdot p)^{a_7} (k_3 \cdot p)^{a_8} (k_4 \cdot p)^{a_9}}{(-k_1^2)^{a_1} (-k_2^2)^{a_2} (m^2 - k_3^2)^{a_3} (m^2 - k_4^2)^{a_4} (m^2 - (\sum k_i + p)^2)^{a_5}} \\
\times (k_1 \cdot k_2)^{a_{10}} (k_1 \cdot k_3)^{a_{11}} (k_1 \cdot k_4)^{a_{12}} (k_2 \cdot k_3)^{a_{13}} (k_2 \cdot k_4)^{a_{14}},
\]

with \( x = p^2/m^2 \).

There are four master integrals in this family.
Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines

\[
F_{a_1, \ldots, a_{14}} = \\
\int \cdots \int \frac{d^D k_1 \cdots d^D k_4 (k_1 \cdot p)^{a_6} (k_2 \cdot p)^{a_7} (k_3 \cdot p)^{a_8} (k_4 \cdot p)^{a_9}}{(-k_1^2)^{a_1} (-k_2^2)^{a_2} (m^2 - k_3^2)^{a_3} (m^2 - k_4^2)^{a_4} (m^2 - (\sum k_i + p)^2)^{a_5}} \times (k_1 \cdot k_2)^{a_{10}} (k_1 \cdot k_3)^{a_{11}} (k_1 \cdot k_4)^{a_{12}} (k_2 \cdot k_3)^{a_{13}} (k_2 \cdot k_4)^{a_{14}},
\]

with \( x = p^2/m^2 \).

There are four master integrals in this family. We choose

\[
J_0 = \{ F_{1,1,1,1,1,0,\ldots,0}, F_{1,1,2,1,1,0,\ldots,0}, F_{1,2,1,1,1,0,\ldots,0}, F_{1,2,1,1,2,0,\ldots,0} \}.\]
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.

The singular points are $x_0 = 0, x_1 = 1, x_2 = 9$ and $x_3 = x_{-1} = \infty$. 
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.

The singular points are $x_0 = 0, x_1 = 1, x_2 = 9$ and $x_3 = x_{-1} = \infty$.

The variable changes corresponding to the singular points are:

$f_0 = x/(2 - x), f_1 = (x - 1)/(1 + 7x/9), f_2 = (9 - x)/(7 + x), f_3 = -9/(2x - 9)$. 
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.

The singular points are $x_0 = 0, x_1 = 1, x_2 = 9$ and $x_3 = x_{-1} = \infty$.

The variable changes corresponding to the singular points are

\[ f_0 = \frac{x}{2 - x}, \quad f_1 = \frac{x - 1}{1 + \frac{7x}{9}}, \]
\[ f_2 = \frac{9 - x}{7 + x}, \quad f_3 = -\frac{9}{2x - 9}. \]

In new variables, the radii of convergence are equal to 1.
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.

The singular points are $x_0 = 0$, $x_1 = 1$, $x_2 = 9$ and $x_3 = x_{-1} = \infty$.

The variable changes corresponding to the singular points are $f_0 = x/(2 - x)$, $f_1 = (x - 1)/(1 + 7x/9)$, $f_2 = (9 - x)/(7 + x)$, $f_3 = -9/(2x - 9)$.

In new variables, the radii of convergence are equal to 1.

For adjacent regions $i$ and $i + 1$ we search the best possible matching point which is such $x$ that it lies between $x_i$ and $x_{i+1}$ and that $|f_i(x)| = |f_{i+1}(x)|$. 
We turn to the basis $J = T^{-1} \cdot J_0$ where DE are in a global normalized Fuchsian form.

The singular points are $x_0 = 0, x_1 = 1, x_2 = 9$ and $x_3 = x_{-1} = \infty$.

The variable changes corresponding to the singular points are

$f_0 = x/(2 - x), f_1 = (x - 1)/(1 + 7x/9), f_2 = (9 - x)/(7 + x), f_3 = -9/(2x - 9)$.

In new variables, the radii of convergence are equal to 1.

For adjacent regions $i$ and $i + 1$ we search the best possible matching point which is such $x$ that it lies between $x_i$ and $x_{i+1}$ and that $|f_i(x)| = |f_{i+1}(x)|$.

Matching points are $\{-3, 3(3 - 2\sqrt{2}), 3, 3(3 + 2\sqrt{2})\}$. 
To fix boundary conditions we choose the point \( x = 0 \) where the integrals of the given family become vacuum integrals.
To fix boundary conditions we choose the point $x = 0$ where the integrals of the given family become vacuum integrals.

To evaluate the four master integrals at $x = 0$ we derive onefold Mellin-Barnes representations for them and obtain the possibility to achieve a high precision for any given coefficient in the $\varepsilon$-expansion.
To fix boundary conditions we choose the point $x = 0$ where the integrals of the given family become vacuum integrals.

To evaluate the four master integrals at $x = 0$ we derive onefold Mellin-Barnes representations for them and obtain the possibility to achieve a high precision for any given coefficient in the $\varepsilon$-expansion.

Using matching we perform an analytic continuation and obtain convergent series expansion in each region.
To fix boundary conditions we choose the point $x = 0$ where the integrals of the given family become vacuum integrals.

To evaluate the four master integrals at $x = 0$ we derive onefold Mellin-Barnes representations for them and obtain the possibility to achieve a high precision for any given coefficient in the $\varepsilon$-expansion.

Using matching we perform an analytic continuation and obtain convergent series expansion in each region.

The code `DESS.m` as well files with input data can be downloaded from https://bitbucket.org/feynmanintegrals/dess.
For example, at \( x_0 = 25 \), we obtain the following result (shown with a truncation to 10 digits) for the first primary integral:

\[
\begin{align*}
&- \frac{0.25}{\epsilon^4} + \frac{2.125}{\epsilon^3} - \frac{0.2391337000}{\epsilon^2} - \frac{5.2663306926}{\epsilon} \\
&\quad - 185.9464179437 + 6.5261388472 i \\
&\quad - (1825.1476432369 - 48.9550593728 i)\epsilon \\
&\quad - (8406.8551978029 - 176.0638485153 i)\epsilon^2 \\
&\quad - (58330.4283767260 - 401.9617475893 i)\epsilon^3.
\end{align*}
\]
For example, at $x_0 = 25$, we obtain the following result (shown with a truncation to 10 digits) for the first primary integral:

\[-\frac{0.25}{\epsilon^4} + \frac{2.125}{\epsilon^3} - \frac{0.2391337000}{\epsilon^2} - \frac{5.2663306926}{\epsilon} - 185.9464179437 + 6.5261388472 i - (1825.1476432369 - 48.9550593728 i)\epsilon - (8406.8551978029 - 176.0638485153 i)\epsilon^2 - (58330.4283767260 - 401.9617475893 i)\epsilon^3.\]

We checked results at sample points (between singular point and matching points) with FIESTA [A.V. Smirnov’16].
We presented an algorithm for the numerical evaluation of a set of master integrals depending nontrivially on one variable at a given real point with a required accuracy.
We presented an algorithm for the numerical evaluation of a set of master integrals depending nontrivially on one variable at a given real point with a required accuracy.

The algorithm is oriented at situations where canonical form of the DE is impossible.
We presented an algorithm for the numerical evaluation of a set of master integrals depending nontrivially on one variable at a given real point with a required accuracy.

The algorithm is oriented at situations where canonical form of the DE is impossible.

We provided a computer implementation of the algorithm in a simple example.
We presented an algorithm for the numerical evaluation of a set of master integrals depending nontrivially on one variable at a given real point with a required accuracy.

The algorithm is oriented at situations where canonical form of the DE is impossible.

We provided a computer implementation of the algorithm in a simple example.

This code is similar in spirit to the well-known existing codes to evaluate harmonic polylogarithms and multiple polylogarithms, where the problem of evaluation reduces to summing up appropriate series.
We presented an algorithm for the numerical evaluation of a set of master integrals depending nontrivially on one variable at a given real point with a required accuracy.

The algorithm is oriented at situations where canonical form of the DE is impossible.

We provided a computer implementation of the algorithm in a simple example.

This code is similar in spirit to the well-known existing codes to evaluate harmonic polylogarithms and multiple polylogarithms, where the problem of evaluation reduces to summing up appropriate series.

Our public package includes tools for a decomposition of the real axis into domains, a subsequent mapping and an introduction of appropriate new variables.
We are thinking of a more general package which would include an automation of as many steps of the presented algorithm as possible. The future package needs at least an implementation of the algorithm to solve difference equations for series expansions at the singular points.
We are thinking of a more general package which would include an automation of as many steps of the presented algorithm as possible. The future package needs at least an implementation of the algorithm to solve difference equations for series expansions at the singular points.

One can hardly construct a general algorithm to fix boundary conditions because, usually, the choice of the corresponding point and the way to obtain data for the boundary conditions is done in every situation in a special way.
We are thinking of a more general package which would include an automation of as many steps of the presented algorithm as possible. The future package needs at least an implementation of the algorithm to solve difference equations for series expansions at the singular points.

One can hardly construct a general algorithm to fix boundary conditions because, usually, the choice of the corresponding point and the way to obtain data for the boundary conditions is done in every situation in a special way.

Still we can suggest a format for including information about the boundary conditions for using it in our future package.
Our future package would check if a given system of DE is already in a global Fuchsian normalized form, with singularities on the real axis, and, if this is true, the package would automatically construct the evolution operator in an expansion up to a required order.
Our future package would check if a given system of DE is already in a global Fuchsian normalized form, with singularities on the real axis, and, if this is true, the package would automatically construct the evolution operator in an expansion up to a required order.

One can apply DE even in the case of one-scale integrals by introducing an extra scale, solving DE with the respect to the ratio of the two scales, $x$, and then picking a contribution to the expansion at the point where $x$ tends to its primary value [J.M. Henn, A.V. Smirnov & V.S.’13].
From the point with boundary conditions (e.g. $x = 0$) to the given point (e.g. $x = 1$),
From the point with boundary conditions (e.g. \( x = 0 \)) to the given point (e.g. \( x = 1 \)), with the help of the operator \( \tilde{U}^{-1}(1/2)U(1/2) \).
From the point with boundary conditions (e.g. \( x = 0 \)) to the given point (e.g. \( x = 1 \)), with the help of the operator \( \hat{U}^{-1}(1/2)U(1/2) \).

An example of using this strategy in concrete situation: [B. A. Kniehl, A. F. Pikelner & O. L. Veretin’17].
From the point with boundary conditions (e.g. $x = 0$) to the given point (e.g. $x = 1$), with the help of the operator $\tilde{U}^{-1}(1/2)U(1/2)$.

An example of using this strategy in concrete situation: [B. A. Kniehl, A. F. Pikelner & O. L. Veretin’17].

We are thinking about adjusting our package to this case in general situation.