One-loop amplitudes with off-shell gluons

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• Factorized cross section calculation
• Off-shell amplitudes
• KaTie: for parton-level event generation with $k_T$-dependent initial states
• Off-shell one-loop amplitudes
Four jets with $k_T$-factorization

- $\Delta S$ is the azimuthal angle between the sum of the two hardest jets and the sum of the two softest jets.
- This variable has no distribution at LO in collinear factorization: pairs would have to be back-to-back.
- $k_T$-factorization allows for the necessary momentum inbalance.

$p p \to 4$ jets $X$

CMS data

$\sqrt{s} = 7$ TeV

$|y| < 4.7$

$p_T^{1^{st}, 2^{nd}} > 50$ GeV

$p_T^{3^{rd}, 4^{th}} > 20$ GeV

$\Delta S$ vs. $1/\sigma \, d\sigma/\Delta S$
General formula for cross section with $\pi^* \in \{g^*, q^*, \bar{q}^*\}$:

$$d\sigma(h_1(p_1)h_2(p_2) \rightarrow Y) = \sum_{a,b} \int d^4k_1 P_{1,a}(k_1) \int d^4k_2 P_{2,b}(k_2) d\hat{\sigma}(\pi^*_a(k_1)\pi^*_b(k_2) \rightarrow Y)$$

Collinear factorization: $P_{i,a}(k) = \int_0^1 \frac{dx}{x} f_{i,a}(x, \mu) \delta^4(k - xp_i)$

$k_T$-factorization: $P_{i,a}(k) = \int \frac{d^2k_T}{\pi} \int_0^1 \frac{dx}{x} F_{i,a}(x, |k_T|, \mu) \delta^4(k - x p_i - k_T)$

- The *parton level* cross section $d\hat{\sigma}(\pi^*_a(k_1)\pi^*_b(k_2) \rightarrow Y)$ can be calculated within perturbative QCD.
- The *parton distribution functions* $f_{i,a}$ and $F_{i,a}$ must be modelled and fit against data.
- Unphysical scale $\mu$ is a price to pay, but its dependence is calculable within perturbative QCD via *evolution equations.*
Factorization

To separate a perturbatively calculable partonic process from universal models for the hadrons in hadron scattering.

Different factorization formulas are applicable for different kinematical regions in terms of the hard scale $P_T$, the transverse momentum inbalance $k_T$, and the saturation scale $Q_s$.

Collinear factorization

$$d\sigma_{AB \rightarrow X} = \int dx_A \sum_a \int dx_B \sum_b f_{a/A}(x_A, \mu) f_{b/B}(x_B, \mu) d\hat{\sigma}_{ab \rightarrow X}(x_A, x_B, \mu)$$

Central scattering: $x_A \approx x_B \sim 1$, and $P_T \gg k_T \gg Q_s$.
Partonic cross section $d\hat{\sigma}_{ab}$ is calculated with on-shell initial-state partons.

High Energy Factorization

Catani, Ciafaloni, Hautmann 1991
Ellis, Collins 1991

$$d\sigma_{AB \rightarrow X} = \int dk_T^2 \int dx_A \int dx_B \sum_b \mathcal{F}_{g^*/A}(x_A, k_T, \mu) f_{b/B}(x_B, \mu) d\hat{\sigma}_{g^*b \rightarrow X}(x_A, x_B, k_T, \mu)$$

Eg. forward-central scattering: $x_B \gg x_A$, and $P_T \sim k_T \gg Q_s$.

Unintegrated gluon density $\mathcal{F}_{g^*/A}(x_A, k_T, \mu)$ evolved following BFKL or similar.
Partonic cross section $d\hat{\sigma}_{g^*b}$ is calculated with an off-shell initial-state gluon.
Factorization

To separate a perturbatively calculable partonic process from universal models for the hadrons in hadron scattering.

Generalized TMD factorization

\[ d\sigma_{AB \rightarrow X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu) f_{b/B}(x_B, \mu) d\hat{\sigma}_{gb \rightarrow X}^{(i)}(x_A, x_B, k_T, \mu) \]

For \( x_A \ll 1 \) and \( P_T \gg k_T \sim Q_s \).

Applicable to \( p - A \) (dilute-dense) collisions.

TMD gluon distributions \( \phi_{gb}^{(i)}(x_A, k_T, \mu) \) satisfy non-linear evolution equations, and admit saturation.

Partonic cross section \( d\hat{\sigma}_{gb}^{(i)} \) depends on color-structure \( i \), and is calculated with on-shell initial-state partons.

Improved generalized TMD factorization

Model interpolating between High Energy Factorization and Generalized TMD factorization: \( P_T \gtrsim k_T \gtrsim Q_s \).

Partonic cross section \( d\hat{\sigma}_{gb}^{(i)} \) depends on color-structure \( i \), and is calculated with off-shell initial-state partons.
Factorization for hadron scattering

General formula for cross section with $\pi^* \in \{g^*, q^*, \bar{q}^*\}$:

$$d\sigma(h_1(p_1)h_2(p_2) \rightarrow Y) = \sum_{a,b} \int d^4k_1 \mathcal{P}_{1,a}(k_1) \int d^4k_2 \mathcal{P}_{2,b}(k_2) \ d\hat{\sigma}(\pi^*_a(k_1)\pi^*_b(k_2) \rightarrow Y)$$

Collinear factorization: $\mathcal{P}_{i,a}(k) = \int_0^1 \frac{dx}{x} f_{i,a}(x, \mu) \delta^4(k - x p_i)$

$k_T$-factorization: $\mathcal{P}_{i,a}(k) = \int \frac{d^2k_T}{\pi} \int_0^1 \frac{dx}{x} \mathcal{F}_{i,a}(x, |k_T|, \mu) \delta^4(k - x p_i - k_T)$

$$\hat{\sigma} = \int d\Phi(1, 2 \rightarrow 3, 4, \ldots, n) |\mathcal{M}(1, 2, \ldots, n)|^2 \ O(p_3, p_4, \ldots, p_n)$$

phase space includes summation over color and spin
squared amplitude calculated perturbatively
observable includes phase space cuts, or jet algorithm
Gauge invariance

In order to be physically relevant, any scattering amplitude following the constructive definition given before must satisfy the following

Freedom in choice of gluon propagator:

\[
\begin{cases}
  \frac{-i}{k^2} \left[ g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right] \\
  \frac{-i}{k^2} \left[ g^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k \cdot n} + (n^2 + \xi k^2) \frac{k^\mu k^\nu}{(k \cdot n)^2} \right]
\end{cases}
\]

Ward identity:

\[\mu \epsilon^\mu(k) \rightarrow \mu k^\mu = 0\]

- Only holds if all external particles are on-shell.
- \(k_T\)-factorization requires off-shell initial-state momenta \(k^\mu = p^\mu + k_T^\mu\).
- How to define amplitudes with off-shell initial-state momenta?
Amplitudes with off-shell gluons
Amplitudes with off-shell gluons

\( n \)-parton amplitude is a function of \( n \) momenta \( k_1, k_2, \ldots, k_n \)
and \( n \) directions \( p_1, p_2, \ldots, p_n \)
Amplitudes with off-shell gluons

\( n \)-parton amplitude is a function of \( n \) momenta \( k_1, k_2, \ldots, k_n \) and \( n \) directions \( p_1, p_2, \ldots, p_n \), satisfying the conditions

\[
\begin{align*}
k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 & \text{momentum conservation} \\
p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 & \text{light-likeness} \\
p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n &= 0 & \text{eikonal condition}
\end{align*}
\]
$n$-parton amplitude is a function of $n$ momenta $k_1, k_2, \ldots, k_n$ and $n$ directions $p_1, p_2, \ldots, p_n$, satisfying the conditions

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\]

With the help of an auxiliary four-vector $q^\mu$ with $q^2 = 0$, we define

\[
k_T^\mu(q) = k^\mu - x(q) p^\mu \quad \text{with} \quad x(q) \equiv \frac{q \cdot k}{q \cdot p}
\]
**Amplitudes with off-shell gluons**

\( n \)-parton amplitude is a function of \( n \) momenta \( k_1, k_2, \ldots, k_n \) and \( n \) directions \( p_1, p_2, \ldots, p_n \), satisfying the conditions

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\]

With the help of an auxiliary four-vector \( q^\mu \) with \( q^2 = 0 \), we define

\[
k^\mu_T(q) = k^\mu - \chi(q)p^\mu \quad \text{with} \quad \chi(q) \equiv \frac{q \cdot k}{q \cdot p}
\]

Construct \( k^\mu_T \) explicitly in terms of \( p^\mu \) and \( q^\mu \):

\[
k^\mu_T(q) = -\frac{\kappa}{2} \varepsilon^\mu - \frac{\kappa^*}{2} \varepsilon^{*\mu} \quad \text{with} \quad \begin{cases} 
\varepsilon^\mu = \frac{\langle p|\gamma^\mu|q \rangle}{[pq]} \\
\varepsilon^{*\mu} = \frac{\langle q|\gamma^\mu|p \rangle}{\langle qp \rangle} \\
\kappa = \frac{\langle q|k|p \rangle}{\langle qp \rangle} \\
\kappa^* = \frac{\langle p|k|q \rangle}{[pq]}
\end{cases}
\]

\( k^2 = -\kappa \kappa^* \) is independent of \( q^\mu \), but also individually \( \kappa \) and \( \kappa^* \) are independent of \( q^\mu \).
Example of a 4-gluon amplitude

\[ \mathcal{A}(1^*, 2^-, 3^*, 4^+) = \frac{\langle 13 \rangle^3 \langle 13 \rangle^3}{\langle 34 \rangle \langle 41 \rangle \langle 1|k_3 + p_3|3 \rangle \langle 3|k_1 + p_4|1 \rangle \langle 32 \rangle \langle 21 \rangle} + \frac{1}{\kappa_1^* \kappa_3} \frac{\langle 12 \rangle^3 \langle 43 \rangle^3}{\langle 2|k_3|4 \rangle \langle 1|k_3 + p_4|3 \rangle (k_3 + p_4)^2} + \frac{1}{\kappa_1 \kappa_3^*} \frac{\langle 23 \rangle^3 \langle 14 \rangle^3}{\langle 2|k_1|4 \rangle \langle 3|k_1 + p_4|1 \rangle (k_1 + p_4)^2} \]

- Eventual matrix element needs factor \( k_1^2 k_3^2 = |\kappa_1|^2 |\kappa_3|^2 \). This must not be included at the amplitude level not to spoil analytic structure.

- Last two terms dominate for \( |k_1| \to 0 \) and \( |k_3| \to 0 \), and give the on-shell helicity amplitudes in that limit.

\[ \mathcal{A}(1^*, 2^-, 3^*, 4^+) \xrightarrow{|k_1|, |k_3| \to 0} \frac{1}{\kappa_1^* \kappa_3} \mathcal{A}(1^-, 2^-, 3^+, 4^+) + \frac{1}{\kappa_1 \kappa_3^*} \mathcal{A}(1^+, 2^-, 3^-, 4^+) \]

- Coherent sum of amplitudes becomes incoherent sum of squared amplitudes via angular integrations for \( \vec{k}_{1T} \) and \( \vec{k}_{3T} \).
Amplitudes with off-shell gluons

AvH, Kutak, Kotko 2013:
Embed the process in an on-shell process with auxiliary partons

\[ p_\mu^A = \Lambda p_\mu^1 - \frac{\kappa_1^*}{2} \epsilon_{1*}^\mu \]

\[ p_{\mu A'} = - (\Lambda - x_1) p_\mu^1 - \frac{\kappa_1}{2} \epsilon_1^\mu \]
Amplitudes with off-shell gluons

AvH, Kutak, Kotko 2013:
Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.

\[
\begin{align*}
p_A^\mu &= \Lambda p_1^\mu - \frac{\kappa_1^*}{2} \varepsilon_1^{*\mu} \\
p_A'^\mu &= -(\Lambda - x_1) p_1^\mu - \frac{\kappa_1}{2} \varepsilon_1^{\mu} \quad (\Lambda \to \infty)
\end{align*}
\]

\[
\begin{align*}
&= i T_{i,j}^a \ p_i^\mu \\
&\xrightarrow{K} \delta_{i,j} \frac{i}{p_1 \cdot K}
\end{align*}
\]
Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013, AvH, Kutak, Salwa 2013:
Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.

\[ j \rightarrow i = -i \delta_{i,j} u(p_1) \]

\[ j \rightarrow i = -i T^a_{i,j} p_1^\mu \]

\[ K = \delta_{i,j} \frac{i}{p_1 \cdot K} \]
Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013, AvH, Kutak, Salwa 2013:
Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.

The BCFW recursion formula becomes

\[
2 \ldots n - 1 = \sum_{i=2}^{n-2} \sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D,
\]

\[
A_{i,h} = \frac{1}{K_{1,i}^2} \left[ \begin{array}{c} \hat{1} \\
\end{array} \right] \frac{1}{2} - h \left[ \begin{array}{c} \hat{i} \\
\end{array} \right] \hat{n} \]

\[
B_i = \frac{1}{2p_i \cdot K_{i,n}} \left[ \begin{array}{c} \hat{i} \\
\end{array} \right] \hat{n} \]

\[
C = \frac{1}{\kappa_1} \left[ \begin{array}{c} \hat{i} \\
\end{array} \right] \hat{n} \]

\[
D = \frac{1}{\kappa_1^\ast} \left[ \begin{array}{c} \hat{i} \\
\end{array} \right] \hat{n} \]

The hatted numbers label the shifted external gluons.
• parton level event generator, like Alpgen, Helac, MadGraph, etc.

• arbitrary processes within the standard model (including effective Hg) with several final-state particles.

• 0, 1, or 2 off-shell initial states.

• produces (partially un)weighted event files, for example in the LHEF format.

• requires LHAPDF. TMD PDFs can be provided as files containing rectangular grids, or with TMDlib Hautmann, Jung, Krämer, Mulders, Nocera, Rogers, Signori 2014.

• a calculation is steered by a single input file.

• employs an optimization phase in which the pre-samplers for all channels are optimized.

• during the generation phase several event files can be created in parallel.

• can generate (naively factorized) MPI events.

• event files can be processed further by parton-shower program like CASCADE
pp → Z + j in the forward direction

Ngroup = 1
Nfinst = 3

process = g u → mu+ mu− u  factor = 1  groups = 1  pNonQCD = 2 0 0
process = g u̅ → mu+ mu− u̅  factor = 1  groups = 1  pNonQCD = 2 0 0
process = g d → mu+ mu− d  factor = 1  groups = 1  pNonQCD = 2 0 0
process = g d̅ → mu+ mu− d̅  factor = 1  groups = 1  pNonQCD = 2 0 0

lhaSet = MSTW2008nlo68cl
offshell = 1 0
tmdTableDir = /home/user0/kTfac/tables/krzysztof02/
tmdpdf = g KMR-gluon.dat
tmdpdf = u KMR-u.dat
tmdpdf = u̅ KMR-ubar.dat
tmdpdf = d KMR-d.dat
tmdpdf = d̅ KMR-dbar.dat
tmdpdf = s KMR-s.dat
tmdpdf = s̅ KMR-sbar.dat
tmdpdf = c KMR-c.dat
tmdpdf = c̅ KMR-cbar.dat
tmdpdf = b KMR-b.dat
tmdpdf = b̅ KMR-bbar.dat

Nflavors = 5
helicity = sampling
Noptim = 1,000,000
Ecm = 7000
Esoft = 20

switch = withQCD Yes
switch = withQED Yes
switch = withWeak Yes
switch = withHiggs No
switch = withHG No
coupling = Gfermi 1.16639d-5
Initial steps have already been taken in the *parton reggeization approach* employing Lipatov’s effective action.

Hentschinski, Sabio Vera 2012  
Chachamis, Hentschinski, Madrigal, Sabio Vera 2012  
Nefedov, Saleev 2017  

The main problem is caused by linear denominators in loop integrals and the divergences they cause. In particular one would like to use a regularization that:

- is manifestly Lorentz covariant
- manifestly preserves gauge invariance
- can be used in combination with dimensional regularization
- is practical
Off-shell one-loop amplitudes

\[ k^\mu = xp^\mu + k_T^\mu \]

\[ p_A^\mu = \Lambda p^\mu + \alpha q^\mu + \beta k_T^\mu \]

\[ p_A'^\mu = k^\mu - p_A^\mu \]

where \( p, q \) are light-like with \( p \cdot q > 0 \), where \( p \cdot k_T = q \cdot k_T = 0 \), and where

\[
\alpha = \frac{-\beta^2 k_T^2}{\Lambda(p + q)^2}, \quad \beta = \frac{1}{1 + \sqrt{1 - x/\Lambda}} \quad \Rightarrow \quad \begin{cases} p_A^2 = p_A'^2 = 0 \\ p_A^\mu + p_A'^\mu = xp^\mu + k_T^\mu \end{cases}
\]

for any value of the parameter \( \Lambda \). Auxiliary quark propagators become eikonal for \( \Lambda \to \infty \):

\[
i \frac{p_A + K}{(p_A + K)^2} = \frac{i p'}{2p \cdot K} + \mathcal{O}(\Lambda^{-1})
\]
Off-shell one-loop amplitudes

\[ k^\mu = x p^\mu + k_T^\mu \]

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where \( p, q \) are light-like with \( p \cdot q > 0 \), where \( p \cdot k_T = q \cdot k_T = 0 \), and where

\[ \alpha = -\beta^2 k_T^2 \frac{\Lambda(p + q)^2}{\Lambda(p + q)^2} , \quad \beta = \frac{1}{1 + \sqrt{1 - x/\Lambda}} \]

\[ \Rightarrow \begin{cases} \quad p_A^2 = p_A'^2 = 0 \\ p_A^\mu + p_A'^\mu = xp^\mu + k_T^\mu \end{cases} \]

for any value of the parameter \( \Lambda \). Auxiliary quark propagators become eikonal for \( \Lambda \to \infty \):

\[ i \frac{p_A + K}{(p_A + K)^2} = i \frac{p}{2p \cdot K} + O(\Lambda^{-1}) \]

- \( \Lambda \)-parametrization provides natural regularization for linear denominators in loop integrals.

- Taking this limit after loop integration will lead to singularities \( \log \Lambda \).
Off-shell one-loop amplitudes

\begin{align*}
k^\mu &= xp^\mu + k_T^\mu \\
p_A^\mu &= \Lambda p^\mu + \alpha q^\mu + \beta k_T^\mu \\
p_A'^\mu &= k^\mu - p_A^\mu \\
k^\mu &= xp^\mu + k_T^\mu
\end{align*}

Well-known decomposition for on-shell one-loop amplitudes in terms of master integrals still holds for finite $\Lambda$.

\begin{align*}
A^{(1)} &= \int d^{4-2\epsilon} \ell \frac{N(\ell)}{\prod_i D_i(\ell)} = \sum_{i,j,k,l} c_4(i, j, k, l) I_4(i, j, k, l) + \sum_{i,j,k} c_3(i, j, k) I_3(i, j, k) \\
& \quad + \sum_{i,j} c_2(i, j) I_2(i, j) + \sum_i c_1(i) I_1(i) + R + O(\epsilon)
\end{align*}

The coefficients $c_4, c_3, c_2, c_1$ are rational functions of the external momenta and $\Lambda$, the master integrals $I_4, I_3, I_2$ contain powers of $\log \Lambda$. 

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Off-shell one-loop amplitudes

\[ k^\mu = xp^\mu + k_T^\mu \]

\[ p_A^\mu = \Lambda p^\mu + \alpha q^\mu + \beta k^\mu \]

\[ p_A'^\mu = k^\mu - p_A^\mu \]

Well-known decomposition for on-shell one-loop amplitudes in terms of master integrals still holds for finite \( \Lambda \).

\[ A^{(1)} = \int d^{4-2\varepsilon} \ell \frac{\mathcal{N}(\ell)}{\prod_i D_i(\ell)} = \sum_{i,j,k,l} c_4(i, j, k, l) I_4(i, j, k, l) + \sum_{i,j,k} c_3(i, j, k) I_3(i, j, k) \]

\[ + \sum_{i,j} c_2(i, j) I_2(i, j) + \sum_i c_1(i) I_1(i) + \mathcal{R} + \mathcal{O}(\varepsilon) \]

Am I allowed to take \( \Lambda \to \infty \) in the integrand before reduction, and just replace

\[ \frac{1}{2p \cdot (\ell + K)} \to \frac{\Lambda}{(\ell + \Lambda p + K)^2} \]

in the master integrals?
Off-shell one-loop amplitudes

\[ k^{\mu} = x p^{\mu} + k_T^{\mu} \]

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Am I allowed to take \( \Lambda \to \infty \) in the integrand before reduction, and just replace

\[ \frac{1}{2 p \cdot (\ell + \Lambda p + K)^2} \]

in the master integrals?

**NO**
Off-shell one-loop amplitudes

Integrand-based reduction methods cannot be applied with naïve limit $\Lambda \to \infty$ on integrand. For example, the integrand of the following graph (Feynman gauge) vanishes in that limit, but the integral does not:

$$\Lambda p + K \to = \int d^{4-2\epsilon} \ell \frac{\langle p|\gamma^\mu (\ell + \Lambda p + K)|\gamma^\mu |p\rangle}{\ell^2 (\ell + \Lambda p + K)^2}$$

$$= 2p \cdot K \left[ \log \Lambda - \frac{1}{\epsilon} - 1 + \log \left( -\frac{2p \cdot K}{\mu^2} \right) + O(\epsilon) \right]$$

But $\langle p|\gamma^\mu \gamma^\mu |p\rangle = 0$, so naïve power counting in $\Lambda$ does not work.
Behavior of the scalar integrals

\[ \int \frac{d^{4-2\epsilon} \ell}{\ell^2(\ell + K_1)^2(\ell + \Lambda p + K_2)^2(\ell + \Lambda p + K_3)^2} = \frac{a \log^2 \Lambda + b \log \Lambda + c + \mathcal{O}(\Lambda^{-1})}{\Lambda^2} \]

\[ \int \frac{d^{4-2\epsilon} \ell}{\ell^2(\ell + K_1)^2(\ell + K_2)^2} = \frac{a \log^2 \Lambda + b \log \Lambda + c + \mathcal{O}(\Lambda^{-1})}{\Lambda} \]

\[ \int \frac{d^{4-2\epsilon} \ell}{\ell^2(\ell + \Lambda p + K_1)^2(\ell + \Lambda p + K_2)^2} = b \log \Lambda + c + \mathcal{O}(\Lambda^{-1}) \]
Am I allowed to apply partional fractioning?

\[
\frac{\Lambda^2}{(\ell + \Lambda p + K_1)^2(\ell + \Lambda p + K_2)^2} = \frac{1}{2p \cdot (K_2 - K_1)} \left[ \frac{\Lambda}{(\ell + \Lambda p + K_1)^2} - \frac{\Lambda}{(\ell + \Lambda p + K_2)^2} \right] + O \left( \frac{1}{\Lambda} \right)
\]
Partional factioning

Am I allowed to apply partional fractioning?

\[
\frac{\Lambda^2}{(\ell + \Lambda p + K_1)^2(\ell + \Lambda p + K_2)^2} = \frac{1}{2p \cdot (K_2 - K_1)} \left[ \frac{\Lambda}{(\ell + \Lambda p + K_1)^2} - \frac{\Lambda}{(\ell + \Lambda p + K_2)^2} \right] + O\left(\frac{1}{\Lambda}\right)
\]

NO
Boxes with two $\Lambda$-dependent denominators decompose into 4 triangles:

$$\Lambda^2 \int \frac{d^{4-2\varepsilon} \ell}{(\ell + K_0)^2(\ell + K_1)^2(\ell + \Lambda p + K_2)^2(\ell + \Lambda p + K_3)^2} = \frac{\Lambda}{2p \cdot (K_3 - K_2)} \int \frac{d^{4-2\varepsilon} \ell}{(\ell + K_0)^2(\ell + K_1)^2(\ell + \Lambda p + K_2)^2} + \frac{\Lambda}{2p \cdot (K_2 - K_3)} \int \frac{d^{4-2\varepsilon} \ell}{(\ell + K_0)^2(\ell + K_1)^2(\ell + \Lambda p + K_3)^2} + \frac{\Lambda}{2p \cdot (K_0 - K_1)} \int \frac{d^{4-2\varepsilon} \ell}{(\ell + K_1)^2(\ell + \Lambda p + K_2)^2(\ell + \Lambda p + K_3)^2} + \frac{\Lambda}{2p \cdot (K_1 - K_0)} \int \frac{d^{4-2\varepsilon} \ell}{(\ell + K_0)^2(\ell + \Lambda p + K_2)^2(\ell + \Lambda p + K_3)^2} + O \left( \frac{1}{\Lambda} \right) + O \left( \frac{1}{\Lambda^2} \right) + \ldots$$
By analysing the residues for the graphs with the highest possible rank in the numerator, we find:

\[
\begin{align*}
K_{n+m+2} &+ K_{n+4} + K_{n+3} \\
\Lambda p + K_1 &\rightarrow \Lambda
\end{align*}
\]

For determining the coefficients for boxes and triangles, the eikonal Feynman rules can be applied on the integrand, with the scalar integrals interpreted following

\[
\frac{1}{2p \cdot (\ell + K)} \rightarrow \frac{\Lambda}{(\ell + \Lambda p + K)^2}
\]

The coefficients for the bubbles, as well as the rational terms, are finite for \( \Lambda \rightarrow \infty \).
Conclusions

- $k_T$-factorization allows for the parton-level description of kinematical situations inaccessible with LO collinear factorization, e.g. $\Delta S$ for four jets.

- Factorization prescriptions with explicit $k_T$ dependence in the pdfs ask for hard matrix elements with off-shell initial-state partons.

- The necessary amplitudes can be defined in a manifestly gauge invariant manner.

- KaTie generates parton-level events with $k_T$-dependent initial states.

- One-loop amplitudes necessary for NLO in the pipeline.
It has to be stated that the mechanism that makes this work is rather non-trivial. One can see this for example by calculating the graph

\[ \Lambda p + K_1 \rightarrow -\Lambda p + K_3 \]

using standard Passarino-Veltman reduction and keeping the full \( \Lambda \) dependence. One will find contributions from all possible master integrals, including the box. Only when decomposing this box into triangles the contribution of the triangles with two denominators coming form the auxiliary quark line vanish for \( \Lambda \rightarrow \infty \), and it appears that one could have applied partional fractioning from the start.
Backup