Improvements of the sector-improved residue subtraction scheme

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September 26th, 2017 – RADCOR 2017 – St. Gilgen
Outline

Introduction

New phase space construction

’t Hooft Veltman scheme

C++ implementation of STRIPPER

Conclusions
NNLO subtraction schemes

Handling real radiation contribution in NNLO calculations
cancellation of infrared divergences

Increasing number of available NNLO calculations with a variety of schemes

- **qT-slicing**  \cite{CataniGrazzini2007, FerreraGrazziniTramontano2011, CataniCieriDeFlorianFerreraGrazzini2012, GehrmannGrazziniKallweitMaierhoferManteuffelRathlevTorre201415, BoncianiCataniGrazziniSargsyanTorre201415}

- **N-jettiness slicing**  \cite{GauntStahlhofenTackmannWalsh2015, BoughezalFockeGieleLiuPetriello201516, BoughezalCampbellEllisFockeGieleLiuPetriello2015, CampbellEllisWilliams2016}

- **Antenna subtraction**  \cite{GehrmannGehrmannDeRidderGloverHeinrich200508, Weinzierl200809, CurrieGehrmannGehrmannDeRidderGloverPires201317, BernreutherBognerDekkers201114, AbelofDekkersGehrmannDeRidder201115, AbelofGehrmannDeRidderMaierhoferPozzorini2014, ChenGehrmannGloverJaquier2015}

- **Colorful subtraction**  \cite{DelDucaSomogyiTroscanyi200513, DelDucaDuhrSomogyiTramontanoTroscanyi2015}

- **Sector-improved residue subtraction (STRIPPER)**  \cite{Czakon201011, CzakonFiedlerMitov201315, CzakonHeymes2014, CzakonFiedlerHeymesMitov201617, BoughezalCaolaMelnikovPetrielloSchulze201314, BoughezalMelnikovPetriello2011, CaolaCzerneckiLiangMelnikovSzafron2014, BrucherseiferCaolaMelnikov201314, CaolaMelnikovRoetsch2017}
How to improve the STRIPPER subtraction scheme?

- Subtraction kinematics
- Missed binning
- Numerical stability
- Computing time

Alternative phase space parametrisation
- 't Hooft Veltman scheme
- event/histogram-smearing,
  On-the-fly stability-check

Fewer sectors

**Idea:** Optimisation through minimisation
Formulation

Hadronic cross section:

\[ \sigma_{h_1 h_2}(P_1, P_2) = \]

\[ \sum \int \int_0^1 dx_1 \ dx_2 \ f_{a/h_1}(x_1, \mu_F^2) f_{b/h_2}(x_2, \mu_F^2) \hat{\sigma}_{ab}(x_1 P_1, x_2 P_2; \alpha_S(\mu_R), \mu_R, \mu_F) \]

Partonic cross section:

\[ \hat{\sigma}_{ab} = \hat{\sigma}_{ab}^{(0)} + \hat{\sigma}_{ab}^{(1)} + \hat{\sigma}_{ab}^{(2)} + O(\alpha_S^3) \]

Contributions with different final state multiplicities and convolutions:

\[ \hat{\sigma}_{ab}^{(2)} = \hat{\sigma}_{ab}^{RR} + \hat{\sigma}_{ab}^{RV} + \hat{\sigma}_{ab}^{VV} + \hat{\sigma}_{ab}^{C2} + \hat{\sigma}_{ab}^{C1} \]

\[ \hat{\sigma}_{ab}^{RR} = \frac{1}{2\hat{s}} \int d\Phi_{n+2} \left\langle M_{n+2}^{(0)} | M_{n+2}^{(0)} \right\rangle F_{n+2} \]

\[ \hat{\sigma}_{ab}^{RV} = \frac{1}{2\hat{s}} \int d\Phi_{n+1} 2 \text{Re} \left\langle M_{n+1}^{(0)} | M_{n+1}^{(1)} \right\rangle F_{n+1} \]

\[ \hat{\sigma}_{ab}^{VV} = \frac{1}{2\hat{s}} \int d\Phi_{n} \left[ 2 \text{Re} \left\langle M_{n}^{(0)} | M_{n}^{(2)} \right\rangle + \left\langle M_{n}^{(1)} | M_{n}^{(1)} \right\rangle \right] F_{n} \]

\[ \hat{\sigma}_{ab}^{C1} = (\text{single convolution}) F_{n+1} \]

\[ \hat{\sigma}_{ab}^{C2} = (\text{double convolution}) F_{n} \]
Sector decomposition

Several layers of decomposition

Selector functions

\[ 1 = \sum_{i,j} \left[ \sum_{k} S_{ij,k} + \sum_{k,l} S_{i,k,j,l} \right] \]

Factorization of double soft limits

\[ \theta(u_1^0 - u_2^0) + \theta(u_2^0 - u_1^0) \]

Sector parametrisation

Parametrisation of \( u_i \) with respect to the reference parton \( r \):
Angles: \( \hat{\eta}_i = \frac{1}{2}(1 - \cos \theta_{ir}) \in [0, 1] \)
Energies: \( \hat{\xi}_i = \frac{u_i^0}{u_{\text{max}}^0} \in [0, 1] \)

Triple collinear factorisation
originally: 5 sub-sectors

\[ \xi_1 > \xi_2 \]
\[ \xi_2 \rightarrow \xi_2 \xi_{2\text{max}} \xi_1 \]
\[ \xi_3 > \xi_2 \]
\[ \xi_2 \rightarrow \xi_2 \xi_1 \xi_3 \]
\[ \xi_4 > \xi_2 \]
\[ \xi_2 \rightarrow \xi_1 \xi_3 \xi_4 \]
\[ \xi_5 > \xi_2 \]
\[ \xi_2 \rightarrow \xi_1 \xi_3 \xi_5 \]
Sector decomposition

Several layers of decomposition

Selector functions

\[ 1 = \sum_{i,j} \left[ \sum_k S_{ij,k} + \sum_{k,l} S_{i,k,j,l} \right] \]

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Triple collinear factorisation

now: 4 sub-sectors

[Caola, Melnikov, Röntsch '17]

→ yesterday’s talk by Raoul Röntsch
New phase space construction: Idea

Goal
Phase space construction with a minimal \# of subtraction kinematics

Old construction
- Start with unresolved partons
- Fill remaining phase space with Born configuration
  \rightarrow Non-minimal \# kinematic configurations
  (e.g. single soft and collinear limits yield different configurations)

New construction
- Start with Born configuration
- Add unresolved partons \((u_i)\)
- Cleverly adjust Born configuration to accommodate the \(u_i\)
New phase space construction

Mapping from $n + 2$ to Born configuration: $\{P, r_j, u_k\} \rightarrow \{\tilde{P}, \tilde{r}_j\}$

Modification of [Frixione, Webber'02]/[Frixione, Nason, Oleari'07]

Requirements:

- Keep direction of reference $r$ fixed
- Invertible for fixed $u_i$: $\{\tilde{P}, \tilde{r}_j, u_k\} \rightarrow \{P, r_j, u_k\}$
- Preserve $q^2 = \tilde{q}^2$, $\tilde{q} = \tilde{P} - \sum_{j=1}^{n_{fr}} \tilde{r}_j$

Main steps:

- Generate Born phase space configuration
- Generate unresolved partons $u_i$
- Rescale reference momentum
- Boost non-reference momenta of the Born configuration
New phase space construction

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Main steps:

- Generate Born phase space configuration
- Generate unresolved partons $u_i$
- Rescale reference momentum
- Boost non-reference momenta of the Born configuration
Behaviour in singular limits

**Collinear limit of** \( u_2 \)

(sector 1, \( \eta_2 \to 0 \))

**Soft limit of** \( u_2 \)

(sector 1, \( \xi_2 \to 0 \))

→ Both singular limits approach the same kinematic configuration
Behaviour in singular limits

Collinear limit of $u_2$
(sector 1, $\eta_2 \to 0$)

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Soft limit of $u_2$
(sector 1, $\xi_2 \to 0$)

→ Both singular limits approach the same kinematic configuration
Behaviour in singular limits

Triple collinear limit of $u_1$ & $u_2$
(sector 1, $\eta_1 \to 0$)

Double soft limit of $u_1$ & $u_2$
(sector 1, $\xi_1 \to 0$)

$\rightarrow$ Both double unresolved limits approach the Born configuration
Behaviour in singular limits

Triple collinear limit of $u_1$ & $u_2$
(sector 1, $\eta_1 \to 0$)

Double soft limit of $u_1$ & $u_2$
(sector 1, $\xi_1 \to 0$)

→ Both double unresolved limits approach the Born configuration
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**Triple collinear limit of \( u_1 \) & \( u_2 \)**

(sector 1, \( \eta_1 \rightarrow 0 \))

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(sector 1, $\eta_1 \to 0$)

Double soft limit of $u_1$ & $u_2$
(sector 1, $\xi_1 \to 0$)

→ Both double unresolved limits approach the Born configuration
Consequences

Features

- Minimal number of subtraction kinematics
- Only one DU configuration
  → pole cancellation for each Born phase space point
- Expected improved convergence of invariant mass distributions, since $\tilde{q}^2 = q^2$

Unintentional features

- Construction in lab frame
- Original construction of ’t Hooft Veltman corrections [Czakon,Heymes’14] is spoiled
’t Hooft Veltman scheme

Treat resolved particles in 4 dimensions (momenta and polarisations)
- Avoid unnecessary $\epsilon$-orders of the matrix elements
- Avoid growth of dimensionality of phase space integrals

Make resolved phase space 4-dim. using measurement function, e.g.

$$F_n \rightarrow F_n \mathcal{N}^{-(n-1)\epsilon} \prod_{i=1}^{n-1} \delta(-2\epsilon)(q_i)$$

Finite parts:
- $\sigma_{FR}^{RR}$
- $\sigma_{FR}^{RV}$
- $\sigma_{FR}^{VV}$

Finite remainder parts:
- $\sigma_{FR}^{FR}$, $\sigma_{FR}^{VV}$, $\sigma_{FR}^{C2}$

Single (SU) and double (DU) unresolved parts:
- $\sigma_{SU}^{RR}$, $\sigma_{SU}^{RV}$, $\sigma_{SU}^{C1}$
- $\sigma_{DU}^{RR}$, $\sigma_{DU}^{RV}$, $\sigma_{DU}^{C1}$, $\sigma_{DU}^{VV}$, $\sigma_{DU}^{C2}$

Finite parts:
't Hooft Veltman scheme

**Goal:** Make SU and DU separately finite

**Idea:** Move “divergent parts” of SU to DU before applying 'tHV scheme

- **SU contribution:** \( \sigma_{SU} = \sigma_{SU}^{RR} + \sigma_{SU}^{RV} + \sigma_{SU}^{C1} \) with
  \[
  \sigma_{SU}^{C1} = \int d\Phi_{n+1} (I_{n+1}^c F_{n+1} + I_n^c F_n)
  \]

- We know: NLO cross section is finite
  \( \rightarrow F_{n+1} \) part of SU is finite: Poles cancel between RR, RV and C1
  (with NLO measurement function)

- With NNLO measurement function: Additional poles arise
  \( \rightarrow SU \) no longer finite by itself

- Non-cancelling \( \epsilon \) poles are generated by terms with \( F_n \)
  \( \rightarrow \) can be moved to DU

\( \rightarrow \) **Task:** Identify non-cancelling parts of SU
‘t Hooft Veltman scheme

Task: Identify non-cancelling parts of SU

- Use parametrised measurement functions
  \[ F_{n+1}^\alpha = F_{n+1} \theta \left( \min_{i,j} \eta_{ij} - \alpha \right) \theta \left( \min_i \frac{u_i^0}{E_{\text{norm}}} - \alpha \right) \]

- Construct:
  \[ \sigma_{SU}^c - I_c^\alpha = \int d\Phi_{n+1} \left( I_{n+1} F_{n+1} + I_n F_n - [I_{n+1}]_{1/\epsilon^2,1/\epsilon} \right) F_{n+1}^\alpha \]

- Rearrangements allow to extract the non-cancelling part:
  \[ N^c(\alpha) = \int d\Phi_{n+1} [I_n]_{1/\epsilon^2,1/\epsilon} F_n \theta^\alpha \]

- Analytically extract divergences \((\ln^k \alpha)\) and cancel them exactly
- Take limit \(\alpha \to 0\) to remove dependence on \(\alpha\)
- Subtract from \(\sigma_{SU}\) and add to \(\sigma_{DU}\)
  \(\to\) separately finite SU and DU contributions
  \(\to\) ready for application of ‘tHV scheme

Looks like slicing, but it is slicing only for divergences
\(\to\) no actual slicing parameter in the result
C++ implementation of STRIPPER

Features of the implementation

- General subtraction framework
  - Provides a general set of subtraction terms
  - Tree-level amplitudes are calculated automatically using a Fortran library [van Hameren '09] [Bury, van Hameren '15]
  - User has to provide the 1- and 2-loop amplitudes

- Separate evaluation of coefficients of scales and PDFs
  → Cheaper calculations with several scales and PDFs

- FastNLO interface
  - Allows to produce tables for fast fits
  - FastNLO tables for $t\bar{t}$ differential distributions released this spring [Czakon, Heymes, Mitov '17]
Example: Driver program

```cpp
// define initial state and (multiple) PDFs
InitialState initial("p","p",Ecms,Emin);
initial.include(LHAPDFsetName);
// define (multiple) scales
ScalesList scales(FixedScales(mt,mt,"\(\mu_R = mt, \mu_F = mt\)));
scales.include(DynamicalScalesHT4(1.,1.));
// set up observables to be calculated
Measurement measurement;
measurement.include(TransverseMomentum({"t"}),
    {{Histogram::bins(40,0.,2000.)}});
// initialise MC generator and specify contribution to calculate
Generator generator(incoming,scales,measurement);
generator.include({{"g","g"},{"t","t\sim","g","g"}},{2,2,0,0,false});
// run integration with 10^6 points
generator.run(1000000);
// write results
ofstream xml("ttbar.xml");
generator.measurement().print(xml);
xml.close();
```
Conclusions

- Minimization of the STRIPPER scheme
- Fewer subsectors in triple collinear sectors
- Alternative phase space parametrisation
- New formulation of ’t Hooft Veltman scheme
- Implementation of STRIPPER as a C++ library
- Currently performing tests for a variety of processes: $pp \rightarrow t\bar{t}$, $e^+e^- \rightarrow 2/3j$, $t$ decay, DIS, Drell-Yan, Higgs decay, dijets
Backup

- Phase space → 19
- Factorisation and subtraction terms → 20
- SU contribution → 21
- SU finiteness → 22
Common starting point for all phase spaces:

\[
d\Phi_n = dQ^2 \left[ \prod_{j=1}^{n_{fr}} \mu_0(r_j) \prod_{k=1}^{n_u} \mu_0(u_k) \delta_+ \left( \left( P - \sum_{j=1}^{n_{fr}} r_j - \sum_{k=1}^{n_u} u_k \right)^2 - Q^2 \right) \right] \\
\prod_{i=1}^{n_q} \mu_{m_i}(q_i) (2\pi)^d \delta^{(d)} \left( \sum_{i=1}^{n_q} q_i - q \right)
\]

with

\[
\mu_m(k) \equiv \frac{d^d k}{(2\pi)^d} 2\pi \delta(k^2 - m^2) \theta(k^0),
\]

\(n\): # final state particles, \(n_{fr}\): # final state references, \(n_u\): # additional partons
Factorization and subtraction terms

**SU phase space**

\[
\int \int_0^1 d\eta \, d\xi \, \eta^{a_1 - b_1 \epsilon} \xi^{a_2 - b_2 \epsilon}
\]

**DU phase space**

\[
\int \int \int \int_0^1 d\eta_1 \, d\xi_1 \, d\eta_2 \, d\xi_2 \, \eta_1^{a_1 - b_1 \epsilon} \xi_1^{a_2 - b_2 \epsilon} \eta_2^{a_3 - b_3 \epsilon} \xi_2^{a_4 - b_4 \epsilon}
\]

**Factorized singular limits**

\[
\int d\Phi_n \prod d x_i \, x_i^{-1 - b_i \epsilon} \tilde{\mu}(\{x_i\})
\]

\[
\prod x_i^{a_i+1} \left\langle M_{n+2} \middle| M_{n+2} \right\rangle F_{n+2}
\]

**Regularisation**

\[
x^{-1 - b \epsilon} = \frac{-1}{b \epsilon} + \left[ x^{-1 - b \epsilon} \right]_{\text{reg. + sub.}}
\]

\[
\int_0^1 dx \left[ x^{-1 - b \epsilon} \right]_+ f(x) = \int_0^1 \frac{f(x) - f(0)}{x^{1+b \epsilon}}
\]
The single unresolved (SU) contribution

- SU contribution
  \[ \sigma_{SU} = \sigma_{SU}^{RR} + \sigma_{SU}^{RV} + \sigma_{SU}^{C1} \]
  with
  \[ \sigma_{SU}^{c} = \int d\Phi_{n+1} (I_{n+1}^{c} F_{n+1} + I_{n}^{c} F_{n}) \]

- NLO measurement function \((\alpha \neq 0)\)
  \[ \int d\Phi_{n+1} \left( I_{n+1}^{RR} + I_{n+1}^{RV} + I_{n+1}^{C1} \right) F_{n+1}^{\alpha} = \text{finite in 4 dim.} \]

- All divergences cancel in \(d\)-dimensions
  \[ \sum_{c} \int d\Phi_{n+1} \left[ \frac{I_{n+1}^{c,(-2)}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)}}{\epsilon} \right] F_{n+1}^{\alpha} \equiv \sum_{c} I_{n+1}^{c} = 0 \]
\[ \sigma_{_{SU}} = \sigma_{_{SU}}^{RR} + \sigma_{_{SU}}^{RV} + \sigma_{_{SU}}^{C1} - \mathcal{I}_{_{RR}} - \mathcal{I}_{_{RV}} - \mathcal{I}_{_{C1}} = 0 \]

\[ \sigma_{_{SU}}^{c} - \mathcal{I}_{c} = \int d\Phi_{n+1} \left\{ \left[ \frac{I_{n+1}^{c,(-2)}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)}}{\epsilon} + I_{n+1}^{c,(0)} \right] F_{n+1} + \left[ \frac{I_{n}^{c,(-2)}}{\epsilon^2} + \frac{I_{n}^{c,(-1)}}{\epsilon} + I_{n}^{c,(0)} \right] F_{n} \right\} \]

\[ = \int d\Phi_{n+1} \left[ \frac{I_{n+1}^{c,(-2)} F_{n+1} + I_{n+1}^{c,(-2)} F_{n}}{\epsilon^2} + \frac{I_{n+1}^{c,(-1)} F_{n+1} + I_{n+1}^{c,(-1)} F_{n}}{\epsilon} \right] (1 - \theta_{\alpha}(\{\alpha_i\})) \]

\[ + \int d\Phi_{n+1} \left[ I_{n+1}^{c,(0)} F_{n+1} + I_{n}^{c,(0)} F_{n} \right] + \int d\Phi_{n+1} \left[ \frac{I_{n}^{c,(-2)}}{\epsilon^2} + \frac{I_{n}^{c,(-1)}}{\epsilon} \right] F_{n} \theta_{\alpha}(\{\alpha_i\}) \]

\[ =: \quad Z_{c}^{c}(\alpha) + C_{c}^{c} + N_{c}^{c}(\alpha) \]

integrable, zero volume for \( \alpha \to 0 \)

no divergences

only \( F_{n} \to DU \)
The function $N^c(\alpha)$

Looks like slicing, but it is slicing only for divergences → no actual slicing parameter in result

Power-log-expansion

\[ N^c(\alpha) = \sum_{k=0}^{\ell_{\text{max}}} \ln^k(\alpha) N_k^c(\alpha) \]

- all $N_k^c(\alpha)$ regular in $\alpha$
- start expression independent of $\alpha$ ⇒ all logs cancel
- only $N_0^c(0)$ relevant

Putting parts together

\[ \sigma_{SU} - \sum_c N_0^c(0) \text{ and } \sigma_{DU} + \sum_c N_0^c(0) \]

are finite in 4 dimension

\[ \Downarrow \]

SU contribution

\[ \sigma_{SU} - \sum_c N_0^c = \sum_c C^c \]

original expression $\sigma_{SU}$ in 4-dim without poles, no further $\epsilon$ pole cancellation
Calculation of $N_0^c(0)$

For each sector/contribution:

1. extraction of $d\Phi_{n+1}$ from $d\Phi_{n+2}|_{SU \; pole}$ (only for $RR$ contribution)

\[
\left. d\Phi_{n+2} \right|_{SU \; pole} = \left( d\Phi_n d^d\mu(u_1) d^d\mu(u_2) \right) \bigg|_{u_2 \colon \text{col/soft}}
\]

2. expansion in $\epsilon$ up to $\epsilon^{-1}$ (except $d\Phi_{n+1}$): $d^d\Phi_{n+1} \left( \frac{\cdot}{\epsilon^2} + \frac{\cdot}{\epsilon} \right)$

3. Identifying $\ln^k(\alpha)$'s from $x_i$ integrations over $\theta$ function

\[
\theta_\alpha(\hat{\eta}, u^0) = \theta(\hat{\eta} - \alpha)\theta(\hat{\xi}u_{max}/E_{norm} - \alpha)
\]

\[\rightarrow\] discard them

4. perform integration over $\theta$-functions of non-cancelling and non-vanishing (in $\alpha \to 0$ limit) terms