First Two-Loop Amplitudes with the Numerical Unitarity Method

Ben Page

Albert-Ludwigs-Universität Freiburg

RADCOR 2017
St. Gilgen, Austria
26\textsuperscript{th} September 2017

Based on work with S. Abreu, F. Febres-Cordero, H. Ita, M. Jaquier and M. Zeng.

LHC Era Phenomenology

- High-luminosity run of LHC will substantially improve experimental precision.
- Need for NNLO predictions for many processes.
- \( \Rightarrow \) Two-loop multiparton QCD amplitudes.
### The Canonical Approach to Two-loop Amplitudes

#### Feynman diagrams

- **Tensor reduction**
  - [Passarino, Veltman '79]
  - IBPs
  - [Tkachov, Chetyrkin '81]

- **Sum of master integrals**

\[
A = \sum_{\Gamma \in \Delta} \sum_{i \in M_{\Gamma}} c_{\Gamma,i} (D) I_{\Gamma,i}.
\]

- **Differential equations**
  - [Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01; Henn '13]

- **Integrated form**

- Standard procedure.
- Large intermediate expressions.
- Can we repeat the leap made at NLO? [OPP '07]
The Numerical Unitarity Approach

- Take an **ansatz** for loop-amplitude integrand, decomposing into master ($M_\Gamma$) and surface ($S_\Gamma$) integrands.

\[
A(\ell_i) = \sum_{\text{Topologies } \Gamma} \sum_{i \in M_\Gamma \cup S_\Gamma} c_{\Gamma,i} m_{\Gamma,i}(\ell_i) \prod_{\text{props } j} \rho_j.
\]

- **Numerically** determine $c_{\Gamma,i}$ from on-shell information.

\[
\sum_{\text{states } i \in T_\Gamma} \prod_{\text{ancestors } \Gamma', i \in M_{\Gamma'} \cup S_{\Gamma'}} A^\text{tree}_i(\ell_\Gamma) = \sum_{\text{ancestors } \Gamma', i \in M_{\Gamma'} \cup S_{\Gamma'}} c_{\Gamma',i} m_{\Gamma',i}(\ell_\Gamma) \prod_{\text{props } j} \rho_j.
\]

[BDDK '94, '95]

- Physical coefficients **directly**. Avoids intermediate complexity.

- Insert master integrals, expand $\Rightarrow$ **integrated amplitude**.
Setting Up Numerical Unitarity @ 2-Loops

\[ \sum \prod \text{states } \ell \in T_{\Gamma} \text{ } A_{i}^{\text{tree}}(\ell_{i}) = \sum \prod \text{ancestors } \ell \in M_{\Gamma} \cup S_{\Gamma} \text{ } c_{\Gamma',i} m_{\ell},(\ell_{i}) \prod \text{props } \rho_{j}(\ell_{i}). \]

New challenges:
- Master/surface decomposition.
- Sub-leading poles.

Old friends:
- Fast products of trees, $T_{\Gamma}$.
- Colour decomposition.
- Coefficient determination, $c_{\Gamma',i}$.

4-gluon Amplitude Topologies $\Gamma$
Master/Surface Decomposition

- Surface terms naturally produced by IBPs.

- Generate complete set for a topology $\Gamma$ by controlling powers of $\rho_j$.

- Constrained coordinates: irreducible scalar products & propagators.

- Express vectors in the new coordinates.

\[ 0 = \int \prod_{l=1,2} d^D \ell_l \frac{\partial}{\partial \ell_j^\nu} \left[ \frac{u^\nu_j}{\prod_{\text{props } k} \rho_k} \right]. \]

\[ u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j. \]

\[ \ell_i^\mu \rightarrow (\rho_i, \alpha_j, \mu_{nm}), \]

\[ \mu_{ll} = (\mu_l)^2 = \rho_{l0} - \sum_{\nu=0}^3 \ell_l^\nu \ell_l^\nu. \]

\[ u = f_i \rho_i \frac{\partial}{\partial \rho_i} + u_j \frac{\partial}{\partial \alpha_j} + \sum_{l,l'=1,2} f_{l,l'} \vec{\mu}_l \cdot \frac{\partial}{\partial \vec{\mu}_{l'}}. \]

\[ = \bar{u} + u_\epsilon. \]
Master/Surface Decomposition

- Vectors satisfy compatibility conditions - syzygy relations.
- Generating set of solutions \( \{ f_i, u_j, f'_l \} \) found with SINGULAR. related [Larsen, Zhang '16]
- Total derivatives using vectors → surface terms.
- Complete decomposition with arbitrary choice of master integrands.

\[
\begin{align*}
\bar{u}(\mu_{11}) &= 2\mu_{11}f_1^1 + 2\mu_{12}f_1^2, \\
\bar{u}(\mu_{22}) &= 2\mu_{22}f_2^2 + 2\mu_{12}f_2^1, \\
\bar{u}(\mu_{12}) &= \mu_{12}(f_1^1 + f_2^2) + \mu_{11}f_2^1 + \mu_{22}f_1^2.
\end{align*}
\]
\[
\Rightarrow 0 = f_i \rho_i \frac{\partial \mu_{nm}}{\partial \rho_i} + u_j \frac{\partial \mu_{nm}}{\partial \alpha_j} - \sum_{l,l'=1,2} f'_l \bar{\mu}_{l'} \frac{\partial \mu_{nm}}{\partial \bar{\mu}_{l'}}.
\]
\[
m_{\Gamma,u}(\ell_i) = \left[ -(\nu_i - 1)f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha_j} + \left( D - \frac{n_\alpha + 1}{2} \right) (f_1^1 + f_2^2) \right].
\]
\[
M_{\Gamma} = \{ \ldots \}
\]
Sub-leading Poles

- Beyond one loop multiple poles can be associated to a given factorization limit.
- Only **leading poles** given by product of trees.
- Some numerators lack associated cut equation.
- Numerators determined from descendant cut equations.

\[
\Gamma \in \Delta \setminus \tilde{\Delta} \\
\Gamma > \Gamma' \\
\prod_{k \in P_k \setminus P_{\Gamma'}} \rho_k(\ell'_{l}) \\
N(\Gamma, \ell'_{l}) \\
= N\left(\begin{array}{c}
\end{array}\right) + \frac{1}{\rho} N\left(\begin{array}{c}
\end{array}\right)
\]
A Proof of Principle - Gluon Gluon Scattering

- **Aim**: To test the Numerical Unitarity approach @ 2 loops for phenomenologically relevant amplitudes.

- **We** recompute the known 4-gluon scattering amplitude in an arbitrary helicity configuration [Anastasiou et al '01; Bern et al '02].

- A gym with **all necessary ingredients** and an analytic target.
Tree Amplitudes and Colour

- Off-shell recursion. [Berends, Giele '87]
- Flexible spin DoF - “$D_s$”.
- Extensive caching across different helicities and orderings.

- Leading colour produced by projecting colour decomposition of product of trees to integrand.

\[
\tilde{N}\left( \begin{array}{c} 1 \\ 2 \end{array} \right) = C\left( \begin{array}{c} 1 \\ 2 \end{array} \right) N\left( \begin{array}{c} 1 \\ 2 \end{array} \right) + C\left( \begin{array}{c} 2 \\ 1 \end{array} \right) N\left( \begin{array}{c} 2 \\ 1 \end{array} \right)
\]

[B.P., Ochirov '16]
Solving for Master Integral Coefficients

\[
\sum_{\Gamma \in \Delta} \frac{c_{\Gamma,i}(D) m_{\Gamma,i}(\ell_{\Gamma'}^i)}{\prod_{k \in P_{\Gamma} \setminus P_{\Gamma'}} \rho_k(\ell_{\Gamma'}^i)} = \sum_{\text{states } i \in T_{\Gamma}} \prod A_{i,\text{tree}}(\ell_{\Gamma}^i) - \sum_{\Gamma \in \Delta \setminus \Delta} \frac{N(\Gamma, \ell_{\Gamma'}^i)}{\prod_{k \in P_{\Gamma} \setminus P_{\Gamma'}} \rho_k(\ell_{\Gamma'}^i)}.
\]

- Sample randomly on-shell phase space $\ell_{\Gamma}^i$ to constrain $c_{\Gamma,i}$.
- PLU factorization for $n \times n$ systems.
- QR factorization for $n \times m$ overdetermined systems.
- LAPACK for double precision. [Anderson et al '99]
- MPACK for high precision. [Nakata '10]

Can determine master integral coefficients for fixed $D$ and $D_s$. 
Regulator Dependence of Coefficients

- Amplitude depends on \((D_s - 2)\), the number of spin states.

  is linear in \(D_s\)  
  is quadratic in \(D_s\)

- Interpolate from products of trees with different \(D_s\). [Giele et al '08]

- \(D\)-dependence of surface terms, \(m_{\Gamma,u}(\ell_l, D)\), means coefficients are rational functions, i.e.:

  \[
  c(D) = \frac{P(D, s, t)}{Q(D)} = \frac{p_0(s, t) + p_1(s, t)D + \ldots + p_i(s, t)D^i}{q_0 + q_1D + \ldots + q_{j-1}D^{j-1} + D^j}.
  \]

- Can reconstruct \(c(D)\) by sampling various \(D\).

related [Peraro '16], [Schabinger, von Manteuffel '14]
The Finished Product

- We thus determine $A(D)$ for a given numerical $(s, t)$.
- $A(D)$ is exact in $D$ and we verify it against analytics.
- We insert the integrals and expand around $D = 4$.

$$A(D) = c_0 \left( \begin{array}{c} \square \\ \square \end{array} \right) I_0 \left( \begin{array}{c} \square \\ \square \end{array} \right) + c_1 \left( \begin{array}{c} \square \\ \square \end{array} \right) I_1 \left( \begin{array}{c} \square \\ \square \end{array} \right) + c_2 \left( \begin{array}{c} \square \\ \square \end{array} \right) I_2 \left( \begin{array}{c} \square \\ \square \end{array} \right) + c_3 \left( \begin{array}{c} \square \\ \square \end{array} \right) I_3 \left( \begin{array}{c} \square \\ \square \end{array} \right) + (s \leftrightarrow t).$$

With, $g_s = 1$, $\mu = 1$, $s = -\frac{1}{4}$ and $t = -\frac{3}{4}$ we find:

<table>
<thead>
<tr>
<th>$A/(A_0 N_C^2)(4\pi)^4$</th>
<th>$\epsilon^{-4}$</th>
<th>$\epsilon^{-3}$</th>
<th>$\epsilon^{-2}$</th>
<th>$\epsilon^{-1}$</th>
<th>$\epsilon^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1_g^-, 2_g^+, 3_g^-, 4_g^+)$</td>
<td>8.00000</td>
<td>55.6527</td>
<td>176.009</td>
<td>332.296</td>
<td>486.502</td>
</tr>
<tr>
<td>$(1_g^-, 2_g^+, 3_g^+, 4_g^+)$</td>
<td>8.00000</td>
<td>55.6527</td>
<td>164.642</td>
<td>222.327</td>
<td>-8.39044</td>
</tr>
</tbody>
</table>
Numerical Stability of MMPP Amplitude

- Double precision comparison to analytics.
- 10000 RAMBO phase space points.
- Coefficient precision often good to $\sim 8$ digits.
- $\sim 4$ digit precision loss when inserting integrals.
- There might exist a basis which better controls the precision.
Analytics from Numerics

- Rescaled integral coefficients are **rational functions** of $x = \frac{t}{s}$.

- **Reconstruct using same techniques** as for the regulator.
  
  related [Peraro '16], [Schabinger, von Manteuffel '14]

- **Exact analytic results** from numerics, e.g (mmpp):

  $$c_0 \left( \begin{array}{c} \vdots \end{array} \right) = 9x + \frac{\epsilon \left( -x^3 - \frac{32x^2}{11} - \frac{97x}{44} - \frac{5}{22} \right)}{\frac{x^2}{33} + \frac{2x}{33} + \frac{1}{33}} + \frac{\epsilon^2 \left( -x^3 - \frac{385x^2}{51} - \frac{937x}{102} - \frac{77}{34} \right)}{-\frac{2x^2}{51} - \frac{4x}{51} - \frac{2}{51}} + \cdots$$

  $$-9 + 66\epsilon - 184\epsilon^2 + 240\epsilon^3 - 144\epsilon^4 + 32\epsilon^5$$

- **Requires 15 evaluations** of the amplitude in high precision.
Conclusions

- We set up a numerical algorithm for two-loop amplitudes.
- Surface terms (IBPs) with algebraic geometry techniques.
- Analytic results can be reconstructed from numerical samples.
- A proof of principle calculation of the 4-gluon amplitude.
- The method shows promise for phenomenological calculations.
Univariate Rational Function Reconstruction

- Thiele’s formula gives $c(D)$ as a continued fraction \cite{Peraro '16}

$$c(D) = a_0 + \frac{D - D_0}{a_1 + \frac{D - D_1}{a_2 + \frac{D - D_2}{\cdots + \frac{D - D_{N-1}}{a_N}}}}.$$  

- Can fix coefficients $a$ from values $c(D_i)$.
- Easy to convert back to canonical $\frac{P(D)}{Q(D)}$.
- Only $P(D)$ is kinematically dependent.
- We fix the rational coefficients of $Q(D)$ once in high precision, recovering the exact form with continued fractions.
Numerical Stability for MPMP

Comparison to analytics over 10000 phase space points.

Rescue system based on accuracy of universal $\frac{1}{\epsilon}$ pole.