FACTORISATION AND SUBTRACTION

Lorenzo Magnea

University of Torino - INFN Torino

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Foreword
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- Exponentiation ties together high orders to low orders.
- Classes of possible virtual poles are absent, with implications for real radiation.
- Virtual corrections suggest soft and collinear limits should `commute'.
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Can one use the structure of virtual singularities as an organising principle for subtraction?

Can the simplifying features of virtual corrections be exported to real radiation?
Outline

- Virtual is easy
- Real is hard
- Local counterterms
- Outlook
• Virtual is easy

• Real is hard

• Local counterterms

• Outlook

In collaboration with
Ezio Maina,
Paolo Torrielli
Sandro Uccirati
VIRTUAL IS EASY
Virtual factorisation: pictorial

A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes
A note on color flow

The soft function is an operator in color space, often written in terms of “gluon insertion operators” $T_a$. One can also pick a basis of color tensors for the selected process. Consider the simple case of quark-antiquark scattering.

At tree level

For this process only two color structures are possible. A basis in the space of available color tensors is

$$c^{(1)}_{abcd} = \delta_{ab}\delta_{cd}, \quad c^{(2)}_{abcd} = \delta_{ac}\delta_{bd}$$

The matrix element is a vector in this space, and the Born cross section is

$$M_{abcd} = M_1 c^{(1)}_{abcd} + M_2 c^{(2)}_{abcd} \rightarrow \sum_{\text{color}} |M|^2 = \sum_{J,L} M_J M^*_L \tr \left( c^{(J)}_{abcd} c^{(L)}_{abcd} \right) = \tr [H S]_0$$

A virtual soft gluon will reshuffle color and mix the components of this vector

QED: $M_{\text{div}} = S_{\text{div}} M_{\text{Born}}$;  \quad QCD: $[M_{\text{div}}]_J = [S_{\text{div}}]_{JL} [M_{\text{Born}}]_L$
Operator Definitions

The precise **functional form** of this graphical factorisation is

$$\mathcal{M}_L \left( p_i/\mu, \alpha_s(\mu^2), \epsilon \right) = \mathcal{S}_{LK} \left( \beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) H_K \left( \frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2) \right) \times \prod_{i=1}^{n} \left[ J_i \left( \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) / J_i \left( \frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right) \right] ,$$

Here we introduced dimensionless **four-velocities** $p_i^\mu = Q\beta_i^\mu$, (for massless particles $\beta_i^2 = 0$), and **factorisation vectors** $n_i^\mu$, $n_i^2 \neq 0$ to define, if needed, the jets.

$$J \left( \frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle .$$

where $\Phi_n$ is the **Wilson line** operator along the direction $n^\mu$,  

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right] .$$

Note: Wilson lines represent **fast** or **very massive** particles, not recoiling against soft radiation.

The vectors $n^\mu$:  

- **Ensure** gauge invariance of the jets.
- **Separate** collinear gluons from wide-angle soft ones.
- **Replace** other hard partons with a **collinear-safe** absorber.
The soft function $S$ is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)^{\{a_k\}} S_{LK} (\beta_i \cdot \beta_j, \epsilon) = \langle 0 | \prod_{k=1}^{n} [\Phi_{\beta_k} (\infty, 0)]_{a_k}^{b_k} | 0 \rangle (c_K)^{\{b_k\}} ,$$

The soft function $S$ obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} S_{LK} (\beta_i \cdot \beta_j, \epsilon) = - S_{LJ} (\beta_i \cdot \beta_j, \epsilon) \Gamma^{S}_{JK} (\beta_i \cdot \beta_j, \epsilon)$$

**NOTE:** $\Gamma^S$ is singular for massless theories, due to overlapping UV and collinear poles.

$S$ is a pure counterterm. In dimensional regularization, using $\alpha_s (\mu^2 = 0, \epsilon < 0) = 0$,

$$S (\beta_i \cdot \beta_j, \alpha_s (\mu^2), \epsilon) = P \exp \left[- \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^S (\beta_i \cdot \beta_j, \alpha_s (\xi^2), \epsilon) \right].$$

This exemplifies a pattern of exponentiation, which is common to all soft and collinear virtual corrections.

- Soft and collinear effects are governed by anomalous dimensions, such as $\Gamma_s$.
- The soft anomalous dimension has non-trivial color and kinematic structure.
- Collinear effects are `color singlet’ and can be extracted from two-parton scatterings.
Simple structure at NNLO

All soft and collinear singularities can be collected in a multiplicative operator $Z$

$$\mathcal{M} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = Z \left( \frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon \right) \mathcal{H} \left( \frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon \right),$$

$Z$ contains both soft singularities from $S$, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d \ln \mu} Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = - Z \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \Gamma \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right).$$

The matrix $\Gamma$, up to NNLO, has a surprisingly simple dipole structure, as at one loop.

$$\Gamma_{\text{dip}} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right) = - \frac{1}{4} \frac{\hat{\gamma}_K (\alpha_s(\mu^2))}{\mu^2} \sum_{j \neq i} \ln \left( \frac{-2 p_i \cdot p_j}{\mu^2} \right) T_i \cdot T_j + \sum_{i=1}^{n} \gamma_{J_i} (\alpha_s(\mu^2)).$$

All singularities are generated by integration over the scale of the coupling.

Tripole corrections, which could naturally appear at two loops, vanish.

Quadrupole corrections arise at three loops, depending only on conformal cross ratios of external momenta (Almelid, Duhr, Gardi 2015).
REAL IS HARD
The computation of a generic IRC-safe observable at NLO requires the combination

\[ \langle O \rangle_{NLO} = \lim_{d \to 4} \left\{ \int d\Phi_n \left[ B_n + V_n \right] O_n + \int d\Phi_{n+1} R_{n+1} O_{n+1} \right\} \]

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

\[ \langle O \rangle_{ct} = \int d\Phi_n d\hat{\Phi}_1 K_{n+1} O_n \]

\[ I_n = \int d\hat{\Phi}_1 K_{n+1} \]

Add and subtract the same quantity to the observable: each contribution is now finite.

\[ \langle O \rangle_{NLO} = \int d\Phi_n \left[ B_n^{(4)} + (V_n + I_n)^{(4)} \right] O_n + \int d\Phi_n \left[ \int d\Phi_1^{(4)} R_{n+1}^{(4)} O_{n+1} - \int d\hat{\Phi}_1^{(4)} K_{n+1}^{(4)} O_n \right] \]

Search for the simplest integrand \( K_{n+1} \) with the correct singular limits.

[See Paolo Torrielli's talk]
The pattern of cancellations is more intricate at higher orders

\[
\langle O \rangle_{\text{NNLO}} = \lim_{d \to 4} \left\{ \int d\Phi_n \left[ B_n + V_n + VV_n \right] O_n + \int d\Phi_{n+1} \left[ R_{n+1} + RV_{n+1} \right] O_{n+1} + \int d\Phi_{n+2} RR_{n+2} O_{n+2} \right\}.
\]

More counterterm functions need to be defined

\[
I_{n+1}^{(1)} = \int d\hat{\Phi}_1^' K_{n+2}^{(1)},
\]

\[
I_{n+1}^{(2)} = \int d\hat{\Phi}_2 K_{n+2}^{(2)} = \int d\hat{\Phi}_1 d\hat{\Phi}_1^' K_{n+2}^{(2)},
\]

A finite expression for the observable in $d=4$ must combine several ingredients

\[
\langle O \rangle_{\text{NNLO}} = \int d\Phi_n \left[ B_n^{(4)} + (V_n + I_n^{(4)}) + (VV_n + I_n^{(2)} + I_n^{(RV)})^{(4)} \right] O_n
\]

\[
+ \int d\Phi_n \left[ \int d\Phi_1^{(4)} R_{n+1}^{(4)} O_{n+1} - \int d\hat{\Phi}_1^{(4)} K_{n+1}^{(4)} O_n + \int d\Phi_1^{(4)} \left(RV_{n+1} + I_n^{(1)}\right)^{(4)} O_{n+1} - \int d\hat{\Phi}_1^{(4)} K_{n+1}^{(RV)(4)} O_n \right]
\]

\[
+ \int d\Phi_n \left[ \int d\Phi_2^{(4)} RR_{n+2}^{(4)} O_{n+2} - \int d\Phi_1^{(4)} d\Phi_1^{(4)} K_{n+2}^{(1)(4)} O_{n+1} - \int d\Phi_2^{(4)} K_{n+2}^{(2)(4)} O_n \right]
\]
A multi-year effort

The subtraction problem at NLO is completely solved, with efficient algorithms applicable to any process for which matrix elements are known. At NNLO after fifteen years of efforts several groups have working algorithms, successfully applied to `simple' process with up to four legs. Heavy computational costs.

- Antenna Subtraction (T. Gehrmann et al.).
- Sector-Improved Residue Subtraction (M. Czakon et al.)
- Nested Soft-Collinear Subtractions (F. Caola et al.)
- ColourfulNNLO (Z. Trocsanyi et al.)
- N-Jettiness Subtraction (F. Petriello et al.).
- QT Subtraction (S. Catani et al.).
- Projection to Born (G. Salam et al.)
- Unsubtraction (G. Rodrigo et al.)
- ....
We partition the phase space in sectors, each containing at most two singularities (FKS).

A momentum mapping is needed for the factorised Born configuration to be physical.

The factorised kernels are the familiar eikonal and DGLAP functions.

Upon subtracting soft and collinear poles, the soft-collinear overlap must be added back.

The explicit form of NLO counterterms highlights some of the practical difficulties.
LOCAL COUNTERTERMS
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At amplitude level poles factorise and exponentiate.
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We need to build cross-section level quantities.
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We need to build cross-section level quantities.

• Inclusive eikonal cross sections are finite.
• They are building blocks for threshold and $Q_T$ resummations.
• They are defined by gauge-invariant operator matrix elements.
• Fixing the quantum numbers of particles crossing the cut one obtains local IR counterterms.
Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.
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At amplitude level poles factorise and exponentiate.
Consider next collinear final state divergences. They are associated with individual partons. At amplitude level poles factorise and exponentiate. Soft-collinear poles can be subtracted.
Consider next collinear final state divergences. They are associated with individual partons.

At amplitude level poles factorise and exponentiate.

- Inclusive jet cross sections are finite.
- They are building blocks for threshold and $Q_T$ resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.
Soft counterterms: all orders

To be precise, we introduce soft matrix elements for the emission of $m$ soft partons.

$$
S_{\lambda_1 \ldots \lambda_m} (k_1, \ldots, k_m; \beta_i) \equiv \langle k_1, \lambda_1; \ldots; k_m, \lambda_m | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) | 0 \rangle
\equiv g^m \varepsilon^{*\mu_1}_{\lambda_1} (k_1) \ldots \varepsilon^{*\mu_m}_{\lambda_m} (k_m) J_{\mu_1 \ldots \mu_m}^{S} (k_1, \ldots, k_m; \beta_i)
\equiv g^m \sum_{p=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^p S_{\lambda_1 \ldots \lambda_m}^{(p)} (k_1, \ldots, k_m; \beta_i).
$$

These matrix elements define soft gluon emission currents. They are gauge invariant and they contain loop corrections to all orders.

Existing finite order calculations and all-order arguments are consistent with the factorisation

$$
M_{\lambda_1 \ldots \lambda_m} (k_1, \ldots, k_m; p_i) \simeq S_{\lambda_1 \ldots \lambda_m} (k_1, \ldots, k_m; \beta_i) \mathcal{H}(p_i).
$$

with corrections that are finite in dimensional regularisation, and integrable in the soft gluon phase space. It is a working assumption: a formal all-order proof is still lacking.
Soft counterterms: all orders

The factorisation is reflected at **cross-section level**, for **fixed** final state **quantum numbers**.

\[
\sum_{\lambda_1 \ldots \lambda_m} |\mathcal{M}_{\lambda_1 \ldots \lambda_m}(k_1, \ldots, k_m; p_i)|^2 \simeq S_m(k_1, \ldots, k_m; \beta_i) |\mathcal{H}(p_i)|^2.
\]

The cross-section level **“radiative soft functions”** are Wilson-line squared matrix elements

\[
S_m(k_1, \ldots, k_m; \beta_i) = \sum_{\lambda_1 \ldots \lambda_m} \langle 0 | \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty) |k_1, \lambda_1; \ldots; k_m, \lambda_m \rangle \langle k_1, \lambda_1; \ldots; k_m, \lambda_m \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) |0 \rangle \\
= (4\pi \alpha_s)^m \sum_{p=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^p S_{m}^{(p)}(k_1, \ldots, k_m; \beta_i).
\]

These functions provide a complete list of **local soft subtraction counterterms**, to all orders.

Indeed, **summing** over particle numbers and **integrating** over the soft phase space one finds

\[
\sum_{m=0}^{\infty} \int d\text{LIPS}_m(k_j) S_{m}^{(p)}(k_1, \ldots, k_m; \beta_i) = \langle 0 | \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) |0 \rangle.
\]

This is a **finite fully inclusive soft cross section**, order by order in perturbation theory.
At **NLO**, only the tree-level single-emission current is required, simply defined by

\[ g \epsilon^*_\lambda(k) \cdot J^{S}_\lambda(k, \beta_i) = \langle k, \lambda \left| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \right| 0 \rangle \]  

Tree level

One obviously recovers all the well-known results for the leading-order soft gluon current

\[ M^{(0)}(k, p_i) = g \epsilon^*_\lambda(k) \cdot J^{S}_\lambda(k, \beta_i) H^{(0)}(p_i) + \mathcal{O}(k^0). \]

\[ J^{S, \mu}_\lambda(k, \beta_i) = \sum_{i=1}^{n} \frac{\beta_i^\mu}{\beta_i \cdot k} T_i. \]

For the cross-section, the tree-level single-radiation soft function acts as a local counterterm.

\[ \left| M^{(0)}_\lambda(k, p_i) \right|^2 \simeq -4\pi \alpha_s \sum_{i,j=1}^{n} \frac{\beta_i \cdot k \beta_j \cdot k}{\beta_i \cdot k} M^{(0)}(p_i) T_i \cdot T_j M^{(0)}(p_i) = S^{(0)}_1(k; \beta_i) \left| H^{(0)}(p_i) \right|^2. \]

- The single-radiative soft function acts as a color operator on the color-correlated Born.
- Beyond **NLO**, tree-level multiple gluon emission currents also follow from this definition.
At one loop, our definition of the soft currents begins to differ from the classic Catani-Grazzini.

\[ M_\lambda(k; p_i) \simeq S_\lambda(k; \beta_i) \mathcal{H}(p_i). \quad M_\lambda(k; p_i) \simeq g \epsilon_\lambda^*(k) \cdot J_{CG}(k, \beta_i) \mathcal{M}(p_i). \]

The difference is however easily bridged: expanding our factorised expression to NLO …

\[ M_\lambda^{(1)}(k; p_i) \simeq S_\lambda^{(0)}(k; \beta_i) \mathcal{H}^{(1)}(p_i) + S_\lambda^{(1)}(k; \beta_i) \mathcal{H}^{(0)}(p_i). \]

… and using the factorisation of the non-radiative amplitude, we build the NLO hard part

\[ \mathcal{M}(p_i) \simeq S(\beta_i) \mathcal{H}(p_i) \quad \rightarrow \quad \mathcal{H}^{(1)}(p_i) = \mathcal{M}^{(1)}(p_i) - S^{(1)}(\beta_i) \mathcal{M}^{(0)}(p_i). \]

Recombining, we get an explicit eikonal expression for the CG one-loop soft current

\[ g^3 \epsilon_\lambda^*(k) \cdot J_{CG}^{(1)}(k, \beta_i) \mathcal{M}^{(0)}(p_i) = \left[ S_\lambda^{(1)}(k; \beta_i) - S_\lambda^{(0)}(k; \beta_i) S^{(1)}(\beta_i) \right] \mathcal{M}^{(0)}(p_i). \]

The two calculations are easily matched: same diagrammatic content, cancellations and result.
Soft currents beyond NLO

The procedure is easily generalised to generic higher orders. At two loops one finds

$$\mathcal{M}_\lambda^{(2)}(k; p_i) \simeq S_\lambda^{(0)}(k; \beta_i) \mathcal{H}^{(2)}(p_i) + S_\lambda^{(1)}(k; \beta_i) \mathcal{H}^{(1)}(p_i) + S_\lambda^{(2)}(k; \beta_i) \mathcal{H}^{(0)}(p_i).$$

To map to the CG definition, express the two-loop hard part in terms of the amplitude

$$\mathcal{H}^{(2)}(p_i) = \mathcal{M}^{(2)}(p_i) - S^{(1)}(\beta_i) \mathcal{M}^{(1)}(p_i) + \left[\left(S^{(1)}(\beta_i)\right)^2 - S^{(2)}(\beta_i)\right] \mathcal{M}^{(0)}(p_i).$$

Recombining, we get an explicit eikonal expression for the two-loop single-gluon soft current

$$g^5 \epsilon^*_\lambda(k) \cdot J^{(2)}_{CG}(k, \beta_i) = S_\lambda^{(2)}(k; \beta_i) - S_\lambda^{(1)}(k; \beta_i) S^{(1)}(\beta_i) - S_\lambda^{(0)}(k; \beta_i) \left[S^{(2)}(\beta_i) - \left(S^{(1)}(\beta_i)\right)^2\right].$$

For the two-leg case, this was computed in (Badger, Glover 2004) to $O(\epsilon^0)$ and by (Duhr, Gehrmann 2013) to $O(\epsilon^2)$, by taking soft limits of full matrix elements. This definition allows to extend the calculation to the general case.

A similar definition emerges for the double-gluon soft current at one and two loops. Based on eikonal Feynman rules, one can begin the process of systematising these calculations.
Collinear counterterms: all orders

For collinear poles, we introduce jet matrix elements for the emission of $m$ collinear partons

\[
J_{m}^{s,\lambda_{j}} (k_{j}; p, n) \equiv \langle p, s; k_{j}, \lambda_{j} | \bar{\psi}(0) \Phi_{n}(\infty, 0) | 0 \rangle \equiv g^{m} \sum_{p=0}^{\infty} \left( \frac{\alpha_{s}}{\pi} \right)^{p} J_{m,s,\lambda_{j}}^{(p)} (k_{j}; p, n) .
\]

At cross-section level, “radiative jet functions” can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

\[
J_{m}^{s,\lambda_{j}} (k_{j}; l, p, n) \equiv \int d^{d}x \, e^{i l \cdot x} \langle 0 | \Phi_{n}(\infty, x) \psi(x) | p, s; k_{j}, \lambda_{j} \rangle \langle p, s; k_{j}, \lambda_{j} | \bar{\psi}(0) \Phi_{n}(0, \infty) | 0 \rangle \\
\equiv (4 \pi \alpha_{s})^{m} \sum_{p=0}^{\infty} \left( \frac{\alpha_{s}}{\pi} \right)^{p} J_{m,s,\lambda_{j}}^{(p)} (k_{j}; l, p, n) .
\]

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

\[
\sum_{m=0}^{\infty} \int d\text{LIPS}_{m+1} (p, k_{j}) \sum_{\lambda_{j}} J_{m,s,\lambda_{j}} (k_{j}; l, p, n) = \text{Disc} \left[ \int d^{d}x \, e^{i l \cdot x} \langle 0 | \Phi_{n}(\infty, x) \psi(x) \bar{\psi}(0) \Phi_{n}(0, \infty) | 0 \rangle \right] .
\]

A “two-point function”, finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^{2} \to 0$), unlike the case of radiative soft functions.
- Working with $n^{2} \neq 0$ eliminates spurious collinear poles, but is cumbersome in practice.
Collinear counterterms: NLO

At NLO, only tree-level single-emission contributes, resulting (for quarks) in three diagrams.

Summing over helicities, and taking the $n^2 \to 0$ limit, one finds a spin-dependent kernel:

$$\sum_{s,\lambda} J_{1}^{s,\lambda}(k; l, p, n) = \frac{4\pi \alpha_s C_F}{l^2} (2\pi)^d \delta^d(l - p - k) \left[ -i \gamma_\mu \psi \gamma_\mu l + \frac{1}{k \cdot n} (\bar{\psi} l + \psi \bar{l}) \right].$$

With a Sudakov decomposition ($n^\mu$ as reference vector) and taking $l_\perp \to 0$, one recovers the full unpolarised DGLAP LO splitting kernel.

- The three diagrams map precisely to the axial gauge calculation by Catani, Grazzini.
- The triple-collinear kernel for quark splitting also emerges. Gluons require more care.
The outlines of a subtraction algorithm emerge. Begin by expanding the virtual matrix element

\[
\mathcal{M}(p_i) = S^{(0)}(\beta_i)\mathcal{H}^{(0)}(p_i) + \frac{\alpha_s}{\pi} \left[ S^{(1)}(\beta_i)\mathcal{H}^{(0)}(p_i) + S^{(0)}(\beta_i)\mathcal{H}^{(1)}(p_i) \right] \\
+ \sum_i \left( J^{(1)}(p_i) - J^{(1)}_{E,0}(\beta_i) \right) S^{(0)}(\beta_i)\mathcal{H}^{(0)}(p_i) \right] + \mathcal{O}(\alpha_s^2).
\]

Construct an expression for the virtual poles of the cross section in terms of the kernels

\[
V_n \equiv \text{2 Re} \left[ M_n^{(0)*} M_n^{(1)} \right] = \frac{\alpha_s}{\pi} S_0^{(1)}(\beta_i) \left| \mathcal{H}_n^{(0)}(p_i) \right|^2 + \frac{\alpha_s}{\pi} \sum_i \left( J_0^{(1)}(p_i) - J^{(1)}_{E,0}(\beta_i) \right) \left| \mathcal{M}_n^{(0)}(p_i) \right|^2 + \text{finite}.
\]

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

\[
S_0^{(1)}(\beta_i) + 4\pi^2 \int d\text{LIPS}_2(d) S_1^{(0)}(k, \beta_i) = \text{finite}
\]

\[
J_0^{(1)}(l, p, n) + 4\pi^2 \int d\text{LIPS}_2(d) J_1^{(0)}(k; l, p, n) = \text{finite}
\]

Construct the appropriate local functions.

\[
K_{n+1}\text{soft} = 4\pi\alpha_s S_1^{(0)}(k, \beta_i) \left| \mathcal{H}_n^{(0)}(p_i) \right|^2 .
\]

\[
K_{n+1}\text{coll} = 4\pi\alpha_s \sum_i J_1^{(0)}(k_i; l, p_i, n_i) \left| \mathcal{M}_n^{(0)}(p_1, \ldots, p_{i-1}, l, p_{i+1}, \ldots, p_n) \right|^2 .
\]

All the fun with sectors, mappings and parametrisations is, unfortunately, still needed …
Tracing soft and collinear at NNLO

As an example of the detailed structure of soft and collinear subtractions at high orders, consider the “jet factor” in the factorised virtual matrix element.
Tracing soft and collinear at NNLO

As an example of the detailed structure of soft and collinear subtractions at high orders, consider the “jet factor” in the factorised virtual matrix element.

\[
\frac{\prod_{i=1}^{n} \mathcal{J}_0(p_i)}{\prod_{i=1}^{n} \mathcal{J}_0^E(\beta_i)} = 1 + g^2 \sum_{i=1}^{n} \left[ \mathcal{J}_0^{(1)}(p_i) - \mathcal{J}_0^{E(1)}(\beta_i) \right] \\
+ g^4 \sum_{i=1}^{n} \left[ \mathcal{J}_0^{(2)}(p_i) - \mathcal{J}_0^{E(2)}(\beta_i) \right] \\
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- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E(1)}(\beta_i) \left[ \mathcal{J}_0^{(1)}(p_i) - \mathcal{J}_0^{E(1)}(\beta_i) \right]
\]
Tracing soft and collinear at NNLO

As an example of the detailed structure of soft and collinear subtractions at high orders, consider the “jet factor” in the factorised virtual matrix element.

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+ g^4 \sum_{i<j=1}^{n} \left[ J_0^{(1)}(p_i) - J_0^E^{(1)}(\beta_i) \right] \left[ J_0^{(1)}(p_j) - J_0^E^{(1)}(\beta_j) \right] \\
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\]

Independent hard collinear poles
As an example of the detailed structure of soft and collinear subtractions at high orders, consider the “jet factor” in the factorised virtual matrix element.

\[
\frac{\prod_{i=1}^{n} \mathcal{J}_0(p_i)}{\prod_{i=1}^{n} \mathcal{J}_0^E(\beta_i)} = 1 + g^2 \sum_{i=1}^{n} \left[ \mathcal{J}_0^{(1)}(p_i) - \mathcal{J}_0^{E(1)}(\beta_i) \right] \\
+ g^4 \sum_{i=1}^{n} \left[ \mathcal{J}_0^{(2)}(p_i) - \mathcal{J}_0^{E(2)}(\beta_i) \right] \\
+ g^4 \sum_{i<j}^{n} \left[ \mathcal{J}_0^{(1)}(p_i) - \mathcal{J}_0^{E(1)}(\beta_i) \right] \left[ \mathcal{J}_0^{(1)}(p_j) - \mathcal{J}_0^{E(1)}(\beta_j) \right] \\
- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E(1)}(\beta_i) \left[ \mathcal{J}_0^{(1)}(p_i) - \mathcal{J}_0^{E(1)}(\beta_i) \right]
\]

The contributions of a single soft gluon accompanied by a hard collinear one factor out and are automatically taken into account.
A number of successful NNLO subtraction algorithms are available.

They are computationally expensive, either analytically, or numerically, or both.

Extensions to multi-leg processes or higher orders is expected to be useful but hard.

Work on refining existing tools to find the `minimal toolbox’ is necessary and under way.

The factorisation of soft and collinear virtual amplitudes contains important information.

A general all-order definition of soft and/or collinear counterterms has been proposed.

Existing results at NLO and beyond are reproduced and possibly generalised.

Tracing the real emission counterterms starting from virtual poles is a useful strategy.

A parallel effort to construct a detailed analytic subtraction algorithm is under way.

What we have is promising preliminary evidence: a lot of work remains to be done.
VIELEN DANK!