Scalar correlator and the QCD vacuum energy in 5 loops

Konstantin Chetyrkin (UHH & KIT)

in collaboration with Pavel Baikov (MSU)

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Motivations

3 talks at this workshop (Ben Ruijl, Andreas Naier and Hans Kühn):
the program of “complete” renormalization of QCD at 5 loops has been successfully finished. Main features:

• all $\beta$-functions and anomalous dimensions are in full mutual agreement

• by 3 (almost) independent teams

• with 3 independent methods: a massive one and two versions of massless ones (the global and local versions of the $R^*$)

• and with 4 independent computer programs: Crushers and Spades ($C^{++}$ based)
  Baicer and Forcer (FORM based)

The present talk: the (relatively) small calculation of the 5-loop anomalous dimensions of the “vacuum energy” in QCD. This is the last (yet missing piece) of the program at 5 loops
Plan

• Quantum Action Principle

• vacuum energy in QCD, its anomalous dimension $\gamma_0$ and its relation to
  the renormalization of $G_{\mu\nu}^2$ /”gluon condensate” /,
  $<\bar{\psi}\psi>$ /“chiral” condensate/ within optimized perturbation theory
  the correlator of two scalar currents
  the term $\approx y_t^4 (1 + \alpha_s + \ldots)$ in $\beta_\lambda$ (the $\beta$-function for the Higgs self-
  interaction coupling in the SM)

• many different ways to compute $\gamma_0$

• connection of $\gamma_0$ with the scalar correlator and calculation of the term of
  order $\alpha_s^4$ /five loops/ for $\gamma_0$

Note: the talk is an extension on the 5 loop level the paper “Leading QCD-induced four-loop
contributions to the $\beta$-function of the Higgs self-coupling in the SM and vacuum stability”,
K. Ch. and M. Zoller, arXiv:1604.00853v2
Quantum Action principle (due to Y. Schwinger)

The quantum action principle relates properties of (regularized) Lagrangian and the full Green functions. Consider the generating functional of (connected) Green function:

\[ G(\mathcal{L}, J) = \ln \left( \int \mathcal{D}\Phi e^{iS(\Phi) + \Phi \cdot J} \right), \quad S(\Phi) = \int \mathcal{L}(\Phi) dx \]

The Action Principle states that

\[ \frac{\partial}{\partial \lambda} G(\mathcal{L}, J) = \left( \int \mathcal{D}\Phi e^{iS(\Phi) + \Phi \cdot J} \frac{\partial}{\partial \lambda} S(\Phi) \right) / G(\mathcal{L}, J), \]

where \( \lambda \) is any parameter in the Lagrangian \( \mathcal{L} \). The action principle works for DR Green (for both bare and MS-subtracted) functions (modulo axial anomalies)

/J.H. Lowenstein (71); Y.-M. P. Lam (72); Breitenlohner and D. Maison (77)/
An example: the (renormalized) QCD Lagrangian

\[ \mathcal{L}_{QCD} = -\frac{1}{4}Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} g Z_1^3 g (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) (A_\mu \times A_\nu)^a \]

\[ - \frac{1}{4} g^2 Z_1^4 g (A_\mu \times A_\nu)^2 + Z_2 \bar{\psi} (i\partial - m) \psi \]

with properly chosen ren. constants \( Z_i \) produces finite green functions. If one differentiates \( G(\mathcal{L}, J) \) w.r.t. the (renormalized) quark mass \( m \), then the functional

\[ \left( \int D\Phi e^{iS(\Phi) + \Phi \cdot J} \left( \int Z_m (\bar{\psi}_B \psi_B) (x) dx \right) \right) / G(\mathcal{L}, J) \]

should also be finite (\( \psi_B \equiv \sqrt{Z_2} \psi \)). But, let us consider eq. \( \ast \) at \( J = 0 \). It corresponds, obviously, to the VEV of the operator \( (Z_m \bar{\psi}_B \psi_B) (x) \), which is not finite already at order \( \alpha_s^0 \) due to the tadpole diagrams (all proportional to \( m^3 \) at \( D = 4 \)): 
This means that our renormalized QCD Lagrangian is not full: the term responsible for the renormalization of the *vacuum energy* is missing.
The full QCD Lagrangian (written in terms of bare fields) reads

\[
\mathcal{L}_{QCD} = \frac{1}{4}(G_{\mu\nu}^B)^2 + \overline{\psi}_B(i\hat{D} - m^B)\psi_B - E_0^B,
\]

\[
E_0^B = \mu^{-2\epsilon}(\rho_V(\mu) - Z_0(\alpha_s)m^4(\mu)), \quad \epsilon = (4 - D)/2, \quad (1)
\]

here \(\rho_V(\mu)\) is the (renormalized) density of vacuum energy and \(Z_0\) is the corresponding renormalization constant.

Note that the term “vacuum energy” is the right one as according to the Gell-Mann-Low formula (1951) the expression \(\ln G(\mathcal{L}, J = 0)\) which is made finite with the renormalization constant \(E_0\) is equal to

\[
\ln \left( \langle 0 | \hat{U}(\tau_2, \tau_1) \rangle \right) = -i\Delta E(\tau_2 - \tau_1) \quad \text{in the limit of} \quad (\tau_2 - \tau_1) \to \infty
\]

Here \(\Delta E = \rho_V V\) is the shift of the vacuum energy due to interaction, and, thus, \(\rho_V\) is vacuum energy density.
Using the action principle one could express the anomalous dimensions of all 3 physical scalar dim=4 operators appearing in the QCD Lagrangian

\[ O_1 = -\frac{1}{4}(G^B_{\mu\nu})^2, \quad O_2 = m_B\bar{\psi}_B\psi_B, \quad O_3 = m_B^4 \]

as follows \(([O_i] = Z_{ij}O_j)\):

\[
Z_{ij} = \begin{pmatrix}
\frac{1}{1-\beta/\epsilon} & \frac{\gamma m/\epsilon}{1-\beta/\epsilon} & -\mu^{-2\epsilon}Z_m^{-4}\alpha_s\frac{\partial}{\partial\alpha_s}Z_0 \\
0 & 1 & -4\mu^{-2\epsilon}Z_m^{-4}Z_0 \\
0 & 0 & Z_m^{-4}
\end{pmatrix}
\]

\[
\gamma_{ij} = \left(\mu^2\frac{d}{d\mu^2}Z_{ik}\right)Z^{-1}_{kj} = \begin{pmatrix}
-\alpha_s\frac{\partial\beta}{\partial\alpha_s} & -\alpha_s\frac{\partial\gamma m}{\partial\alpha_s} & -\alpha_s\frac{\partial\gamma_0}{\partial\alpha_s} \\
0 & 0 & -4\gamma_0 \\
0 & 0 & 4\gamma_m
\end{pmatrix}
\]

* V. Spiridonov (‘84), V. Spiridonov & K.Ch. (‘88)
Other useful applications of $\gamma_0$

1. If we differentiate the generating functional $\ln G(\mathcal{L}, J = 0)$ two times wrt the quarks mass $m$, then we will produce, obviously, the scalar correlator (at zero momentum transfer):

$$\Pi^S(q^2, m) = \int dx \ e^{iqx} \langle 0 | T [j_s(x)j_s(0)] | 0 \rangle$$

The correlator meets the following evolution eq.:

$$\mu^2 \frac{d}{d\mu^2} \Pi^S = \gamma_q^S q^2 + \gamma_m^S m^2$$

Clearly, $\gamma_m^S$ is obtained by two more differentiation wrt $m$ and essentially is proportional to $\gamma_0$. The knowledge of $\gamma_m^S$ helps to find the power suppressed correction to the Higgs decay rate into $b$-quarks.
2. As $\gamma_0$ fully describes the mixing of $m\bar{\psi}\psi$ to $m^4$, it could be effectively employed to find the quartic mass corrections to $R(s)$ (K. Ch. & J. Kühn, 1994, term of order \(m_q^4/s^2\alpha_s^2\); K. Ch., R. Harlander, & J. Kühn, 2001, term of order \(m_q^4/s^2\alpha_s^3\))

3. $\gamma_0$ as well as the (perturbatively computed with all quark massive) VEV $\langle \bar{\psi}\psi \rangle$ are main ingredients of the so-called variationally and RG optimized perturbation theory as applied to the chiral condensate (in the massles limit!)

See, the paper J. L. Kneur and A. Neveu, “Chiral condensate from renormalization group optimized perturbation,” Phys. Rev. D 92, no. 7, 074027 (2015)/

Using 4-loop $\gamma_0$ and 3-loop $\langle \bar{\psi}\psi \rangle$ they arrived at

$$-\langle \bar{\psi}\psi \rangle^{1/3} (2\text{ GeV}) = 281 \pm 4 \pm 7\text{MeV}$$

which is in agreement to other independent determinations
4. If we differentiate the generating functional $\ln G(\mathcal{L}, J = 0)$ four times wrt the quarks mass $m$, we find that $\gamma_0$ is directly related to an important contribution to the $\beta$-function of the Higgs self-interaction $\beta_\lambda$ in the SM /K. Ch, M. Zoller (16)/. Namely, the “maximally QCD” term in the $\beta_\lambda$ which has the structure

$$
\beta^{QCD}_\lambda = y_t^4 \left( \sum_{i \geq 0} c_i (a_s)^i \right)
$$

happens to be equal

$$
\beta^{QCD}_\lambda \equiv y_t^4 \gamma_0
$$

which is, of course, a direct consequence of the action principle and, as such, must be valid in all orders in $\alpha_s$! /Checked by direct comparison up to and including 4 loops./
Example for free theory \((\alpha_s = 0)\)

\[
(\frac{d}{dm})^4 \quad \Rightarrow
\]
There are remarkably many different ways to compute the anom. dim. $\gamma_0$: it is now well-known at 4-loop level, the calucaltions were performed in 3 different ways (with agreeing results for $D=4$).

0. by directly renormalizing the vacuum energy diagrams (was done only for $D=3$ QCD by Y. Schroder in 2002 in order to find free energy in the effective high temperature QCD)

1. by renormalizing the scalar correlator (K. Ch. ('98) with MINCER and IR-tricks); later it was confirmed /and a little bit corrected/ by P. Baikov and K. Ch (2000) with BAICER and without any IR-tricks)

2. by renormalizing the VEV of the scalar current /C. Sturm '06/ (that is computing massive tadpoles)

3. by computing and renormalizing OPE in x-space for the quark propagators (via MINCER, that is dealing with 3-loop massless propgators). /A. Maier '08/

4. by computing the $\beta_\lambda$ / S. Martin (2015); K. Ch. and M. Zoller (2016) /both with massive tadpoles/. See, also, the talk by A. Pikelner.
We have computed the 5-loop contribution to the $\gamma_0$ within the massless approach (that is with the use of the global $R^*$-operation, the FORM program BAICER (and the computer facilities of the KIT) for a general case of arbitrary many quark flavours with different masses.

The corresponding vacuum anomalous dimension $\hat{\gamma}_0$ is conveniently defined as: ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\hat{\gamma}_0 = \left( \sum_i m_i^4 \right) \gamma_{d_0}^d(a_s) + \left( \sum_{i \neq j} m_i^2 m_j^2 \right) \gamma_{n_0}^{n_d}(a_s)$$

(for one massive quark we have $\hat{\gamma}_0 = m_q^4 \gamma_{0}^{d_0}(a_s)$ and $\gamma_0 \equiv \gamma_{0}^{d_0}$)

A convenient way to compute $\hat{\gamma}_0$ is to start from the (non-diagonal) scalar correlator

$$\Pi^{SS}(Q^2, m_u, m_d, m, \mu, a_s) = \int e^{iqx} \langle 0 | T \left[ j^S(x) (j^S)^\dagger \right] (0) | 0 \rangle dx$$

which meets the evolution equation:

$$\mu^2 \frac{d}{d\mu^2} \Pi^{SS} = Q^2 \gamma^{SS}(a_s) + \gamma_{m}^{SS}(a_s, m_u, m_d, m_q)$$

(2)

with

$$\gamma_{m}^{SS} = \gamma_{1}^{SS}(a_s) (m_u^2 + m_d^2) + \gamma_{2}^{SS}(a_s) m_u m_d + \gamma_{3}^{SS}(a_s) \sum_i m_i^2$$
To connect $\gamma^{SS}_m$ with $\hat{\gamma}_0$ we use the well-known Ward identity /Broadhurst (1975)/

$$q_\mu q_\nu \Pi^V_{\mu\nu} = (m_u - m_d)^2 \Pi^{SS} + (m_u - m_d)(\langle \bar{\psi}_u \psi_u \rangle \langle \bar{\psi}_d \psi_d \rangle)$$

With $q = 0$ and

$$\mu^2 \frac{d}{d\mu^2} m_d \bar{\psi}_u \psi_u = -m_d \frac{\partial}{\partial m_u} \hat{\gamma}_0$$

we arrive after simple algebra at:

$$\mu^2 \frac{d}{d\mu^2} \Pi^{SS} = \frac{1}{m_u - m_d} \left( \frac{\partial}{\partial m_u} - \frac{\partial}{\partial m_d} \right) \hat{\gamma}_0$$

$$= 4 (\gamma^d_0 - \gamma^{nd}) \left( m^2_u + m^2_d + m_u m_d \right) + 4 \gamma^{nd} \sum_i m^2_i$$

Thus, we can easily restore the full vacuum anom. dim. $\hat{\gamma}_0$ from $\gamma^{SS}_m$ with one self-consistency check, namely:

$$\gamma^{SS}_1 \equiv \gamma^{SS}_2$$
\[ \gamma_0^{di} = \frac{1}{16\pi^2} \left( -3 - 4 \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ -\frac{313}{24} + \frac{5}{4} n_f + 2 \zeta_3 \right] \right. \\
+ \left( \frac{\alpha_s}{\pi} \right)^3 \left[ -\frac{14251}{432} + \left( \frac{341}{486} - \frac{2}{2} \zeta_3 \right) n_f^2 + \frac{231}{2} \zeta_3 \right] \\
+ \left( \frac{4109}{648} + \frac{35}{18} \zeta_3 + \frac{16}{3} \zeta_4 \right) n_f - \frac{19}{2} \zeta_4 - \frac{1975}{18} \zeta_5 \right] \\
+ \left( \frac{\alpha_s}{\pi} \right)^4 \left[ -\frac{303061}{3072} + \frac{882061}{432} \zeta_3 + \frac{20083}{288} \zeta_3^2 - \frac{124511}{192} \zeta_4 - \frac{11543}{3} \zeta_5 + \frac{632375}{576} \zeta_6 + \frac{36883}{32} \zeta_7 \right] \\
+ n_f \left( \frac{6286061}{62208} + \frac{593}{864} \zeta_3 - \frac{2327}{144} \zeta_3^2 + \frac{50867}{576} \zeta_4 + \frac{1571}{144} \zeta_5 - \frac{27125}{288} \zeta_6 - \frac{147}{16} \zeta_7 \right) \\
+ n_f^2 \left( -\frac{530837}{373248} - \frac{4817}{432} \zeta_3 + \frac{179}{96} \zeta_4 + \frac{83}{9} \zeta_5 \right) + n_f^3 \left( \frac{373}{3456} + \frac{13}{216} \zeta_3 - \frac{1}{6} \zeta_4 \right) \]
\[
\gamma_{0}^{nd} = \frac{1}{16\pi^2} \left( 6 \left( \frac{\alpha_s}{\pi} \right)^2 + \left( \frac{\alpha_s}{\pi} \right)^3 \left[ \frac{176}{3} + 33\zeta_3 - 15\zeta_5 + n_f \left( -\frac{4}{9} \right) \right] \right)
\]
\[
+ \left( \frac{\alpha_s}{\pi} \right)^4 \left[ \left( \frac{14147}{24} + \frac{36691}{72} \zeta_3 + \frac{967}{16} \zeta_2^3 - \frac{4851}{32} \zeta_4 - \frac{6890}{9} \zeta_5 + \frac{3675}{32} \zeta_6 + \frac{3829}{12} \zeta_7 \right) \right. \\
+ n_f \left( -\frac{779}{27} - 24\zeta_3 - \frac{9}{8} \zeta_2^2 + \frac{99}{16} \zeta_4 + \frac{15}{2} \zeta_5 - \frac{75}{16} \zeta_6 \right) - \frac{1}{18} n_f^2 \right] \right)
\]

Numerically, we have:

\[
\gamma_{0}^{di} = \frac{-3}{16\pi^2} \left( 1 + 1.333 a_s + (3.546 - 0.4167 n_f) a_s^2 \right.
\]
\[
+ (6.069 - 4.8170 n_f + 0.03327 n_f^2) a_s^3 \right. \]
\[
+ (-14.658 - 26.779 n_f + 1.0816 n_f^2 + 0.000038 n_f^3) a_s^4 \left. \right) 
\]
\[
= 1. + 1.3333 a_s + 1.04585 a_s^2 - 21.636 a_s^3 - 136.384 a_s^4 \quad \text{if } n_f = 6 
\]
\[
\gamma_{0}^{nd} = \frac{3}{8\pi^2} \left( a_s^2 + (13.7968 - 0.07407 n_f) a_s^3 + (128.339 - 8.2703 n_f - 0.0093 n_f^2) a_s^4 \right) 
\]
Conclusions

- We have computed the QCD vacuum anomalous dimension at 5 loops and, thus, have finished the program of the 5-loop renormalization of the QCD Lagrangian