Closed Form Solutions

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What is a closed form solution?

**Example:** Solve this equation for $y = y(x)$.

$$y' = \frac{4 - x^3}{(1 - x)^2} e^x$$

**Definition**

A closed form solution is an expression for an **exact** solution given with a **finite amount of data**.

This is not a closed form solution:

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 + \cdots$$

because making it exact requires infinitely many terms.

The Risch algorithm finds a closed form solution:

$$y = \frac{2 + x^2}{1 - x} e^x$$
Previous slide: A closed form solution is an expression for an exact solution with only a finite amount of data.

Risch algorithm finds (if it exists) a closed form solution $y$ for:

$$y' = f$$

To make that well-defined, specify which expressions are allowed:

Define $E_{\text{in}}$ and $E_{\text{out}}$ such that:

- Any $f \in E_{\text{in}}$ is allowed as input.
- Output: a solution iff $\exists$ solution $y \in E_{\text{out}}$.

Risch: $E_{\text{in}} = E_{\text{out}} = \{\text{elementary functions}\}$

$$= \{\text{expressions with } \mathbb{C}(x) \ \exp \ \log \ + - \cdot \div \text{ composition and algebraic extensions}\}.$$
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Liouvillian solutions

Kovacic’ algorithm (1986)

1. Solves homogeneous differential equations of order 2

\[ a_2 y'' + a_1 y' + a_0 y = 0 \]

(Risch: inhomogeneous equations of order 1)

2. It finds solutions in a larger class:

\[ E_{\text{out}} = \{ \text{Liouvillian functions} \} \supset \{ \text{elementary functions} \} \]

3. but it is more restrictive in the input:

\[ a_0, a_1, a_2 \in \{ \text{rational functions} \} \subset \{ \text{elementary functions} \} \]

Remark

\[ \exists \text{ common functions that are not Liouvillian.} \]

Allow those as closed form \( \sim \) need other solvers.
A non-Liouvillean example

Let

\[ y := \oint_\gamma \exp \left( \frac{t^2 + x + 1}{x - t} \right) \frac{t}{x^2 + x + 1} \, dt \]

Zeilberger's telescoping algorithm \( \rightsquigarrow \) an equation for \( y \):

\[(x^4 - x)y'' + (4x^4 + 2x^3 - 3x^2 - 7x + 1)y' + (6x^3 - 9x^2 - 12x + 3)y = 0\]

Closed form solutions were thought to be rare.

But (for order 2) telescoping equations often (always?) have closed form solutions:

\[ \exp(-2x) \cdot \left( I_0 \left( 2\sqrt{x^2 + x + 1} \right) - \frac{x + 1}{\sqrt{x^2 + x + 1}} I_1 \left( 2\sqrt{x^2 + x + 1} \right) \right) \]

and

\[ \exp(-2x) \cdot \left( K_0 \left( 2\sqrt{x^2 + x + 1} \right) + \frac{x + 1}{\sqrt{x^2 + x + 1}} K_1 \left( 2\sqrt{x^2 + x + 1} \right) \right) \]
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\]

and

\[
\exp(-2x) \cdot \left( K_0 \left( 2\sqrt{x^2 + x + 1} \right) + \frac{x + 1}{\sqrt{x^2 + x + 1}} K_1 \left( 2\sqrt{x^2 + x + 1} \right) \right)
\]
Take a rational function in several variables, for example:

\[ F = \frac{1}{1 - x - 2y - 3z - 4yz - 5xyz - 6xyz^2} \in \mathbb{Q}(x, y, z) \]

and write it as a multivariate power series

\[ F = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} x^i y^j z^k \in \mathbb{Q}[[x, y, z]] \]

The \textbf{diagonal} is

\[ \text{diag}(F) = \sum_{i=0}^{\infty} a_{iii} x^i \in \mathbb{Q}[[x]] \]

\textbf{Fact:} Diagonals of rational functions are \textbf{D-finite} (holonomic); they satisfy a homogeneous linear differential equation with polynomial coefficients.

\textbf{Conjecture:} If such an equation has order 2 then it has closed form solutions.
Diagonals of rational functions

For the example, the diagonal $D = \text{diag}(F)$ satisfies:

$$a_2 D'' + a_1 D' + a_0 D = 0$$

where

$$a_2 = x(2x + 1)(40x^3 + 4x^2 - 72x - 5)(500x^4 - 149x^3 + 939x^2 - 1061x + 5)$$

$$a_1 = 120000x^8 + 32160x^7 - 288416x^6 - 74344x^5 - 341206x^4 + 135372x^3 + 93397x^2 + 10510x - 25$$

$$a_0 = 40000x^7 + 10240x^6 - 210464x^5 + 4944x^4 - 58610x^3 + 15752x^2 - 3715x + 1225$$

To solve order 2 equations with $a_0, a_1, a_2 \in \mathbb{Q}[x]$ download:

www.math.fsu.edu/~eimamoglu/hypergeometricsols

Result:

$$2F_1 \left( \begin{array}{cc} \frac{1}{12} & \frac{5}{12} \\ 1 & \end{array} \bigg| f \right) \cdot (625x^4 + 140x^3 + 1158x^2 - 196x + 1)^{-1/4}$$

where

$$f = 124416 x^3 (2x + 1)^2 (500x^4 - 149x^3 + 939x^2 - 1061x + 5)$$

$$\frac{1}{(625x^4 + 140x^3 + 1158x^2 - 196x + 1)^3}$$
Globally bounded equations

Random equations rarely have closed form solutions.

So-called **globally bounded** equations are common in:
- combinatorics (generating functions of sequences in oeis.org)
- physics (Ising model, Feynman integrals, etc.)
- Period integrals, creative telescoping, diagonals.

**Conjecture**

*Globally bounded equations (of order 2) have closed form solutions.*

In other words: **Closed form solutions are common**

(for non-random equations of order 2)

If Maple + Mathematica don’t find them  ⇔  download our code!
Local to global strategy

**Risch:** Given elementary function $f$, solve:

$$y' = f$$

$\{\text{poles of } y\} \subseteq \{\text{poles of } f\} = \text{known}$

**Kovacic:** Given polynomials $a_0, a_1, a_2 \in \mathbb{C}[x]$, solve:

$$a_2y'' + a_1y' + a_0y = 0$$

$\{\text{poles of } y\} \subseteq \{\text{roots of } a_2\} = \text{known}$

Local to global strategy

$\{\text{poles of } y\} + \{\text{terms in polar parts}\} \ (\text{+ other data}) \rightsquigarrow y$
Local data: Classifying singularities

Example:

\[ y = \exp(r) \quad \text{where} \quad r = \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x - 1} + 5 + 7x \]

\( y \) has **essential singularities** at the poles of \( r \).

**Definition**

\( y_1, y_2 \neq 0 \) have an **equivalent singularity** at \( x = p \)
when \( y_1/y_2 \) is meromorphic at \( x = p \).

Equivalence class of \( y \) at \( x = 0, \ x = 1, \ x = \infty \) (local data)

\[ \leadsto \quad \text{Polar part of } r \text{ at } x = 0, \ x = 1, \ x = \infty \] (local data)

\[ \leadsto \quad \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x - 1} + 7x \] (local data)

\[ \leadsto \quad r \quad \text{(up to a constant term)} \] (global data)

\[ \leadsto \quad y \quad \text{(up to a constant factor)} \] (global data)
Recall: $y_1, y_2$ have **equivalent singularity** at $x = p$ if $y_1/y_2$ is meromorphic at $x = p$.

Hence:

$y_1, y_2$ equivalent at every $p \in \mathbb{C} \cup \{\infty\}$

$\iff$

$y_1/y_2$ meromorphic at every $p \in \mathbb{C} \cup \{\infty\}$

$\iff$

$y_1/y_2 \in \mathbb{C}(x)$

Hence:

${\{\text{Eq. class of } y \text{ at all } p}\}} \iff y \text{ up to a rational factor}$

For a differential equation $L$ can compute:

${\{\text{generalized exponents of } L \text{ at } p}\} \approx {\{\text{Eq. classes of solutions}\}}$

Choose the right one at each $p \leadsto$ a solution (up to $\approx$)
Example: generalized exponents

Example: let $L$ have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp\left(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x - 3} + \frac{e_{4,1}}{x - 4}\right)$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp\left(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x - 3} + \frac{e_{4,2}}{x - 4}\right)$$

where $e_{p,i} \in \mathbb{C}\left[\frac{1}{x-p}\right]$ encodes the polar part $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the **generalized exponents** of $L$ at $x = p$ and can be computed from $L$:

$$E_0 = \{e_{0,1}, e_{0,2}\}, \quad E_3 = \{e_{3,1}, e_{3,2}\}, \quad E_4 = \{e_{4,1}, e_{4,2}\}$$

To find $y_1$ we need to **choose the correct element** of each $E_p$.

The example has $2^3 = 8$ combinations.
One combination $\mapsto y_1$, another $\mapsto y_2$, other six $\mapsto$ nothing.

Can reduce #combinations (e.g. Fuchs’ relation)
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Can reduce \#combinations (e.g. Fuchs’ relation)
Generalized exponents $\rightsquigarrow$ hyper-exponential solutions:

Let $a_0, \ldots, a_n \in \mathbb{C}[x]$ and $L(y) := a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$.

Hyper-exponential solution: $y = \exp(\int r)$ for some $r \in \mathbb{C}(x)$.

\{\text{generalized exponent of such } y \text{ at all singularities } p \text{ of } L\} \\
$\rightsquigarrow$ \\
y up to a \textbf{polynomial} factor (generalized exponent $\approx$ eq. class)

Algorithm hyper-exponential solutions:

1. Compute generalized exponents $\{e_{p,1}, \ldots, e_{p,n}\}$ at each singularity $p \in \mathbb{C} \cup \{\infty\}$ of $L$.
2. For each combination $e_p \in \{e_{p,1}, \ldots, e_{p,n}\}$ (for all $p$) compute polynomial solutions of a related equation.
Generalized exponents $\leadsto$ hyper-exponential solutions:

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\{ \text{generalized exponent of such } y \text{ at all singularities } p \text{ of } L \} \leadsto y \text{ up to a polynomial factor} \quad (\text{generalized exponent } \approx \text{ eq. class})

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1. **Compute generalized exponents** \( \{e_{p,1}, \ldots, e_{p,n}\} \) at each singularity \( p \in \mathbb{C} \cup \{\infty\} \) of \( L \).
2. For each **combination** \( e_p \in \{e_{p,1}, \ldots, e_{p,n}\} \) (for all \( p \)) compute **polynomial solutions** of a related equation.
**Goal:** define, then find, **closed form solutions** of:

\[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{with} \quad a_0, \ldots, a_n \in \mathbb{C}(x). \quad (1) \]

The **order** is \( n \) (we assume \( a_n \neq 0 \)).

Consider **closed form** expressions in terms of **functions** that are:

1. **well known,** and
2. **D-finite:** satisfies an equation of form (1).

**D-finite of order 1 = hyper-exponential function.**

**Well known D-finite functions of order 2:**

- Airy functions, Bessel functions, Kummer, Whittaker, \ldots
- Gauss hypergeometric function \( _2 F_1(a, b; c \mid x) \)

Klein’s theorem: Liouvillian solutions are \( _2 F_1 \) expressible.
Closed form solutions of linear differential equations:

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(1)

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Idea for constructing a Bessel-solver:

- Bessel functions have an essential singularity at $x = \infty$.
- Just like the function $\exp(x)$.
- So the strategy for hyper-exponential solutions may work for Bessel-type solutions as well.
- It also works for Airy, Kummer, Whittaker, and hypergeometric $\binom{p}{q}$ functions if $p + 1 \neq q$.

Later: Other strategies for Gauss hypergeometric $\binom{2}{1}$ function (to solve globally bounded equations of order 2).

Question: Which Bessel expressions should the solver look for? Which Bessel expressions are D-finite?
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Later: Other strategies for Gauss hypergeometric $2F_1$ function (to solve globally bounded equations of order 2).

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Let $B_\nu(x)$ be one of the Bessel functions, with parameter $\nu$.

**Bessel type closed form expressions** should allow:
- algebraic functions
- exp and log
- composition
- field operations
- differentiation and integration
- and of course $B_\nu(x)$.

**Example:** $B_0(\exp(x))$ is a **Bessel type closed form expression** but is **not relevant** for (1) since it is **not D-finite**.

**Question:** which Bessel type expressions are D-finite?
D-finite functions:

A function $y = y(x)$ is **D-finite of order** $n$ if it satisfies a
differential equation of order $n$ with rational function coefficients.

**Operations that don’t increase the order:**

1. $y(x) \mapsto y(f)$ for some $f \in \mathbb{C}(x)$ called **pullback function**.
2. $y \mapsto r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}$ for some $r_i \in \mathbb{C}(x)$.
3. $y \mapsto \exp(\int r) \cdot y$ for some $r \in \mathbb{C}(x)$.

**Operations that can increase the order:**

4. Same as (1),(2),(3) but with algebraic functions $f$, $r_i$, $r$.
5. $y_1, y_2 \mapsto y_1 + y_2$ \quad order $n_1, n_2 \leadsto$ order $\leq n_1 + n_2$
6. $y_1, y_2 \mapsto y_1 \cdot y_2$ \quad order $n_1, n_2 \leadsto$ order $\leq n_1 \cdot n_2$
   Special case: $y \mapsto y^2$ \quad order $n$ \leadsto order $\leq \frac{n(n+1)}{2}$

Have algorithms to recover any combination of: (2), (3), (5), and part of (6).
D-finite functions:

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Have algorithms to recover any combination of: (2), (3), (5), and part of (6).
Bessel type solutions of second order equations

Let $B_\nu(x)$ be one of the Bessel functions.  
$B_\nu(\sqrt{x})$ is D-finite of order 2.  Transformations (1), (2), (3) $\leadsto$

$$\exp(\int r) \cdot \left( r_0 \cdot B_\nu(\sqrt{f}) + r_1 \cdot B_\nu(\sqrt{f})' \right)$$  (2)

is D-finite of order 2 for any $r, r_0, r_1, f \in \mathbb{C}(x)$.

**Theorem (Quan Yuan 2012)**

Let $k$ be a subfield of $\mathbb{C}$ and let $L$ be a linear homogeneous differential equation over $k(x)$ of order 2.  
If $\exists$ solution of form (2) with algebraic functions $r, r_0, r_1, f$ then $\exists$ solution with rational functions $r, r_0, r_1, f \in k(x)$.

**Bessel-type solutions of higher order equations:**

$\leadsto$ Add transformations (4),(5),(6).
Finding Bessel type solutions

\[ a_2 y'' + a_1 y' + a_0 y = 0 \quad \text{where} \quad a_0, a_1, a_2 \in \mathbb{C}[x]. \]

**Goal:** Find Bessel-type solutions.

**Idea:** Recover the pullback function \( f \) in transformation (1) from data that is invariant under transformations (2),(3).

**Hyper-exponential solutions:**
Generalized exponents \( \rightsquigarrow \{ \text{polar parts of } f \} \rightsquigarrow f \)

**Bessel-type solutions:**
Generalized exponents \( \rightsquigarrow \{ \text{[half] of terms of polar parts of } f \} \rightsquigarrow \text{need more data to find } f. \)

**More data:** regular singularities \( \rightsquigarrow \text{roots of order } \not\in \text{denom}(\nu) \cdot \mathbb{Z} \)

**Combine data** \( \rightsquigarrow f \) except in one case: \( \text{denom}(\nu) = 2 \) that “happens” to be solvable with Kovacic
Finding Bessel type solutions

\[ a_2y'' + a_1y' + a_0y = 0 \quad \text{where} \quad a_0, a_1, a_2 \in \mathbb{C}[x]. \]

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**Hyper-exponential solutions:**

Generalized exponents \( \rightsquigarrow \) \{polar parts of \( f \)\} \( \rightsquigarrow f \)

**Bessel-type solutions:**

Generalized exponents \( \rightsquigarrow \) \{[half] of terms of polar parts of \( f \)\} \( \rightsquigarrow \) need **more data** to find \( f \).

**More data:** regular singularities \( \rightsquigarrow \) roots of order \( \notin \text{denom}(\nu) \cdot \mathbb{Z} \)

**Combine data** \( \rightsquigarrow f \) **except in one case:** \( \text{denom}(\nu) = 2 \) that “happens” to be solvable with Kovacic
Local to global strategy for difference equations

Use local data that is **invariant** under the difference analogue of transformations (2),(3):

- Giles Levy (Ph.D 2009)
- Yongjae Cha (Ph.D 2010)

Example: oeis.org/A000179 (Ménage numbers)

**Recurrence operator:**

\[(\tau + 1) \circ (n\tau^2 - (n^2 + 2n)\tau - n - 2)\]

*where \(\tau\) is the **shift-operator**.*

**_solver**  \[\sim c_1 \cdot n \cdot I_n(-2) + c_2 \cdot n \cdot K_n(2) + c_3 \cdot \epsilon(n)\]

*where \(I_n(x)\) and \(K_n(x)\) are **Bessel functions** and \(\epsilon(n)\) is a **complicated expression** that converges to 0 as \(n \to \infty\).*

**Result:**

\[A000179(n) = \text{round}\left(\frac{2n}{e^2} \cdot K_n(2)\right) \quad \text{(for } n > 0)\]
The Gauss hypergeometric function is:

\[
\mathop{2F_1}\nolimits\left(\begin{array}{c}a, b \\ c\end{array}\right | x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n
\]

where \((a)_n = a \cdot (a + 1) \cdots (a + n - 1)\).

If \(L(y) = 0\) is a globally bounded equation of order 2 then it conjecturally has algebraic or \(2F_1\)-type solutions:

\[
y = \exp\left(\int r \cdot \left( r_0 \cdot \mathop{2F_1}\nolimits\left(\begin{array}{c}a, b \\ c\end{array}| f \right) + r_1 \cdot \mathop{2F_1}\nolimits\left(\begin{array}{c}a, b \\ c\end{array}| f \right)'\right)\right)
\]

Problem: The local to global strategy:

\[
\text{invariant local data } \rightsquigarrow \text{ pullback function } f \rightsquigarrow y
\]

works for many functions, but \(2F_1\) can be problematic because \(f\) can be large even if the amount of local data is small.
The **Gauss hypergeometric function** is:

\[
_{2}F_{1}\left(\begin{array}{c}
\frac{a}{c}, \frac{b}{c} \\
\frac{c}{c}
\end{array} \middle| \frac{x}{x}\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n
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where \((a)_n = a \cdot (a + 1) \cdots (a + n - 1)\).

If \(L(y) = 0\) is a **globally bounded** equation of order 2 then it conjecturally has algebraic or \(_2F_1\)-type solutions:

\[
y = \exp\left(\int r \cdot \left( r_0 \cdot _2F_1\left(\begin{array}{c}
a, b \\
c
\end{array} \middle| f\right) + r_1 \cdot _2F_1\left(\begin{array}{c}
a, b \\
c
\end{array} \middle| f\right)\right)\right)
\]

**Problem**: The local to global strategy:

\[
\text{invariant local data } \leadsto \text{ pullback function } f \leadsto y
\]

works for many functions, but \(_2F_1\) can be problematic because \(f\) can be large even if the amount of local data is small.
Small equation:

\[ 4x(x^2 - 34x + 1)y'' + (8x^2 - 204x + 4)y' + (x - 10)y = 0 \]

The smallest solution:

\[ \frac{\sqrt{3 - 3x - \sqrt{x^2 - 34x + 1}}}{x + 1} \cdot _2F_1 \left( \frac{1}{3}, \frac{2}{3} \mid f \right) \]

has

\[ f = \frac{(x^3 + 30x^2 - 24x + 1) - (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x + 1)^3} \]

How to construct \( f \) from a \textbf{small amount} of invariant local data:

- Exponent-differences: \( 0, \ 0, \ \frac{1}{2} \) (mod \( \mathbb{Z} \))
- at the singularities: \( x = 0, \ x = \infty, \ x^2 - 34x + 1 = 0 \)
$2F_1$-type solutions and related topics:

- Tingting Fang (Ph.D 2012)
  - Compute $D$-module automorphisms $\rightsquigarrow$ descent.
    (also useful for non $2F_1$ cases and for order $> 2$)

- Vijay Kunwar (Ph.D 2014)
  - Small $f$: Construct from invariant local data.
  - Large $f$: Tabulate and use combinatorial objects (such as dessins d’enfant) to prove completeness.

- Erdal Imamoglu (Ph.D 2017)
  - If transformation (2) is not needed: quotient method.
  - Otherwise: Simplify equations using “integral basis”. Then use quotient method.

- Wen Xu (Ph.D in progress)
  - Multivariate generalizations of $2F_1$ such as Appell $F_1$. 
The following equation came from lattice path combinatorics \( \implies \textbf{globally bounded} \), conjecturally implies \( \exists \ 2F_1\)-type solutions

\[
x(8x^2 - 1)(8x^2 + 1)(896x^5 - 512x^4 + 832x^3 - 127x^2 - 6x - 12) \cdot y''
- (8x^2 + 1)(71680x^7 - 36864x^6 + 46080x^5 - 3528x^4 - 5280x^3
+ 155x^2 + 24x + 36) \cdot y' + (1720320x^8 - 786432x^7 + 1078272x^6
- 183360x^5 + 48384x^4 - 12464x^3 - 4560x^2 - 928x - 96) \cdot y = 0
\]

\url{www.math.fsu.edu/~eimamogl/hypergeometricsols} does this:

- Take the **differential module** for this equation.
- Compute its **integral basis**.
- Construct integral element \( Y \) with **minimal degree** at infinity.
- Then \( Y \) satisfies a small equivalent equation:

\[
x(8x^2 - 1)(8x^2 + 1) \cdot Y'' + (320x^4 - 1) \cdot Y' + 192x^3 \cdot Y = 0
\]

- Quotient method \( \rightsquigarrow \) closed form for \( Y \) \( \rightsquigarrow \) closed form for \( y \).
Implementation of Erdal Imamoglu for $2F_1$-type solutions

The following equation came from lattice path combinatorics

$\Longrightarrow$ **globally bounded**, conjecturally implies $\exists\ 2F_1$-type solutions

$$x(8x^2 - 1)(8x^2 + 1)(896x^5 - 512x^4 + 832x^3 - 127x^2 - 6x - 12) \cdot y'' - (8x^2 + 1)(71680x^7 - 36864x^6 + 46080x^5 - 3528x^4 - 5280x^3 + 155x^2 + 24x + 36) \cdot y' + (1720320x^8 - 786432x^7 + 1078272x^6 - 183360x^5 + 48384x^4 - 12464x^3 - 4560x^2 - 928x - 96) \cdot y = 0$$

[www.math.fsu.edu/~eimamogl/hypergeometricsols](http://www.math.fsu.edu/~eimamogl/hypergeometricsols) does this:

- Take the **differential module** for this equation.
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- Construct integral element $Y$ with **minimal degree** at infinity.
- Then $Y$ satisfies a small equivalent equation:

$$x(8x^2 - 1)(8x^2 + 1) \cdot Y'' + (320x^4 - 1) \cdot Y' + 192x^3 \cdot Y = 0$$

- Quotient method $\leadsto$ closed form for $Y \leadsto$ closed form for $y$. 
Example from the Ising model

\[ x(1-11x)(1+4x^2)y'' + (1-22x+8x^2-132x^3)y' + (-3+x-33x^2)y = 0 \]

Solutions near \( x = 0 \):

\[ S_1 = 1 + 3x + \frac{37}{2}x^2 + \cdots \]
\[ S_2 = \ln x + (3 \ln x + 5)x + \left( \frac{37}{2} \ln x + \frac{73}{2} \right)x^2 + \cdots \]

Solutions near \( x = 1/11 \):

\[ T_1 = 1 - \frac{77}{25}(x - \frac{1}{11}) + \cdots \]
\[ T_2 = \ln \left( \frac{1}{11} - x \right) - \left( \frac{77}{25} \ln \left( \frac{1}{11} - x \right) + \frac{649}{125} \right)(x - \frac{1}{11}) + \cdots \]

Analytic continuation from \( x = 0 \) to \( x = 1/11 \) sends \( S_1 \) to a linear combination of \( T_1 \), \( T_2 \). Which linear combination? Can always find an approximate answer (e.g. by evaluating at intermediate points)

www.math.fsu.edu/~eimamogl/hypergeometricsols

\( \rightsquigarrow \) closed form solution \( \rightsquigarrow \) exact answer.

Thank you