Cristal and Azurite: new tools for integration-by-parts reductions

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and work in progress with Alessandro Georgoudis and Yang Zhang
1 **Azurite** *(A ZURich-bred method for finding master InTEgrals)*: Mathematica/Singular package providing a basis of loop integrals


2 **Cristal** *(Complete Reduction of IntegralS Through All Loops)*: complete integration-by-parts reductions.

[work in progress]
Integration-by-parts reductions

Integration-by-parts identities arise from the vanishing integration of total derivatives,

\[
\int \prod_{i=1}^{L} \frac{d^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{\nu_j^\mu P}{D_1^{a_1} \cdots D_k^{a_k}} = 0 .
\]

where \( P \) and \( \nu_j^\mu \) are polynomials in \( \ell_i, p_j, \) and \( a_i \in \mathbb{N} . \)

Role in perturbative QFT calculations:

- **Reduction.** IBP identities reduce any set of loop integrals to a typically much smaller set of master integrals.
- **Computing master integrals.** Using IBP reduction, the master integrals \( \mathcal{I}_j \) can be computed via differential equations:

\[
\frac{\partial}{\partial x_m} \mathcal{I}(x, \epsilon) = A_m(x, \epsilon) \mathcal{I}(x, \epsilon) ,
\]

where \( x_m \) denotes a kinematical invariant.
IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to large linear systems


Azurite uses unitarity cuts to block-diagonalize system

\[
\begin{pmatrix}
\vdots
\end{pmatrix} \rightarrow \begin{pmatrix}
\vdots
\end{pmatrix}
\]

We use the Baikov representation \( (k = \frac{L(L+1)}{2} + L(n - 1)) \),

\[
I(N; a) = \int \prod_{j=1}^{L} \frac{d^D \ell_j}{i \pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{dz_1 \cdots dz_k}{z_1^{a_1} \cdots z_k^{a_k}} \frac{\text{Gram}(z)}{(\vec{\rho}, \ell)} \frac{D-L-n}{2} N
\]

in which cuts are straightforward to apply,

\[
\int \frac{dz_i}{z_i^{a_i}} \xrightarrow{\text{cut}} \int_{\Gamma_{\xi}(0)} \frac{dz_i}{Z_i^{a_i}} \quad i \in S_{\text{cut}}
\]

Let us find the IBP relations of the double box on its maximal cut.

After cutting \( \frac{1}{z_i} \to \delta(z_i), \ i = 1, \ldots, 7 \), the double-box integral takes the form

\[
I_{\text{cut}}^{\text{DB}}[N] = \int \mathrm{d}z_8 \, \mathrm{d}z_9 \, G(z)^{\frac{D-6}{2}} N(z).
\]

An IBP relation corresponds to a total derivative. The generic total derivative of the form \( I_{\text{cut}}^{\text{DB}} \) is

\[
0 = \int \left[ \sum_{i=8}^9 \frac{\partial}{\partial z_i} \left( a_i(z) G(z)^{\frac{D-6}{2}} \right) \right] \mathrm{d}z_8 \wedge \mathrm{d}z_9
= \int \left[ \sum_{i=8}^9 \left( \frac{\partial a_i}{\partial z_i} + \frac{D - 6}{2G(z)} a_i \frac{\partial G(z)}{\partial z_i} \right) \right] G(z)^{\frac{D-6}{2}} \mathrm{d}z_8 \wedge \mathrm{d}z_9.
\]

The red term corresponds to an integral in \((D - 2)\) dimensions.
To get the generic exact form

$$0 = \int \left[ \sum_{i=8}^{9} \left( \frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \right) \right] G(z) \frac{D-6}{2} dz_8 \wedge dz_9$$

to correspond to an IBP relation in $D$ dimensions, we demand that the red term is polynomial,

$$\sum_{i=8}^{9} \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} = \tilde{b} \quad \implies \quad \sum_{i=8}^{9} a_i \frac{\partial G}{\partial z_i} + bG = 0 \quad (\text{with } b = \frac{2}{6-D} \tilde{b})$$

with $a_i, b$ polynomials in $z$. Such equations, with polynomial solutions, are known in algebraic geometry as \textit{syzygy equations}.

[Gluza, Kajda, Kosower, PRD83(2011)045012], [Schabinger, JHEP01(2012)077], [Ita, PRD94(2016)116015]

Obtain IBPs by plugging $(a_i, b)$ into the top equation.
Note: $(qa_i, qb)$ is also a solution, for polynomial $q$. 
Algorithm of **Azurite**

Given an input set of inverse propagators $D_1, \ldots, D_k$, **Azurite** determines a basis as follows.

1. Find automorphism groups $G$ of the graph $\Gamma$ and its subgraphs.

2. Find a list $C$ of cuts such that no two elements of $C$ are related by a discrete symmetry of a (sub)graph of $\Gamma$.

3. For each cut $c \in C$, construct IBP identities and symmetry relations on $c$ (with $\mathbb{Z}_p$ values for kinematics and $D$).

4. Apply Gauss-Jordan elimination to the system of identities. The non-pivot columns correspond to basis integrals.
Example: determine list of needed cuts $\mathcal{C}$

A priori $2^7 = 128$ cuts to consider for four gluons at two loops. Mod out by discrete symmetries $\implies$ only 34 cuts are needed.
Consider IBP identities on maximal cut \( \{1,2,3,4,5,6,7\} \)

With \( I[m,n] \equiv I[(\ell_1 + p_4)^2m(\ell_2 + p_1)^2n] \) and \( (s \equiv 1, t \equiv 3, D \equiv 8009 \text{ (mod 9001)}) \):

\[
I[0,2] = -1075I[0,0] + 3228I[1,0]
\]

Wrt. \( \{I[0,3], I[1,2], I[2,1], I[3,0], I[0,2], I[1,1], I[2,0], I[0,1], I[1,0], I[0,0]\} \), record the linear relations as

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -3228 & 1075 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -449 & 3477 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -4499 & 4499 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1611 & -536 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -3228 & 1075 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1611 & -536 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -449 & 3477
\end{pmatrix}
\]
Example: IBP relations on the maximal cut

Consider IBP identities on maximal cut \( \{1,2,3,4,5,6,7\} \)

\[
\begin{array}{c}
(n) \\
(m)
\end{array}
\stackrel{\text{\rightarrow}}{\longrightarrow}
\begin{array}{c}
3 \\
4 \\
2 \\
7 \\
5 \\
+ c_2
\end{array}
\]

With \( I[m, n] \equiv I[(\ell_1 + p_4)^2m(\ell_2 + p_1)^2n] \) and \( (s \equiv 1, t \equiv 3, D \equiv 8009 \mod 9001) \):

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0 & 0 & 0 & 0 & 1 & 0 & 0 & -4499 & 4499 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -3228 & 1075 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix}
\]

Add non-pivot elements \( \{I[1, 0], I[0, 0]\} \) to basis.
Example: IBPs on less-than-maximal cuts

On the six-fold cut \( \{1,2,3,4,5,7\} \):
\[
\left[ \left( \ell_1 + p_4 \right)^{2n} \left( \ell_2 + p_1 \right)^{2p} \left( \ell_2 + p_4 \right)^{2q} \right] \rightarrow 0
\]
\[\Rightarrow \text{ no elements added to basis}\]

On the five-fold cut \( \{1,2,4,5,7\} \):
\[
\left[ \left( \ell_1 + P_{12} \right)^{2m} \left( \ell_1 + p_4 \right)^{2n} \left( \ell_2 + p_1 \right)^{2p} \left( \ell_2 + p_4 \right)^{2q} \right] \rightarrow c_{mnpq}
\]
\[\Rightarrow \left[0,0,0,0\right] \text{ added to basis}\]
Further graph simplifications

- Subgraphs with massless tadpoles vanish

\[ \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
\end{array} = 0 \]

- Use momenta \((p_1, p_2)\) of graph under consideration; not input momenta \((p_1, p_2, p_3)\)

\[ \begin{array}{c}
2 \\
3 \\
4 \\
1 \\
\end{array} \]

\(\implies\) decreases \# of scalar products.
For the planar triple-box topology, **AZURITE** finds 26 basis integrals
### Azurite: Further Results and Timings

<table>
<thead>
<tr>
<th>Time</th>
<th>MIs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3s</td>
<td>8 MIs</td>
</tr>
<tr>
<td>2.4s</td>
<td>81 MIs</td>
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<tr>
<td>1.8s</td>
<td>18 MIs</td>
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<tr>
<td>2.2s</td>
<td>31 MIs</td>
</tr>
<tr>
<td>2.7s</td>
<td>73 MIs</td>
</tr>
<tr>
<td>1.3s</td>
<td>12 MIs</td>
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<tr>
<td>2.8s</td>
<td>70 MIs</td>
</tr>
<tr>
<td>2.5s</td>
<td>35 MIs</td>
</tr>
<tr>
<td>3.5s</td>
<td>31 MIs</td>
</tr>
<tr>
<td>6.4s</td>
<td>61 MIs</td>
</tr>
<tr>
<td>170s</td>
<td>42 MIs</td>
</tr>
<tr>
<td>67s</td>
<td>85 MIs</td>
</tr>
</tbody>
</table>
### Azurite: timing compared to Mint

#### Mint: [Lee and Pomeransky, JHEP 1311 (2013) 165]

<table>
<thead>
<tr>
<th>topology</th>
<th>Mint timing</th>
<th>Azurite timing</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Mint topology" /></td>
<td>3.1 s</td>
<td>1.3 s</td>
<td>2.4</td>
</tr>
<tr>
<td><img src="image2.png" alt="Mint topology" /></td>
<td>3.2 s</td>
<td>1.3 s</td>
<td>2.5</td>
</tr>
<tr>
<td><img src="image3.png" alt="Mint topology" /></td>
<td>3.2 s</td>
<td>1.3 s</td>
<td>2.5</td>
</tr>
<tr>
<td><img src="image4.png" alt="Mint topology" /></td>
<td>6.9 s</td>
<td>3.5 s</td>
<td>2.0</td>
</tr>
<tr>
<td><img src="image5.png" alt="Mint topology" /></td>
<td>29.5 s</td>
<td>6.4 s</td>
<td>4.6</td>
</tr>
<tr>
<td><img src="image6.png" alt="Mint topology" /></td>
<td>&gt; 2 \times 10^5 s</td>
<td>170 s</td>
<td>&gt; 1000</td>
</tr>
</tbody>
</table>
**Reduze2:** [Manteuffel and Studerus, 1201.4330]

<table>
<thead>
<tr>
<th>sector</th>
<th>Mint</th>
<th>Reduze2</th>
<th>Azurite</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Mint** undercounting: misses critical point(s) at infinity.

**Advertisement:** new version of **Azurite** to appear soon!  
For four-loop example: 180 s $\rightarrow$ 41 s
**Cristal**: complete IBP reductions.

Idea: construct total derivatives on *spanning set of cuts* $\mathcal{C}$:

where $\mathcal{C} = \{ c \in \mathcal{B} : \nexists b \in \mathcal{B} : b \text{ strict subgraph of } c \}$

**Cristal** can also be used to produce differential equations. For example,
Conclusions and Outlook

- **Azurite** is a versatile tool for computing bases of integrals: any loop order, any multiplicity, internal/external masses, planar/nonplanar, ... 

- **Azurite** is faster than **Mint** by a factor which increases rapidly with the complexity of the problem (up to $>1000$).

- **Azurite** provides an initial step for **Cristal**, a code for complete IBP reductions in current progress.