Towards four-loop Standard Model renormalization in the gaugeless limit

Andrey Pikelner

Hamburg University

in collaboration with A.Bednyakov and B.Kniehl

RADCOR-2017
Higgs potential and its minimum

Metastability condition:

if $\lambda$ becomes negative provided it remains small in absolute magnitude the SM vacuum is unstable but sufficiently long-lived compared to the age of the Universe
From EW to Planck scale

$$\mu^2 \frac{\partial a_i}{\partial \mu^2} = \beta_{a_i}(a_1 \ldots a_n)$$

Experiment or PDG input

Matching

RG-evolution

Matching

Planck scale

$$\mathcal{O}(10^{19})$$

$$\mathcal{O}(M_Z)$$

$$G_F, M_W, M_Z, M_H, M_t$$

$$g'(\mu), g(\mu), y_t(\mu), \lambda(\mu)$$

$$\downarrow$$
LHC Higgs mass and stability

Constraints:

Non-perturbativity
Small $\lambda(\mu)$ for all $\mu < M_{pl}$

Stability
$\lambda(\mu) = 0$, $\beta_\lambda = 0$

Most sensitive to parameters (also with largest uncertainty):

$M_t = 173.1 \pm 0.6 \quad M_H = 125.09 \pm 0.24 \quad \alpha_s(M_Z) = 0.1181 \pm 0.0011$
Motivation: SM vacuum stability analysis at NNLO

- Three loop beta-functions for gauge Yukawa and self-coupling
  [Mihaila, Salomon, Steinhauser’12; Bednyakov, AFP, Velizhanin’12,13; Chetyrkin, Zoller’13]
- Two loop full $\mathcal{O}(\alpha^2)$ threshold corrections
  [Buttazzo et al.’13; Kniehl, AFP, Veretin’15; Martin’15]

$$ V_{\text{eff}}(\phi \gg v) \simeq \frac{\lambda(\mu = \phi)}{4} \phi^4 $$

$$ (4\pi)^2 \frac{d\lambda}{d \ln \mu^2} = 12\lambda + 6y_t^2\lambda - 3y_t^4 + \ldots $$

$$ (4\pi)^2 \frac{dy_t}{d \ln \mu^2} = \frac{9}{4}y_t^3 - 4g_s^2y_t + + \ldots $$

Towards extremely high energies in the SM
SM parameters definition and evolution

- In broken phase as input we use the set of parameters defined in OS scheme:
  
  \[ M_b, M_W, M_Z, M_H, M_t, G_F \]

- In unbroken phase we use the set of running parameters in \( \overline{MS} \) scheme:
  
  \[ g_1(\mu), g_2(\mu), g_s(\mu), y_t(\mu), y_b(\mu), \lambda(\mu) \]

- RG equations are coupled, but main sensitivity is to \( g_s, y_t \)

- Main uncertainty for initial values is in \( g_s, y_t, \lambda \) and so in \( M_t \) and \( M_H \)

- One more complication in SM compared to QCD: gauge group is not simple
  
  \[ \text{SU}(3) \rightarrow \text{SU}(3) \times \text{SU}(2) \times U(1) \]

Gaugeless limit of the Standard Model

\[ g_1 = g_2 = 0, \text{ no gauge boson lines}(W \text{ and } Z), \text{ only gluons} \]
Main steps of beta-functions evaluation

QCD $O(\alpha_s^2)$  [Egorian,...'78]  →  QCD $O(\alpha_s^3)$  [Tarasov,...'80]  →  QCD $O(\alpha_s^4)$  [Larin,...'97]  →  QCD $O(\alpha_s^5)$

$G_1 \times G_2, O(g_1 g_2)$  [Jones'82]  →  $\beta_g, O(g_1 y)$  [Machacek,...'83]  →  $\beta_g, O(g_1 y^2, g_1 y \lambda)$  [Pickering,...'01]

$\beta_y, O(g_1 g_2)$  [Machacek,...'84]  →  SM, $O(\alpha_i^2)$  [Arason,...'92]

$\beta_\lambda, O(\lambda g_2)$  [Machacek...'85]  →  $\beta_\lambda, O(\alpha_i^3)$  [Chetyrkin,...'12]

$\beta_\lambda, O(\lambda^3)$  [Brezin,...'73]  →  $\beta_\lambda, O(\lambda^4)$  [Kazakov,...'79]  →  $\beta_\lambda, O(\lambda^5)$  [Gorishnii,...'83]  →  $\beta_\lambda, O(\lambda^6)$

SM, 3-loop
After Higgs discovery results:

QCD $\mathcal{O}(\alpha_s^5)$

- [Baikov,...'16]
- [Herzog,...'17]
- [Luthe,...'17]

SM, 3-loop

- [Mihaila,...'12]
- [Bednyakov,...'13]
- [Chetyrkin,...'13]

What is next?

$\beta_\lambda, \mathcal{O}(\lambda^6)$

- [Kompaniets, Panzer'17]

Try out to renormalize Standard Model at four-loop level
After Higgs discovery results:

After Higgs discovery results:

\[ \text{QCD } \mathcal{O}(\alpha_s^5) \]

\[ \text{SM, 3-loop} \]

\[ \beta_\lambda, \mathcal{O}(\lambda^6) \]

\[ \text{What is next?} \]

\[ \text{What is next?} \]

Try out to renormalize Standard Model at four-loop level
What is known at four-loops?

- Strong coupling beta-function in the SM gaugeless limit
  - This talk
    - [Bednyakov, AFP’15][Zoller’15]

- Leading QCD corrections to $\beta_\lambda$
  - [Martin’15][Chetyrkin, Zoller’16]

- Generalization of QCD results on reducible fermion representaions
  - [Chetyrkin, Zoller’17]

- Pure QCD corrections to $\beta_y$ and Higgs field anomalous dimension
  - [Larin, . . . ’97][Chetyrkin’97][Chetyrkin’96]

Are these corrections dominating?
Strong coupling — matching and running: known results

- Running in $\overline{\text{MS}}$
  - 1-loop [Gross, Wilczek'73, Politzer'73]
  - 2-loop [Jones'74, Egorian, . . . '78] (QCD), [Machachek, Vaughn'83] (SM)
  - 3-loop [Tarasov, . . . '80] (QCD), [Mihaila, . . . '12, Bednyakov, . . . '12] (SM)
  - 4-loop [van Ritbergen, Vermaseren, Larin'97, Czakon'04] (QCD)
  - 5-loop [Baikov, . . . '16, Luthe, . . . '16, Herzog, . . . '17] (QCD) state-of-the-art!

- Matching in $\overline{\text{MS}}$: we “match” effective five-flavor ($n_f = 5$) QCD and a more fundamental theory (usually, QCD with top quark $n_f = 6$)
  \[ \alpha_s^{(5)}(\mu) = \alpha_s(\mu) \xi \alpha_s(\mu, M) \]
  “integrate out” heavy fields with mass $M$
  - 1/2-loop [Bernreuther, Wetzel’81-83], [Bednyakov’15] (2-loop SM)
  - 3-loop [Chetyrkin, Kniehl, Steinhauser’97-98]
  - 4-loop [Schroder, . . . ’05, Chetyrkin, . . . ’05, Kniehl, . . . ’06] state-of-the-art!

NB: L-loop running + (L-1)-loop matching are self-consistent
Strong coupling — matching and running: known results

- **Running in \( \overline{\text{MS}} \)
  - 1-loop [Gross, Wilczek'73, Politzer'73]
  - 2-loop [Jones'74, Egorian,... '78] (QCD), [Machachek, Vaughn'83] (SM)
  - 3-loop [Tarasov,... '80] (QCD), [Mihaila,... '12, Bednyakov,... '12] (SM)
  - 4-loop [van Ritbergen, Vermaseren, Larin'97, Czakon'04] (QCD)
  - 5-loop [Baikov,... '16, Luthe,... '16, Herzog,... '17] (QCD) state-of-the-art!

- **Matching in \( \overline{\text{MS}} \):** we “match” effective five-flavor \((n_f = 5)\) QCD and a more fundamental theory (usually, QCD with top quark \(n_f = 6\))

\[
\alpha_s^{(5)}(\mu) = \alpha_s(\mu)\xi\alpha_s(\mu, M)
\]

“integrate out” heavy fields with mass \(M\)

- 1/2-loop [Bernreuther, Wetzel'81-83], [Bednyakov'15] (2-loop SM)

- 3-loop [Chetyrkin, Kniehl, Steinhauser'97-98]

- 4-loop [Schroder,... '05, Chetyrkin,... '05, Kniehl,... '06] state-of-the-art!

**NB:** L-loop running + (L-1)-loop matching are self-consistent
Four-loop strong coupling beta-function: from QCD to SM

- **Starting point:** Limit of vanishing $SU(2) \times U(1)$ gauge couplings
  Only the following SM parameters are considered

$$a_i = \left( \frac{g_s^2}{16\pi^2}, \frac{y_t^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G \right)$$

Significantly reduce number of diagrams!
NB: we keep track of gauge-parameter dependence!

- Easier to track $\gamma_5$ in dimensional regularization (see below):
  - no $\gamma_5$ in gauge vertices!
  - $\gamma_5$ appears only in (pseudo/charged) scalar couplings.

- In what follows ($h$ counts powers of $a_i$)

$$\frac{d a_s}{d \log \mu^2} = \beta_{a_s} = -\sum_{i=0}^{3} \beta_i h^{i+2}$$
Further simplifications: background field gauge

- Split gauge fields $V = \tilde{V} + \hat{V}$ in
  - quantum $\tilde{V} = (\tilde{G}, \tilde{W}, \tilde{B}, \ldots)$ and
  - background $\hat{V} = (\hat{G}, \hat{W}, \hat{B}, \ldots)$

- Background fields do not propagate

- Modified Feynman rules [Abbot’80]

- QED-like connection between renormalization constants
  \[
  Z_{a_s} = 1/Z_{\tilde{G}}, \quad Z_{\xi G} = Z_{\tilde{G}}
  \]

- Only two-point functions are required (given all 3-loop Z-factors)

- Multiplicative renormalization by $a_{\text{bare}} = Z_a a_{\text{ren}}$
  \[
  \Gamma_{\text{ren}}^{(l)} = Z_{\Gamma}^{(l)} \left[ 1 + \Gamma_{\text{bare}}^{(1)}(a_{\text{bare}}) + \Gamma_{\text{bare}}^{(2)}(a_{\text{bare}}) + \cdots + \Gamma_{\text{bare}}^{(l)}(a_{\text{bare}}) \right]
  \]

- No need to apply IRR trick, have access to finite parts
RGE from propagator-type integrals

- Three-loop experience (different approaches to IRR [Vladimirov'78])
  - Massless propagators: gauge couplings, field renormalization constants using FORM based MINCER package
    NB: finite parts are available (in massless/unbroken theory)
  - Three-loop massive bubbles: Yukawa and Higgs self-coupling using FORM based MATAD package
    NB: aux mass renormalization

- Four-loop experience
  - QCD beta function
    massive vacuum integrals available, possible to calculate all types of renormalization constants.
  - Massless propagators. Difficult to prepare reduction. Easy to formulate the problem
    Independent tool for two-point Green functions renormalization constants calculation
Setup v.1

- Model file of the SM in BFG tested at lower loop calculations

- **DIANA/QGRAF**
  Diagram generation
  [Nogueira’93; Fleischer, Tentyukov’99]

- Prepared set of mappings to 3 auxiliary topos
  each topo with 11 denominators and 3 irreducible numerators

- Reduction
  - **LiteRed**
    IBP rules preparation
  - **FIRE5, C++ version**
    Integral reduction
  - Master integrals
    4-loop propagators
  [Lee’12; Smirnov’14; Baikov, Chetyrkin’10; Lee, Smirnovs’11]
Four-loop RGE from fully massive tadpoles

\[
\frac{1}{(k+p)^2 - M^2} = \frac{1}{k^2 - m_A^2} + \frac{M^2 - p^2 - 2kp - m_A^2}{k^2 - m_A^2} \frac{1}{(k+p)^2 - M^2}
\]

\[\omega = -2\] 
\[\omega = -2\] 
\[\omega = -3\]

- Four-loop QCD beta-function [Ritbergen, Vermaseren, Larin'97][Czakon'04]
- Anom. dim. of twist-2 operators in QCD and N=4 SYM [Velizhanin'14]
- Renormalization of QCD with extended fermion sector [Chetyrkin, Zoller'17]
Setup v.2: improved reduction

We have four independent ways for reduction:

1. Reduction of massless propagators
   - LiteRed
     IBP rules preparation
   - FIRE5, C++ version
     Integral reduction
   - Master integrals
     4-loop propagators

2. Reduction of massles propagators with FORM package FORCER
   Parametric integral reduction

3. Reduction of fully massive tadpoles
   - LiteRed
     IBP rules preparation
   - FIRE5, C++ version
     Integral reduction
   - Master integrals
     4-loop fully massive tadpoles

4. Reduction of four-loop tadpoles with new FORM code FMFT
   details below
4 loop QCD $\beta$-function and renormalization constants

- IRR with auxiliary mass
  - fully massive four-loop tadpoles, possible to calculate all renormalization constants
  - From $Z_g, Z_c, Z_{ccg}$
    \[ Z_{ag} = \frac{Z_{ccg}^2}{Z_c^2 Z_g} \]  
    \[ Z_{ag} = \frac{Z_{qqg}^2}{Z_q^2 Z_g} \]

- From $Z_g, Z_q, Z_{qqg}$
  \[ Z_{ag} = \frac{Z_{qqg}^2}{Z_q^2 Z_g} \]

- Using 3-loop massless integrals
  - From $Z_c, Z_{ccg}$ and already known $\beta_{ag}$
    Impossible to calculate $Z_g$, but independent calculation of other RCs
    \[ Z_g = \frac{Z_{ccg}^2}{Z_c^2 Z_{ag}} \]

- Using 4-loop massless propagator type integrals and BFG
  \[ Z_{as} = 1/Z_{\tilde{G}}, \quad Z_{\xi_G} = Z_{\tilde{G}} \]
FMFT: four-loop tadpoles reduction

- Topologies $H, X, BMW$ need manual reduction rules

![](image)

- Topology $FG$ and all its subtopologies can be expressed as convolution

\[
J_{FG} = \int d[p] \begin{pmatrix}
    k_1 \quad k_2 \\
    k_1 - p \quad k_2 - p \\
    \end{pmatrix}
\]

- One-loop and two-loop propagator type integrals with massive lines can be reduced separately
**FMFT: topology FG and performance**

- Using dimensional shifts we can reduce numerators of both one- and two-loop subintegrals

- Remaining integrals with arbitrary power of denominator with momenta $p$ reduced using one dimensional recurrence relations

**Available for download:**

http://git.io/fmft

Nonplanar integral $X(-n, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with numerator

<table>
<thead>
<tr>
<th>$n$ =</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>FMFT</td>
<td>0:00:11</td>
<td>0:00:27</td>
<td>0:01:55</td>
<td>0:07:35</td>
<td>0:25:31</td>
<td>01:30:31</td>
</tr>
</tbody>
</table>

Time format **hh:mm:ss**, FIRE used with LiteRed rules
Renormalization constants in gaugeless limit of SM

- $\gamma_H$ - Higgs field anomalous dimension from $Z_H$
- $\beta_\lambda$ - Higgs self-coupling beta-function from $Z_\lambda$

$$Z_\lambda = \frac{Z_{HHHH}}{Z_H^2}$$

- $\beta_{yt}$ - top Yukawa coupling beta-function from $Z_{yt}$

$$Z_{yt} = \frac{Z_{uuH}}{\sqrt{Z_{u,L}Z_{u,R}Z_H}}$$

- $\beta_{m^2}$ - Higgs mass parameter beta-function from $Z_{m^2}$

$$Z_{m^2} = \frac{Z_{HH[HH]}}{Z_H}$$

- Green functions calculated using massless propagators:

$$\Gamma_{HH}, \Gamma_{uu}, \Gamma_{uuH}$$

- Green functions calculated using massive tadpoles:

$$\Gamma_{HH}, \Gamma_{uu}, \Gamma_{uuH}, \Gamma_{HHHH}, \Gamma_{HH[HH]}$$
Traces at three-loop level

For $\gamma_5$ in dimensional regularization we use naive treatment, except diagrams with two fermionic traces with four contracted indices

$$\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5] \text{tr}[\gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_5]$$

$\sim \text{tr}[\gamma^\alpha \gamma^\beta \gamma^5]$  
$\sim 2n + 1$  
$\gamma$ matrices

Substitution is correct for $D = 4$

$$\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5] = -4i\epsilon_{\mu\nu\rho\sigma}$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} = -\mathcal{T}^{[\mu\nu\rho\sigma]}_{[\alpha\beta\gamma\delta]}, \quad \mathcal{T}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} = \delta^\mu_\alpha \delta^\nu_\beta \delta^\rho_\gamma \delta^\sigma_\delta,$$

Apply in $D = 4 - 2\epsilon$, but for diagrams w/o subdivergencies
Number of diagrams up to four loops

- Total number of diagrams and additional diagrams with $\varepsilon \otimes \varepsilon$ contraction

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>$3/\gamma_5$</th>
<th>$4/\gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{\hat{g}\hat{g}}$</td>
<td>4</td>
<td>43</td>
<td>867</td>
<td>25374/72</td>
</tr>
<tr>
<td>$\Gamma_{uu}$</td>
<td>4</td>
<td>39</td>
<td>920</td>
<td>30035/94</td>
</tr>
<tr>
<td>$\Gamma_{HH}$</td>
<td>1</td>
<td>14</td>
<td>276</td>
<td>8822/18</td>
</tr>
<tr>
<td>$\Gamma_{uuH}$</td>
<td>3</td>
<td>102</td>
<td>4030/18</td>
<td>185981/2048</td>
</tr>
<tr>
<td>$\Gamma_{HHHH}$</td>
<td>5</td>
<td>47</td>
<td>1307</td>
<td>46536/74</td>
</tr>
<tr>
<td>$\Gamma_{HH[HH]}$</td>
<td>3</td>
<td>27</td>
<td>616</td>
<td>23044/18</td>
</tr>
</tbody>
</table>

- Number of diagrams for $\Gamma_{HHHH}$ can be reduced due to graph symmetries using GraphState package

Original numbers:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{HHHH}$</td>
<td>15</td>
<td>327</td>
<td>13212</td>
<td>685599</td>
</tr>
</tbody>
</table>
Uncertainty in contributions from lower loops

- No contribution from one-loop diagrams

- Determined contribution from one-loop renormalization ($Z_{yt}$ insertion)

- Undetermined contribution from three-loop renormalization ($Z_g, Z_{yt}$ insertion)
Reading point dependence

\[
\text{tr}[\gamma_\mu p_1 \gamma_\nu p_2 \gamma_\nu p_3 \gamma_\mu p_4] \rightarrow \text{tr}[\gamma_\mu p_1 \gamma_\nu p_2 \gamma_\nu p_3 \gamma_\mu p_4] \rightarrow \text{tr}[\gamma_\mu p_1 \gamma_\nu p_2 \gamma_\nu p_3 \gamma_\mu p_4] \rightarrow \text{tr}[\gamma_\nu p_2 \gamma_\nu p_3 \gamma_\nu p_4 \gamma_\mu p_1]
\]

- Nontrivial dependence on “reading point” selection in each fermion trace
- Final results with info about all possible trace cut options
For quark propagator and $uuH$ vertex situation is more complicated.
Keeping trace of uncertainties

Two generalizations of trace taking procedures:

1. Use tensor reduction and move trace outside the integral

\[
\int d[p_1 \ldots n] \text{tr}[\mathcal{P}_1 \ldots \mathcal{P}_k] \rightarrow \text{tr}[\gamma_\alpha \ldots \gamma_\beta] \int d[p_1 \ldots n] (p_i \cdot p_j)
\]

- Safe to apply \( d = 4 \) rules if integral have no higher poles
- Different reading points corresponds to different integrals combinations

2. Cut fermion line in all possible ways, but take trace under integral sign
Relative contributions: strong coupling

4-loop $\beta_{a_s}$

- Pure QCD corrections dominating

[Source: Larin et al.'97; Czakon'04]
Relative contributions: Yukawa top

► Pure QCD contribution ($\gamma_m$ in QCD) only subleading [Larin et al.'97; Chetyrkin'97]
► Very large relative cancellations
Relative contributions: self-coupling

4-loop $\beta_{a\lambda}$

- Pure QCD correction ($a_s^3 a_t^2$) [Martin’15; Chetyrkin, Zoller’16]
- Comparable to ($a_s a_t^4$) and have opposite sign
Relative contributions: mass parameter

4-loop $\beta_{m^2}$

- Leading QCD ($a_s^3 a_t$) from external leg renormalization [Chetyrkin'96]
- With operator $[\phi\phi]$ insertion into internal line, ($a_s^2 a_t^2$) also subleading
Relative loop contributions

Substituting running couplings at scale $\mu = M_t$ and separating contribution from diagrams with two fermion traces requiring non naive treatment:

- Contribution from part with uncertainty due to $\gamma_5$ in $\beta_{a_s}$ is negligible
- Large cancellations, no uncertainty

$$\frac{\beta_{m^2}}{\beta_0} = h \cdot 1. + h^2 \cdot 0.0264 + h^3 \cdot 0.00266 - h^4 \cdot [0.27 - 0.66] \times 10^{-4}$$

- Small cancellation and small contribution from piece with uncertainty

$$\frac{\beta_{a_\chi}}{\beta_0} = h \cdot 1. + h^2 \cdot 0.0089 - h^3 \cdot 0.00025 - h^4 \cdot [0.1323 \pm 0.0048] \times 10^{-2}$$

- Large cancellations and large contribution from piece with uncertainty

$$\frac{\beta_{a_t}}{\beta_0} = h \cdot 1. + h^2 \cdot 0.1529 + h^3 \cdot 0.00743 + h^4 \cdot [0.019 \pm 0.612] \times 10^{-3}$$
Conclusions

- We calculated universal contribution not affected by $\gamma_5$ definition in dimensional regularization to four-loop SM beta-functions in gaugeless limit.

- Part dependent on $\gamma_5$ definition parametrized for different reading points and need further analysis.

- Four-loop package FMFT for fully massive tadpoles reduction created and successfully applied for calculation of all needed Green functions.