# Evolution equations of neutrino mixing in matter

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*Abstract* A set of rephasing invariant parameters for the neutrino mixing are introduced. Under this parametrization, the squared elements of the neutrino mixing matrix are found to satisfy a set of differential equations as functions of the induced mass. They show clearly the dominance of pole terms when the neutrino induced masses cross. It is found that these equations have very good approximate solutions for all values of the induced mass. The results may be applicable to extracting unknowns from the Long Baseline Experiments (LBL).

*Introduction* It is suggested [1] that there exists a set of rephasing invariants in the neutrino mixing matrix $(V)$ if one demands $\det V = +1$,

$$\Gamma_{ijk} = V_{\alpha i}V_{\beta j}V_{\gamma k} = R_{ijk} - iJ, \tag{1}$$

where the common imaginary part can be identified with the Jarlskog invariant. Their real parts are defined as

$$(R_{123}, R_{231}, R_{312}; R_{132}, R_{213}, R_{321}) = (x_1, x_2, x_3; y_1, y_2, y_3). \tag{2}$$

The $(x, y)$ parameters satisfy two constraints

$$\det V = (x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1, \tag{3}$$

$$(x_1x_2 + x_2x_3 + x_3x_1) - (y_1y_2 + y_2y_3 + y_3y_1) = 0. \tag{4}$$

In addition, it is found that

$$J^2 = x_1x_2x_3 - y_1y_2y_3. \tag{5}$$

The $(x, y)$ parameters are related to $|V_{\alpha i}|^2$ by

$$W = \left[|V_{\alpha i}|^2\right] = \begin{pmatrix} x_1 - y_1, x_2 - y_2, x_3 - y_3 \\ x_3 - y_2, x_1 - y_3, x_2 - y_1 \\ x_2 - y_3, x_3 - y_1, x_1 - y_2 \end{pmatrix}, \tag{6}$$

The matrix of the cofactors of $W$, denoted as $w$ with $w^TW = (detW)I$, are given by replacing the "-" sign in the elements of $W$ with a "+" sign.

*Differential Equations for Matter Effects* When neutrinos propagate in a medium of constant density, their interactions induce a term in the effective Hamiltonian, $H = \frac{1}{2E}M_\nu M_\nu^\dagger$, given by $(\delta H)_{ee} = A = 2\sqrt{2}G_F n_e E$. Thus, the neutrino mass eigenvalues squared $(D_i = m_i^2)$ and mixing matrix are functions of $A$. It was shown [2] that they satisfy a set of differential equations, given by

$$\boxed{\frac{dD_i}{dA} = |V_{ei}|^2 = W_{ei}}, \quad \boxed{\frac{dV_{\alpha i}}{dA} = \sum_{k\neq i}\frac{V_{\alpha k}V_{ei}}{D_i - D_k}V_{ek}^*}. \tag{7}$$

For $W_{\alpha i} = |V_{\alpha i}|^2$, we find

$$\boxed{\frac{d}{dA}W_{\alpha i} = \frac{d}{dA}|V_{\alpha i}|^2 = 2\sum_{k\neq i}\frac{1}{D_i - D_k}Re(\Pi_{ik}^{\alpha e})}, \tag{8}$$

where we have used the definition $\Pi_{ij}^{\alpha\beta} = V_{\alpha i}V_{\alpha j}^*V_{\beta j}V_{\beta i}^* = \Lambda_{\gamma k} + iJ$. As a brief summary, we conclude that the evolution of $W_{\alpha i}$ obeys

$$\boxed{\frac{1}{2}\frac{d}{dA}W_{ei} = \sum_{k\neq i}\frac{W_{ei}W_{ek}}{D_i - D_k}}, \tag{9}$$

$$\boxed{\frac{1}{2}\frac{d}{dA}W_{\alpha i} = \sum_{k\neq i\neq j}\frac{\Lambda_{\beta j}}{D_i - D_k}}, \alpha \neq \beta, (\alpha, \beta) = (\mu, \tau), \tag{10}$$

where $\Lambda_{\gamma k} = Re(\Pi_{ij}^{\alpha\beta})$ are combinations of $W_{\alpha i}$:

$$\Lambda_{\gamma k} = \frac{1}{2}(W_{\alpha i}W_{\beta j} + W_{\alpha j}W_{\beta i} - W_{\gamma k}). \tag{11}$$

In addition, The equation for $J$ was also computed:

$$\boxed{\frac{d}{dA}(\ln J) = \frac{-W_{e1} + W_{e2}}{D_1 - D_2} + \frac{-W_{e2} + W_{e3}}{D_2 - D_3} + \frac{-W_{e3} + W_{e1}}{D_3 - D_1}}. \tag{12}$$

These results are manifestly invariant in form under the exchanges $(\alpha \leftrightarrow \beta)$ and $(i \leftrightarrow j \leftrightarrow k)$, in contrast to the Standard Parametrisation, in which $\theta_{ij}$ have complicated permutation properties.

*Approximate Solutions* The behavior of $W_{\alpha i}$ in matter can be solved numerically [3]. As for the analytical solutions, one notes that different pole terms in $dW_{\alpha i}/dA$ dominate at different ranges of $A$. One may thus solve for the approximate solutions [3] for $W_{\alpha i}$ at various regions of $A$, with the following parametrisation of $[W]$ in vacuum as the initial conditions:

$$[W_0] = \begin{pmatrix} \frac{2}{3} + b - \frac{c}{2}, & \frac{1}{3} - b - \frac{c}{2}, & c \\ \frac{1}{6} - \frac{b}{2} + \frac{c}{4} - \frac{d}{2} + e, & \frac{1}{3} + \frac{b}{2} + \frac{c}{4} - \frac{d}{2} - e, & \frac{1}{2} + d - \frac{c}{2} \\ \frac{1}{6} - \frac{b}{2} + \frac{c}{4} + \frac{d}{2} - e, & \frac{1}{3} + \frac{b}{2} + \frac{c}{4} + \frac{d}{2} + e, & \frac{1}{2} - d - \frac{c}{2} \end{pmatrix}, \tag{13}$$

where $b$, $c$, $d$, and $e$ evolve with $A$ and are of order $\sim 0.01$ or less. However, except for $d$, which should be determined by careful analysis of the experimental data, the behaviours of all other parameters can be properly analyzed based on known results.

*Applications and Outlooks* The "$W-$ centric" parametrisation leads to simple expressions for the equations that relate physical observables. It may help uncover the underlying symmetries in flavor physics. In addition, to within the accuracy of $\sim 0.01$ for the $\nu$-oscillation probability functions under this parametrization, it is hoped that careful analysis of, e.g., the LBL data, may shed light on the parameter $d$, which is closely related to the $\mu - \tau$ asymmetry, $J$, and $\theta_{23}$ (in the Standard Parametrisation).

**References**

[1] T. K. Kuo and T.-H. Lee, Phys. Rev. D **71**, 093001 (2005).

[2] S. H. Chiu, T. K. Kuo and L. X. Liu, Phys. Lett. B **687**, 184 (2010).

[3] S. H. Chiu and T. K. Kuo, Phys. Rev. D **97**, no. 5, 055026 (2018).