Neutrino mixing matrices with prescribed singular values

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Aim of studies
- Investigation of the experimentally determined $3 \times 3$ neutrino mixing matrix from point of view of matrix theory,
- determination of a number of sterile neutrinos by analyzing singular values of three dimensional mixing matrices,
- construction of the complete unitary mixing matrix via the procedure of matrix dilation.

Common parametrizations of non-unitary mixing matrices [1]
- QR decomposition: $V = (I - \alpha)U = TU$
  where $\alpha$ is lower triangular matrix and $U$ is unitary.
- Polar decomposition: $V' = (I - \eta)U'$
  where $\eta$ is a Hermitian matrix and $U'$ is unitary.

Present status [1]
- Current upper bounds on the $\alpha$ matrix for three different neutrino mass scenarios:

  $\begin{array}{lcc}
  m > EW & \alpha_{ee} & \alpha_{\mu\mu} \\
  \sigma_{ee} & 1.3 \cdot 10^{-3} & 2.2 \cdot 10^{-4} \\
  \sigma_{\mu\mu} & 2.8 \cdot 10^{-3} & 6.8 \cdot 10^{-4} \\
  \sigma_{\tau\tau} & 2.7 \cdot 10^{-3} & 1.2 \cdot 10^{-3} \\
  \end{array}$

  $\begin{array}{lcc}
  \Delta m^2 \geq 100 \text{ eV}^2 & \alpha_{ee} & \alpha_{\mu\mu} \\
  \sigma_{ee} & 2.4 \cdot 10^{-2} & 2.2 \cdot 10^{-2} \\
  \sigma_{\mu\mu} & 1.0 \cdot 10^{-1} & 2.5 \cdot 10^{-2} \\
  \sigma_{\tau\tau} & 6.9 \cdot 10^{-2} & 1.2 \cdot 10^{-2} \\
  \end{array}$

  $\begin{array}{lcc}
  \Delta m^2 \sim 0.1 - 1 \text{ eV}^2 & \alpha_{ee} & \alpha_{\mu\mu} \\
  \sigma_{ee} & 1.0 \cdot 10^{-2} & 1.4 \cdot 10^{-2} \\
  \sigma_{\mu\mu} & 1.0 \cdot 10^{-1} & 1.0 \cdot 10^{-1} \\
  \sigma_{\tau\tau} & 4.5 \cdot 10^{-2} & 5.3 \cdot 10^{-2} \\
  \end{array}$

Matrix theory: singular values [2]

- Singular values $\sigma_i$ of a given matrix $A$ are square roots of the eigenvalues $\lambda_i$ of the matrix $AA^T$
  $$\sigma_i(A) = \sqrt{\lambda_i(AA^T)}, \quad i = 1, 2, 3, \ldots$$

  Properties:
  - operator norm (spectral norm):
    $$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sigma_{\text{max}}(A)$$
  - contractions:
    $$\|A\| \leq 1$$

General beyond the Standard Model (BSM) scenario [3]
- Extensions from three neutrinos setting: $V \rightarrow W = \begin{pmatrix} V & V e \end{pmatrix}$
- matrix $W$ is unitary,
- operator norm is bounded by unity:
  $$\|V\| \leq 1$$
- Any deviation of singular value of $V$ from unity $\rightarrow$ BSM scenario.

Matrix dilation:
- Minimal number of additional sterile neutrinos is determined by the number of singular values strictly less than one.

Strategy: $3 + 1$ scenario studies
- We are focusing on one additional neutrino model by studying $T$ matrices with prescribed singular values.
- To produce $T$ matrix that fulfill $3 + 1$ structure, we are setting the singular values to:
  $$\sigma(T) = \{1.00, 1.00, \sigma_3\}, \quad \sigma_3 < 1$$
- Conditions given by constraints on $\alpha$ matrix are met through the random generation of $T$ entries within region given by fixed singular values.

Results: $3 + 1$: diagonal cases
For each mass scheme it is possible to find $3 \times 3$ mixing matrices that agrees with data and can describe mixing with only one sterile neutrino. This is simply satisfied by the following diagonal matrices:

$$\begin{array}{l}
\Delta m^2 \geq 100 \text{ eV}^2 \quad \text{and} \quad \Delta m^2 \approx 0.1 - 1 \text{ eV}^2 \\
T = \begin{pmatrix}
1.0000 & 0 & 0 \\
0 & 1.0000 & 0 \\
0 & 0 & 0.9972
\end{pmatrix}
\end{array}$$

Are these the smallest possible values?

Results: $3 + 1$: non-diagonal cases
To find non-diagonal cases we involve the idea of an inverse eigenvalue problem, i.e., we construct matrix with prescribed singular values [4]:

$$\begin{array}{l}
m > EW \\
\sigma(T) = (1.0000, 1.0000, 0.9917) \\
T = \begin{pmatrix}
0.9987 & 0 & 0 \\
0.0007 & 0.9988 & 0 \\
0.0027 & 0.0012 & 0.9972
\end{pmatrix}
\end{array}$$

$$\begin{array}{l}
\Delta m^2 \geq 100 \text{ eV}^2 \\
\sigma(T) = (1.000, 1.000, 0.899) \\
T = \begin{pmatrix}
1.000 & 0 & 0 \\
0.001 & 1.000 & 0 \\
0.012 & 0.012 & 0.900
\end{pmatrix}
\end{array}$$

$$\begin{array}{l}
\Delta m^2 \approx 0.1 - 1 \text{ eV}^2 \\
\sigma(T) = (1.000, 1.000, 0.888) \\
T = \begin{pmatrix}
0.994 & 0 & 0 \\
0.013 & 0.993 & 0 \\
0.045 & 0.053 & 0.900
\end{pmatrix}
\end{array}$$

Conclusions
- Every mass scheme allows for one additional sterile neutrino,
- based on Weyl inequalities, errors of $T$ entries are estimated on: $0.003$ [3],
- singular values and inverse eigenvalue problem are useful tools for study non-unitary effects in neutrino sector.

Outlook
- Analysis of $3 \times 3$ mixing matrix in the context of two additional neutrinos,
- construction of complete unitary mixing matrix through the method of the unitary dilation for $3 + 1$ and $3 + 2$ scenarios [3].