Renormalization-Group Equations in Neutrino Physics: Radiative Corrections & Matter Effects

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XXVIII International Conference on Neutrino Physics and Astrophysics, 4 - 9 June, 2018, Heidelberg, Germany

Radiative Corrections to Leptonic Flavor Mixing Parameters

New physics at high energy scales may account for both neutrino masses and leptonic flavor mixing. In this case, the theoretical predictions at the high-energy scale have to be confronted with the low-energy data by implementing the renormalization-group (RG) equations [1, 2].

Similar to quarks and charged leptons in the standard model, massive neutrinos can also be Dirac particles. The one-loop RG equations for the Yukawa coupling matrices of quarks and leptons are given by [3, 4, 5]

$$\begin{aligned}
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_{\rm u} &= \left[\alpha_{\rm u} + \frac{3}{2}\left(Y_{\rm u}Y_{\rm u}^\dagger\right) - \frac{3}{2}\left(Y_{\rm d}Y_{\rm d}^\dagger\right)\right] Y_{\rm u}\,, & \alpha_{\rm u} &= -\frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi\,;\\
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_{\rm d} &= \left[\alpha_{\rm d} - \frac{3}{2}\left(Y_{\rm u}Y_{\rm u}^\dagger\right) + \frac{3}{2}\left(Y_{\rm d}Y_{\rm d}^\dagger\right)\right] Y_{\rm d}\,, & \alpha_{\rm d} &= -\frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 + \chi\,;\\
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_\nu &= \left[\alpha_\nu + \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) - \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_\nu\,, & \alpha_\nu &= -\frac{9}{20}g_1^2 - \frac{9}{4}g_2^2 + \chi\,;\\
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_l &= \left[\alpha_l - \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) + \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_l\,, & \alpha_l &= -\frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 + \chi\,,
\end{aligned} \tag{1}$$

where $\boldsymbol{t \equiv \ln(\mu/\Lambda_{\rm EW})}$ with μ being the renormalization scale and $\Lambda_{\rm EW} \sim 10^2~{\rm GeV}$ the electroweak scale, and $\chi \equiv {\rm Tr}\left[3\left(Y_{\rm u}Y_{\rm u}^\dagger\right) + 3\left(Y_{\rm d}Y_{\rm d}^\dagger\right) + \left(Y_\nu Y_\nu^\dagger\right) + \left(Y_l Y_l^\dagger\right)\right]$. In addition, the RG equations of three gauge couplings are $(16\pi^2){\rm d}g_i/{\rm d}t = b_i g_i^3$ with the coefficients $\{b_3, b_2, b_1\} = \{-7, -16/9, 41/10\}$.

Integral Forms in the Top and Tau Dominance

Without loss of generality, we work in the flavor basis where $Y_{\rm u} = {\rm diag}\{y_u, y_c, y_t\} \equiv D_{\rm u}$ and $Y_l = {\rm diag}\{y_e, y_\mu, y_\tau\} \equiv D_l$. Furthermore, in consideration of the strong hierarchy among the quark and lepton Yukawa couplings, it is safe to neglect the terms of $Y_{\rm d}Y_{\rm d}^\dagger$ and $Y_\nu Y_\nu^\dagger$ on the right-hand side of the RG equations in Eq. (1). Therefore, we have

$$\begin{aligned}
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} D_{\rm u} &= \left[\alpha_{\rm u} + \frac{3}{2}D_{\rm u}^2\right] D_{\rm u}\,, & 16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_{\rm d} &= \left[\alpha_{\rm d} - \frac{3}{2}D_{\rm u}^2\right] Y_{\rm d}\,,\\
16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} D_l &= \left[\alpha_l + \frac{3}{2}D_l^2\right] D_l\,, & 16\pi^2 \frac{\mathrm{d}}{\mathrm{d}t} Y_\nu &= \left[\alpha_\nu - \frac{3}{2}D_l^2\right] Y_\nu\,.
\end{aligned} \tag{2}$$

From Eq. (2), one can derive the corresponding Yukawa couplings (with primes) at the electroweak scale $\Lambda_{\rm EW}$:

$$\begin{aligned}
D_{\rm u}' &= I_{\rm u} T_{\rm u}\, D_{\rm u}\,, & I_{\rm f} &= \exp\left[-\frac{1}{16\pi^2}\int_0^{t_0} \alpha_{\rm f}(t)\,{\rm d}t\right]\,, & {\rm f} &= {\rm u}, {\rm d}, \nu, l\,,\\
Y_{\rm d}' &= I_{\rm d} T_{\rm d}\, Y_{\rm d}\,, & \xi_q &= \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_q^2(t)\,{\rm d}t\right]\,, & q &= u, c, t\,,\\
D_l' &= I_l T_l\, D_l\,, & \zeta_\alpha &= \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_\alpha^2(t)\,{\rm d}t\right]\,, & \alpha &= e, \mu, \tau\,,\\
Y_\nu' &= I_\nu T_\nu\, Y_\nu\,,
\end{aligned} \tag{3}$$

where $t_0 \equiv \ln(\Lambda/\Lambda_{\rm EW})$ with Λ a superhigh-energy scale, $T_{\rm d} = T_{\rm u}^{-1} = {\rm diag}\{\xi_u, \xi_c, \xi_t\}$ and $T_\nu = T_l^{-1} = {\rm diag}\{\zeta_e, \zeta_\mu, \zeta_\tau\}$.

Naumov- and Toshev-like Relations

In contrast with the matter effects, the radiative corrections to flavor mixings are dominated by the third generation of quarks and leptons, namely, the top quark and tau lepton. Therefore, it is more convenient to adopt the following parametrization

$$U = \begin{pmatrix} \cos\theta_l & \sin\theta_l & 0\\ -\sin\theta_l & \cos\theta_l & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} e^{-{\rm i}\phi_\ell} & 0 & 0\\ 0 & \cos\theta_\ell & \sin\theta_\ell\\ 0 & -\sin\theta_\ell & \cos\theta_\ell\end{pmatrix}\begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu & 0\\ \sin\theta_\nu & \cos\theta_\nu & 0\\ 0 & 0 & 1\end{pmatrix},$$

and likewise for the quark flavor mixing matrix. Consider the commutators at low and high energy scales [6]

$$\begin{aligned}
\left[D_l'^2, H_\nu'\right] &\equiv {\rm i}X_\ell'\,, & X_\ell' &= {\rm i}\begin{pmatrix} 0 & \Delta_{\mu e}' Z_{e\mu}' & \Delta_{\tau e}' Z_{e\tau}'\\ \Delta_{e\mu}' Z_{\mu e}' & 0 & \Delta_{\tau\mu}' Z_{\mu\tau}'\\ \Delta_{e\tau}' Z_{\tau e}' & \Delta_{\mu\tau}' Z_{\tau\mu}' & 0\end{pmatrix}, & \Delta_{\alpha\beta}^{(\prime)} &= m_\alpha^{(\prime)2} - m_\beta^{(\prime)2},\\
\left[D_l^2, H_\nu\right] &\equiv {\rm i}X_\ell\,, & X_\ell &= {\rm i}\begin{pmatrix} 0 & \Delta_{\mu e} Z_{e\mu} & \Delta_{\tau e} Z_{e\tau}\\ \Delta_{e\mu} Z_{\mu e} & 0 & \Delta_{\tau\mu} Z_{\mu\tau}\\ \Delta_{e\tau} Z_{\tau e} & \Delta_{\mu\tau} Z_{\tau\mu} & 0\end{pmatrix}, & Z_{\alpha\beta}^{(\prime)} &= \sum_i y_i^{(\prime)2}\, U_{\alpha i}^{(\prime)} U_{\beta i}^{(\prime)*},
\end{aligned} \tag{4}$$

with $H_\nu^{(\prime)} \equiv Y_\nu^{(\prime)} Y_\nu^{(\prime)\dagger}$. Using Eqs. (3) and (4) we are ready to derive the Naumov- and Toshev relations

$$J'\Delta_{21}'\Delta_{31}'\Delta_{32}' = I_\nu^6 \zeta_e^2 \zeta_\mu^2 \zeta_\tau^2 J \Delta_{21}\Delta_{31}\Delta_{32}\,, \qquad \sin\phi_\ell\,\sin 2\theta_l = \sin\phi_\ell'\,\sin 2\theta_l'\,.$$

References for Renormalization-Group Equations

[1] E.C.G. Stueckelberg, A. Petermann, Helv. Phys. Acta **26** (1953) 499.
[2] M. Gell-Mann, F.E. Low, Phys. Rev. **95** (1954) 1300.
[3] K.S. Babu, Z. Phys. C **35** (1987) 69.
[4] J.W. Mei, Phys. Rev. D **71** (2005) 073012.
[5] M. Lindner, M. Ratz, M.A. Schmidt, JHEP **09** (2005) 081.
[6] Z.Z. Xing, S. Zhou, **arXiv:1804.01925**.

Matter Effects on Leptonic Flavor Mixing Parameters

When neutrinos are propagating in matter, the coherent forward elastic scattering of neutrinos off the background particles may have a great impact on the neutrino flavor conversions [1, 2]. The effective Hamiltonian for neutrino oscillations in matter reads

$$H_{\rm eff} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger + \begin{pmatrix} \boldsymbol{a} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \equiv \frac{1}{2E} V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0\\ 0 & \widetilde{m}_2^2 & 0\\ 0 & 0 & \widetilde{m}_3^2\end{pmatrix}V^\dagger \tag{1}$$

with $\boldsymbol{a \equiv 2\sqrt{2}G_{\rm F}N_e E}$ being the matter parameter that measures the contribution from the coherent forward scattering. Here U and V stand for the leptonic flavor mixing matrix in vacuum and that in matter, respectively.

Interesting identities between the effective neutrino parameters in matter and the genuine parameters in vacuum can be established [3, 4]

$$\frac{\widetilde{J}}{J} = \frac{\Delta_{12}\Delta_{23}\Delta_{31}}{\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}\widetilde{\Delta}_{31}} = \frac{|V_{e1}|\cdot|V_{e2}|\cdot|V_{e3}|}{|U_{e1}|\cdot|U_{e2}|\cdot|U_{e3}|}\,, \tag{2}$$

where the Jarlskog invariant $\boldsymbol{J \equiv {\rm Im}[U_{e1}U_{e2}^* U_{\mu 1}^* U_{\mu 2}]}$ and the neutrino mass-squared differences $\boldsymbol{\Delta_{ij} \equiv m_i^2 - m_j^2}$ (for $ij = 12, 23, 31$) have been defined, and likewise for the counterparts in matter. The first identity in Eq. (2) is known as the Naumov relation.

RG-like Equations for the Effective Mixing Parameters

Differentiating the effective Hamiltonian $H_{\rm eff}$ with respect to the matter parameter a, one can find the following set of differential equations [5, 6]

$$\frac{\rm d}{{\rm d}a}|V_{\alpha i}|^2 = 2\sum_{j\neq i} {\rm Re}\left[V_{ei}V_{ej}^* V_{\alpha i}^* V_{\alpha j}\right]\widetilde{\Delta}_{ij}^{-1}\,, \qquad \frac{\rm d}{{\rm d}a}\widetilde{m}_i^2 = |V_{ei}|^2\,, \tag{3}$$

for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$. More explicitly, we get

$$\begin{aligned}
\frac{\rm d}{{\rm d}a}|V_{e1}|^2 &= 2|V_{e1}|^2\left(|V_{e2}|^2\widetilde{\Delta}_{12}^{-1} - |V_{e3}|^2\widetilde{\Delta}_{31}^{-1}\right)\,, & \frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{12} &= |V_{e1}|^2 - |V_{e2}|^2\,;\\
\frac{\rm d}{{\rm d}a}|V_{e2}|^2 &= 2|V_{e2}|^2\left(|V_{e3}|^2\widetilde{\Delta}_{23}^{-1} - |V_{e1}|^2\widetilde{\Delta}_{12}^{-1}\right)\,, & \frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{23} &= |V_{e2}|^2 - |V_{e3}|^2\,;\\
\frac{\rm d}{{\rm d}a}|V_{e3}|^2 &= 2|V_{e3}|^2\left(|V_{e1}|^2\widetilde{\Delta}_{31}^{-1} - |V_{e2}|^2\widetilde{\Delta}_{23}^{-1}\right)\,, & \frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{31} &= |V_{e3}|^2 - |V_{e1}|^2\,,
\end{aligned} \tag{4}$$

which have been recast into a symmetric and closed form. Note that only four of the above differential equations are independent. Then it is straightforward to verify

$$\frac{\rm d}{{\rm d}a}\ln\left[\widetilde{J}\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}\widetilde{\Delta}_{31}\right] = 0\,, \qquad \frac{\rm d}{{\rm d}a}\ln\left[|V_{e1}|^2|V_{e2}|^2|V_{e3}|^2\widetilde{\Delta}_{12}^2\widetilde{\Delta}_{23}^2\widetilde{\Delta}_{31}^2\right] = 0\,, \tag{5}$$

indicating that the identities in Eq. (2) can be interpreted as differential invariants.

RG-like Equations in the Standard Parametrization

The effective flavor mixing matrix V in matter can be parametrized in terms of three mixing angles $\{\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}\}$ and one CP-violating phase $\tilde{\delta}$, namely,

$$V = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\tilde{\theta}_{23} & \sin\tilde{\theta}_{23}\\ 0 & -\sin\tilde{\theta}_{23} & \cos\tilde{\theta}_{23}\end{pmatrix}\begin{pmatrix} \cos\tilde{\theta}_{13} & 0 & \sin\tilde{\theta}_{13}\\ 0 & e^{-{\rm i}\tilde{\delta}} & 0\\ -\sin\tilde{\theta}_{13} & 0 & \cos\tilde{\theta}_{13}\end{pmatrix}\begin{pmatrix} \cos\tilde{\theta}_{12} & \sin\tilde{\theta}_{12} & 0\\ -\sin\tilde{\theta}_{12} & \cos\tilde{\theta}_{12} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Due to the electron-flavor dominance in matter effects, we can choose the standard parametrization of V, for which the RG-like equations turn out to be simple [6]

$$\begin{aligned}
\frac{\rm d}{{\rm d}a}\tilde{\theta}_{12} &= \frac{1}{2}\sin 2\tilde{\theta}_{12}\left(\frac{\cos^2\tilde{\theta}_{13}}{\widetilde{\Delta}_{21}} - \frac{\sin^2\tilde{\theta}_{13}\widetilde{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\right)\,,\\
\frac{\rm d}{{\rm d}a}\tilde{\theta}_{13} &= \frac{1}{2}\sin 2\tilde{\theta}_{13}\left(\frac{\cos^2\tilde{\theta}_{12}}{\widetilde{\Delta}_{31}} + \frac{\sin^2\tilde{\theta}_{12}}{\widetilde{\Delta}_{32}}\right)\,,\\
\frac{\rm d}{{\rm d}a}\tilde{\theta}_{23} &= \frac{1}{2}\sin 2\tilde{\theta}_{12}\sin\tilde{\theta}_{13}\cos\tilde{\delta}\,\frac{\widetilde{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\,,\\
\frac{\rm d}{{\rm d}a}\tilde{\delta} &= -\sin 2\tilde{\theta}_{12}\,\frac{\sin\tilde{\theta}_{13}\sin\tilde{\delta}}{\cot 2\tilde{\theta}_{23}}\,\frac{\widetilde{\Delta}_{21}}{\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}\,.
\end{aligned} \tag{6}$$

One can prove that the Toshev relation $\boldsymbol{\sin\tilde{\delta}\,\sin 2\tilde{\theta}_{23} = \sin\delta\,\sin 2\theta_{23}}$ holds exactly [4].

References for Matter Effects on Neutrino Oscillations

[1] L. Wolfenstein, Phys. Rev. D **17** (1978) 2369.
[2] S.P. Mikheyev, A.Yu. Smirnov, Sov. J. Nucl. Phys. **42** (1985) 913.
[3] V.A. Naumov, Int. J. Mod. Phys. D **1** (1992) 379.
[4] S. Toshev, Mod. Phys. Lett. A **6** (1991) 455.
[5] S.H. Chiu, T.K. Kuo, Phys. Rev. D **97** (2018) 055026.
[6] Z.Z. Xing, S. Zhou, Y.L. Zhou, JHEP **05** (2018) 015.