α_s - evolution

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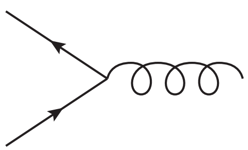
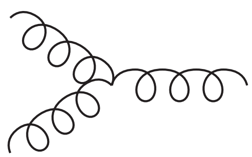
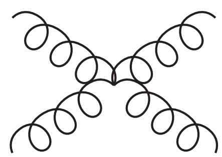
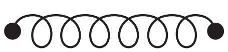
Motivation

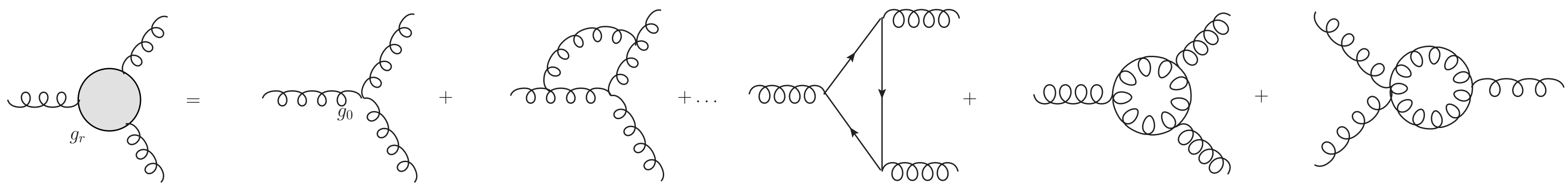
- The strong coupling constant α_s is large at low energies (confinement) and small at large energies (asymptotic freedom).
- QCD does not predict α_s, but its energy dependence.
- Testing QCD demands at least two measurements of α_s at different energies. The knowledge of the energy dependence allows the comparison.

The QCD Langragian

- $\mathcal{L}_{\mathrm{QCD}} = \sum_{k=1}^{n_f} \overline{\Psi}_k (\mathrm{i}\gamma_\mu D^\mu - m_k)\psi_k - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a$
- $D^\mu = \partial^\mu - \mathrm{i}gT^a G_{\mu a}$ with gluon field $G_{\mu a}$ and color matrix T^a
  ψ^k: quark field with flavour k and mass m_k
  $F_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + gf^{abc} G_{\mu b} G_{\nu c}$ (field strength tensor)
- From $\mathcal{L}_{\mathrm{QCD}}$ all QCD interactions arise: quark-quark-gluon vertex, three-gluon vertex, four-gluon vertex, quark and gluon propagators...

Bare and renormalized Quantities



- All diagrams above are formally divergent, introduce Z -factors to reformulate theory, f. e.
- $V^0_{ggg} = Z_1 V^r_{ggg}, \;\; g^0 = Z_g g^r, \;\; \alpha^0 = Z_3 \alpha^r, \;\; G^{a,0}_\mu = Z_3^{1/2} G^{a,r}_\mu$
- $V^0, g^0, \alpha^0, G^{a,0}_\mu$: bare, infinite vertex, coupling constant, gauge fixing parameter, gluon field $V^r, g^r, \alpha^r, G^{a,r}_\mu$: renormalized, quantities
- Slavnov-Taylor-identity: $Z_1 Z_3^{-3/2} = Z_g$

1-loop results

- Perform loop integrations in $D = 4 - 2\varepsilon$ dimensions
- $Z_3 = 1 - \frac{\alpha_s}{4\pi}\left[\frac{4}{3}T_R n_f - \frac{1}{2}C_A\left(\frac{13}{3} - \alpha^r\right)\right]\frac{1}{\varepsilon}$
- $Z_1 = 1 - \frac{\alpha_s}{4\pi}\left[C_A\left(-\frac{17}{12} + \frac{3}{4}\alpha^r\right) + \frac{4}{3}T_R n_f\right]\frac{1}{\varepsilon}$
- $Z_g = 1 - \frac{\alpha_s}{4\pi}\frac{1}{6}\left(11 C_A - 4 T_R n_f\right)\frac{1}{\varepsilon}$
  independent of the gauge fixing parameter
- T_R, C_A: gauge group specific constants, n_f: number of flavours
- QED: $T_R = 1$, $C_A = 0$, $Z_e = 1 + \frac{\alpha_{\mathrm{em}}}{4\pi}\frac{2}{3}\frac{1}{\varepsilon}$
- QCD: $T_R = \frac{1}{2}$, $C_A = N_c = 3$, $Z_g = 1 - \frac{\alpha_s}{4\pi}\frac{1}{2}\underbrace{\left(11 - \frac{2}{3}n_f\right)}_{>0,\ \text{if}\ n_f < \frac{33}{2}}\frac{1}{\varepsilon}$

The scale dependence of the strong coupling

- for dimensional reasons: $g^0 = \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} g^r Z_g$
- but physical quantities are scale-independent: $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g^0 = 0$
- $\Rightarrow \mu \frac{\mathrm{d}g^r}{\mathrm{d}\mu} Z_g \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} + g^r \mu \frac{\mathrm{d}Z_g}{\mathrm{d}\mu} \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} + g^r Z_g \varepsilon \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} = 0$
- Define $\beta = \mu \frac{\mathrm{d}g^r}{\mathrm{d}\mu}$ and rewrite the last equation:

$$\beta = \underbrace{-\varepsilon g^r}_{\to 0} - \frac{1}{(4\pi)^2} \underbrace{\frac{11 C_A - 4 T_R n_f}{3}}_{\beta_0} (g^r)^3 + \mathcal{O}\left((g^r)^5\right)$$

The scale dependence of the strong coupling, cont'd

- Rewrite the last equation in terms of $a_s = \frac{(g^r)^2}{(4\pi)^2}$:

$$\frac{\mathrm{d}\,a_s}{\mathrm{d}\log(\mu^2)} = -\beta_0 a_s^2$$

- and for n loops:

$$\frac{\mathrm{d}\,a_s}{\mathrm{d}\log(\mu^2)} = \sum_{k=0}^{\infty} \beta_k a_s^{2+k}$$

  where the first coefficients β_k are given by

$$\begin{aligned}
\beta_0 &= 11 - \tfrac{2}{3} n_f\,, \\
\beta_1 &= 102 - \tfrac{38}{3} n_f\,, \\
\beta_2 &= -\tfrac{2857}{2} + \tfrac{5033}{18} n_f - \tfrac{325}{54} n_f^2\,, \\
\beta_3 &= \tfrac{149753}{6} + 3564\zeta_3 - \left(\tfrac{1078361}{162} + \tfrac{6508}{27}\zeta_3\right) n_f + \left(\tfrac{50065}{162} + \tfrac{6472}{81}\zeta_3\right) n_f^2 + \tfrac{1093}{739} n_f^3
\end{aligned}$$

The scale dependence of the strong coupling, cont'd

- This differential equation can be solved analytically for $k = 0$ by separation of variables:

$$\int_{a_s(\mu_0^2)}^{a_s(\mu^2)} \frac{\mathrm{d}\,a_s}{a_s^2} = -\beta_0 \int_{\mu_0^2}^{\mu^2} \mathrm{d}\log(\mu^2)$$

$$\Rightarrow a_s(\mu^2) = \frac{a_s(\mu_0^2)}{1 + a_s(\mu_0^2)\beta_0 \log(\mu^2/\mu_0^2)}$$

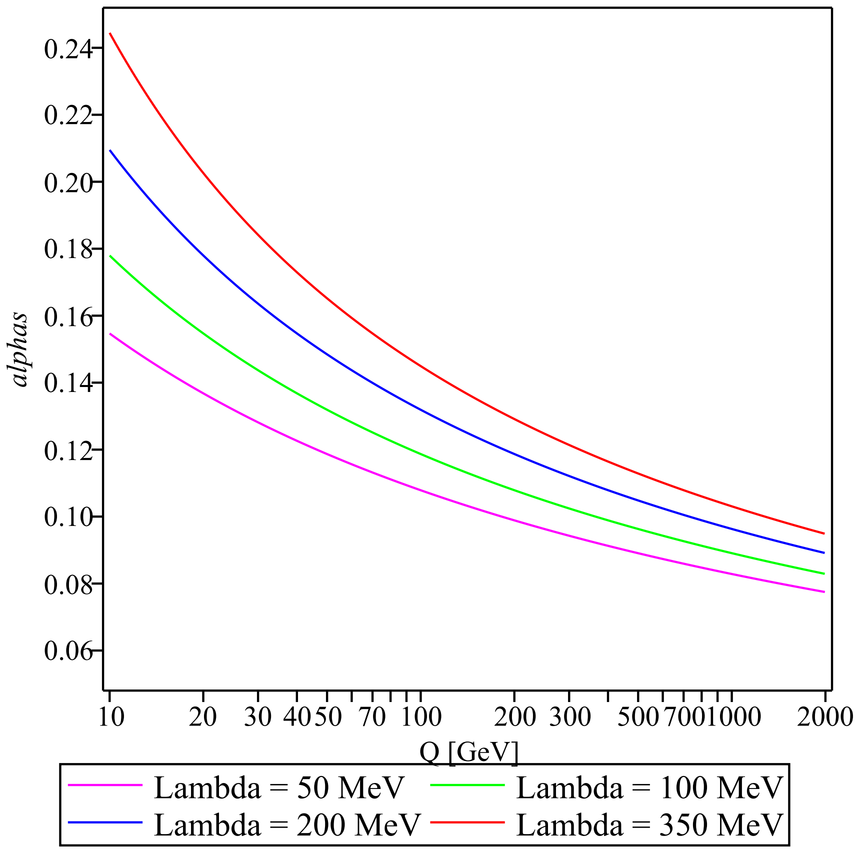
- or generally:

$$\int_{a_s(\mu_0^2)}^{a_s(\mu^2)} \frac{\mathrm{d}\,a_s}{a_s^2(\beta_0 + \beta_1 a_s + \ldots)} = -\int_{\mu_0^2}^{\mu^2} \mathrm{d}\log(\mu^2)$$

The Λ - parameter

- Define $\Lambda^2 = \mu_0^2 \exp\left(-\frac{4\pi}{\beta_0 \alpha_s(\mu_0^2)}\right)$
- and rewrite the evolution equation for a_s:

$$a_s(\mu^2) = \frac{1}{\beta_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)}$$



Properties of the a_s evolution equation

- $a_s(\mu^2) = \frac{1}{\beta_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)}$
- a_s depends only on the parameter Λ (QCD scale parameter)
- $a_s \longrightarrow \infty$ for $\mu \longrightarrow \Lambda$
  $\Lambda \approx 200\,\mathrm{MeV}$,
  QCD perturbation theory is valid for $Q^2 \gtrsim 1\,\mathrm{GeV}^2 > \Lambda^2$, i.e. $\alpha_s \lesssim 0.4$
- a_s is a decreasing function of Q^2 due to the negative sign of the rhs of the differential equation and due to $\beta_0 > 0$ for $n_f < 17$ (we have $n_f \leq 6$ flavours)
  asymptotic freedom: At high energies the particles are free

Maple

- `Maple`: Computer algebra system (CAS), not only numerical calculations, but also algebraic manipulations
- Useful commands:
  - `restart;` clears the internal memory of the `Maple` kernel so that `Maple` works as if it were restarted; hint: every command is closed by a semicolon ";" or a colon ":" (suppresses output)
  - `a := 1;` assigns the variable a with the value 1
  - `dsolve({ODE, inc}, y(x))`: solves the ordinary differential equation `ODE` with the initial condition(s) `inc` in terms of `y(x)`
  - `with plots:`
    `plot(f, x=x0..x1, list_of_options);` plots the function `f` in the interval `x0..x1` with the options `list_of_options`
  - A command is executed by typing [Enter], [Enter] [⇑ Shift] causes a line break

Maple

- Download `alphasevolution.mw`
- Start `xmaple` (provides a graphical user interface) and open that file.

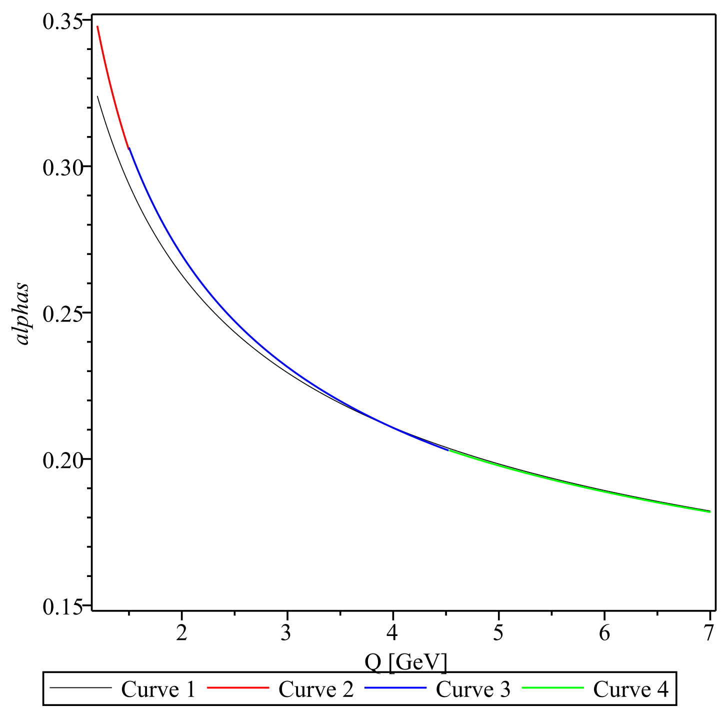
Threshold matching

- So far, effects of finite quark masses are neglected, o.k. for $Q \gg m_b$
- but important for energies $\approx m_c$, m_b (masses of charm and bottom quark)
- $\Rightarrow a_s$ depends indirectly on the quark masses due to the number of quarks n_f with $m_q < \mu$ entering the β - function
- $a_s(\mu, n_f - 1)$ and $a_s(\mu, n_f)$ must be consistent at the quark mass threshold $\mu = m_c$ and $\mu = m_b$
  $\Rightarrow$ matching conditions
- Naively: $a_s(\mu = m_{c,b}, nf - 1) = a_s(\mu = m_{c,b}, nf)$. Correct for matching at 1- or 2-loop level.
  At 3- and 4-loop: $\alpha_s(\mu, n_f - 1) = \alpha_s(\mu, n_f)\left(1 + C_2\left(\frac{\alpha_s}{\pi}\right)^2 + C_3\left(\frac{\alpha_s}{\pi}\right)^3\right)$

Threshold matching, cont'd

- At 3 - loops and higher: discontinuities at $\mu = m_{c,b}$
- These discontinuities are **NOT** measurable, they are artefacts due to truncation of the β - function

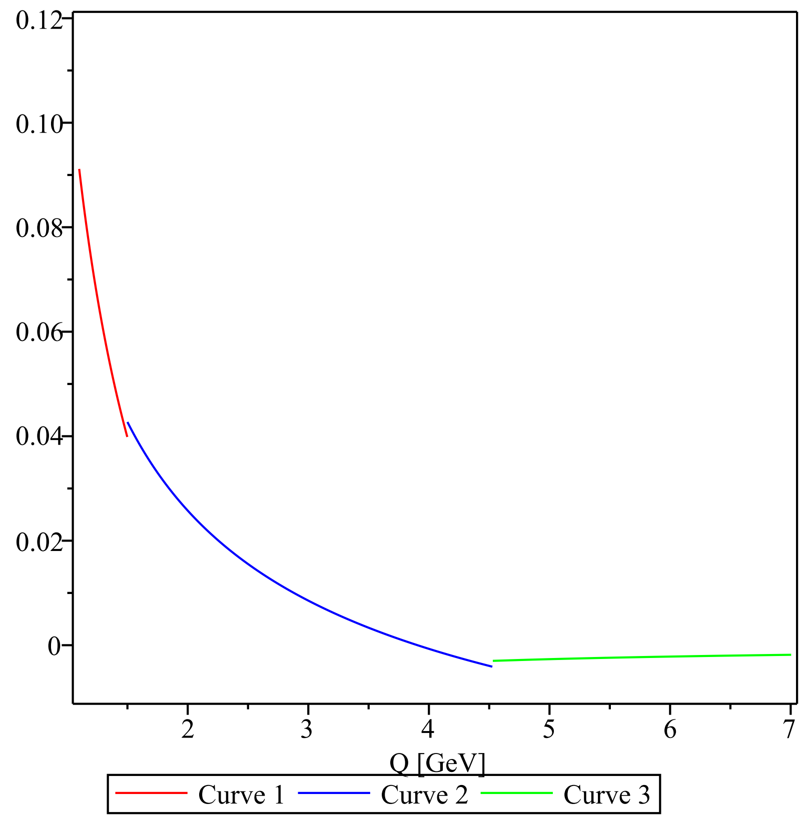
Figure: $\alpha_s^{3-\mathrm{loop}}(\mu)$ with threshold matching



Threshold matching, cont'd

- On the following plot, one clearly sees the discontinuities.

Figure: Fractional differences $1 - \frac{\alpha_s^{3-\text{loop}}(\mu)}{\alpha_s^{4-\text{loop,unm.}}(\mu)}$



α_s from the experiment

- Determine α_s from
  - τ-decay: $R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons } \nu_\tau)}{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}$, $\alpha_s(M_{Z^0}) = 0.1197 \pm 0.0016$
  - bottomonium Υ decay: $R_\tau = \frac{\Gamma(\Upsilon \to \gamma gg)}{\Gamma(\Upsilon \to ggg)}$, $\alpha_s(M_{Z^0}) = 0.119 \pm 0.006$
  - jet production in e^+e^- production: $\alpha_s(M_{Z^0}) = 0.1224 \pm 0.0039$
  - the ratio of the hadronic to the electronic decay width of the Z^0 boson: $R_{Z^0} = \frac{\Gamma(Z^0 \to \text{hadrons})}{\Gamma(Z^0 \to e^+e^-)}$, $\alpha_s(M_{Z^0}) = 0.1193 \pm 0.0028$
- World average 2009: $\alpha_s(M_{Z^0}) = 0.1184 \pm 0.00067$