Recursive Methods For Scattering Amplitudes

Simon Badger (DESY, Zeuthen)

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Outline

Computations for QCD Cross-sections:

- Colour ordering
- Spinor/Helicity Methods
- Feynman techniques
  - $qq \longrightarrow gg$
  - $gg \longrightarrow gg$
  - $gg \longrightarrow ggg$
- On-shell recursion for gluon amplitudes
  - BCFW relations for n-gluon amplitudes

Structure of the Hard Matrix Elements

Schematically:

$$\sigma_{pp\to X} = \sum_{i,j,k} f_i(\mu^2) f_j(\mu^2) \sigma_{ij\to k}(Q^2) D_{k\to X}(\mu^2)$$

$$\sigma_{ij\to k}(Q^2) = \int d\text{LIPS} |\mathcal{A}_{ij\to k}|^2 \delta^{(4)}(p_i + p_j - p_k)$$

Today:

- Techniques for efficient evaluation of $|\mathcal{A}_{ij\to k}|^2$
- Why the fuss?
  - Feynman diagrams are messy!
  - Hidden structures in gauge theory amplitudes

$SU(N)$ **Colour Algebra 1**

Amplitude separates into colour factor multiplied by kinematic amplitude

$$\mathcal{A} = \sum_k \mathcal{D}_k = \sum_l C_l A_l$$

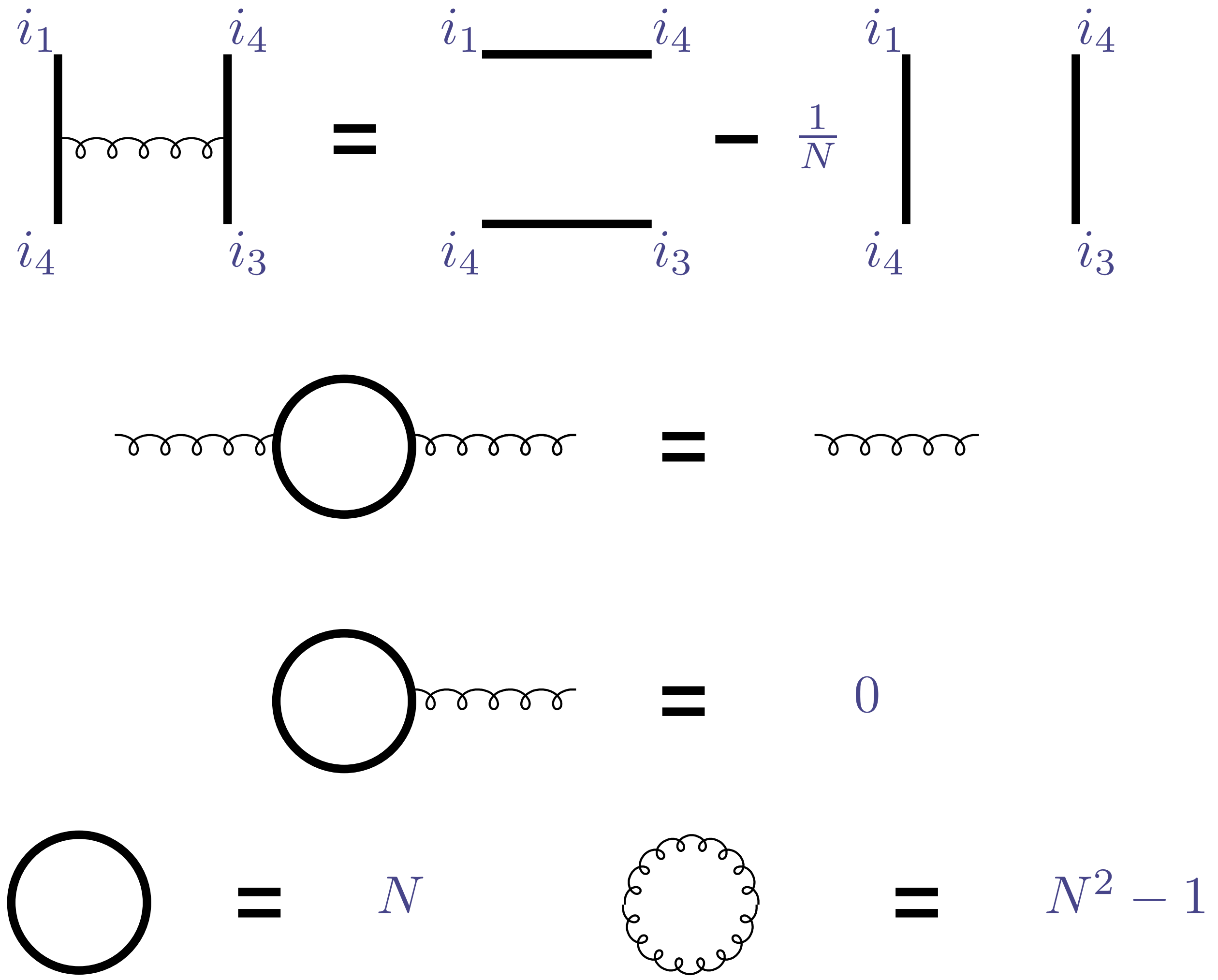
- C_l - Colour factors, f^{abc}, T^a_{ij}
- A_l - Kinematic factors, $\{p_k\}$
- $l \neq k$!

Use group structure to apply symmetry to the kinematic amplitude

$$\mathrm{tr}(T^a T^b) = \delta^{ab} \qquad T^a_{i_1 i_2} T^a_{i_3 i_4} = \delta_{i_1 i_4}\delta_{i_2 i_3} - \frac{1}{N}\delta_{i_1 i_2}\delta_{i_3 i_4}$$

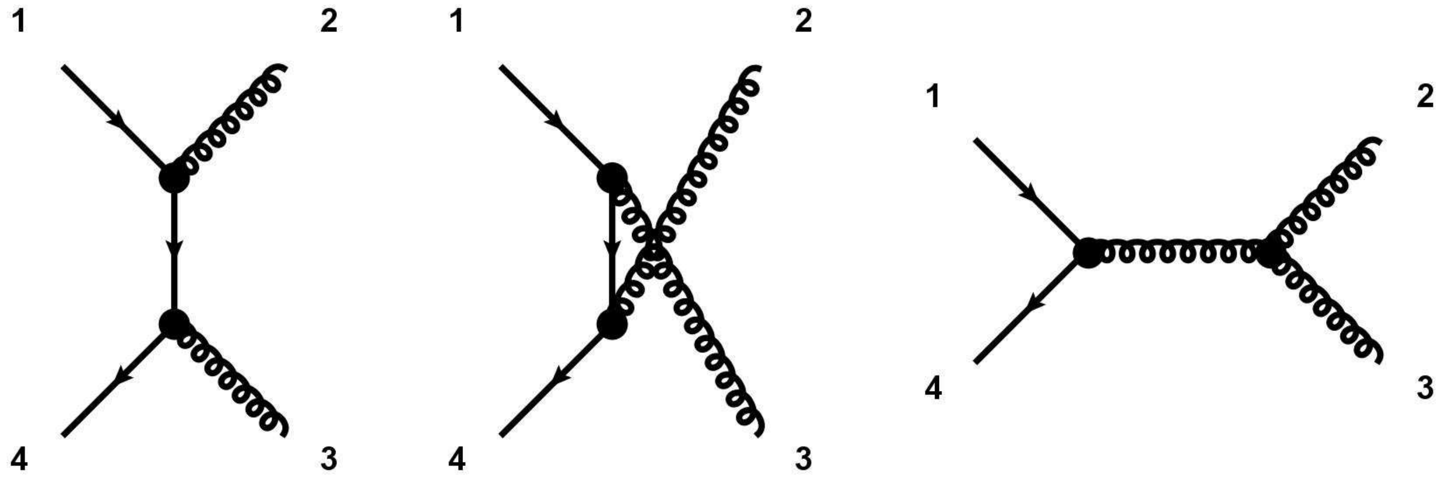
$$\mathrm{tr}(T^a) = 0 \qquad \delta_{ii} = N \qquad \delta^{aa} = N^2 - 1$$

$SU(N)$ Colour Algebra 2



Example 1: $q\bar{q}gg$

3 Feynman diagrams:



Colour factors:

- Diagram 1: $T^{a_2}_{i_1 k} T^{a_3}_{k i_4} = (T^{a_2} T^{a_3})_{i_1 i_4} = C_1$
- Diagram 2: $T^{a_3}_{i_1 k} T^{a_2}_{k i_4} = (T^{a_3} T^{a_2})_{i_1 i_4} = C_2$
- Diagram 3: $T^b_{i_1 i_4} f^{b a_2 a_3} = T^b_{i_1 i_4} \, \mathrm{tr}([T^{a_2} T^{a_3}] T^b) = C_1 - C_2$

Example 1: $q\bar{q}gg$

$$\mathcal{A}_4(1_q,2,3,4_{\bar{q}}) = C_1K_1 + C_2K_2$$

$$K_1 = A_4(1_q,2,3,4_{\bar{q}})$$

$$K_2 = A_4(1_q,3,2,4_{\bar{q}})$$

Compute the squared matrix element summing over colour:

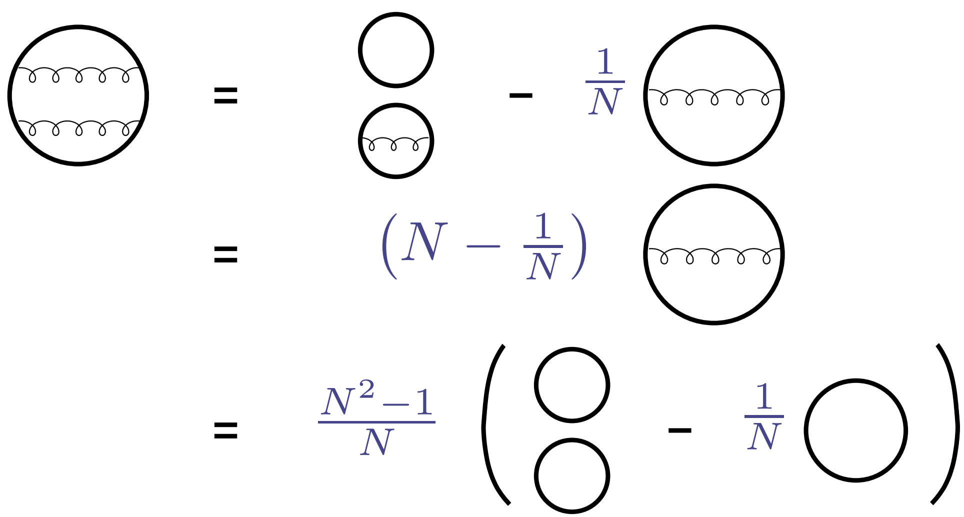
$$\sum_{\text{col.}} |\mathcal{A}_4(1_q,2,3,4_{\bar{q}})|^2 = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} C_1C_1^\dagger & C_1C_2^\dagger \\ C_2C_1^\dagger & C_2C_2^\dagger \end{pmatrix} \begin{pmatrix} K_1^\dagger \\ K_2^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} \mathrm{tr}(T^{a_2}T^{a_3}T^{a_3}T^{a_2}) & \mathrm{tr}(T^{a_2}T^{a_3}T^{a_2}T^{a_3}) \\ \mathrm{tr}(T^{a_2}T^{a_3}T^{a_2}T^{a_3}) & \mathrm{tr}(T^{a_2}T^{a_3}T^{a_3}T^{a_2}) \end{pmatrix} \begin{pmatrix} K_1^\dagger \\ K_2^\dagger \end{pmatrix}$$

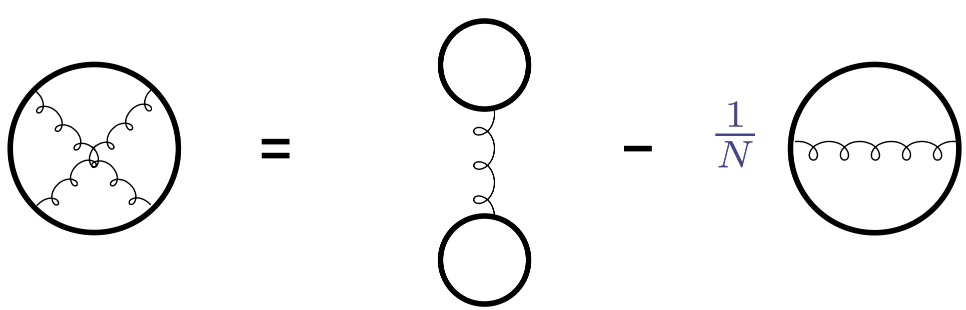
Example 1: $q\bar{q}gg$

Evaluation of colour traces:

- $\mathrm{tr}(T^{a_2}T^{a_3}T^{a_3}T^{a_2}) = \frac{(N^2-1)^2}{N}$



- $\mathrm{tr}(T^{a_2}T^{a_3}T^{a_2}T^{a_3}) = -\frac{(N^2-1)}{N}$



Example 1: $q\bar{q}gg$

Squared matrix elements summed over colour:

$$\sum_{\text{col.}} |\mathcal{A}_4(1_q, 2, 3, 4_{\bar{q}})|^2 = N(N^2-1)\left(|K_1|^2 + |K_2|^2\right) - \frac{N^2-1}{N}|K_1 + K_2|^2$$

- Next step: evaluate K_1 using helicity methods

$$K_1 = A_4(1_q^{h_1}, 2^{h_2}, 3^{h_3}, 4_{\bar{q}}^{h_4})$$

K_1 is a function of:

- helicities $\{h_i\}$: $2^4 = 16$ configurations
- momenta $\{p_i\}$

Spinor/Helicity Methods 1

- Massless momenta can be represented in terms of a 2-component Weyl spinor basis:

$$SO(3,1) \leftrightarrow SU(2) \times SU(2)$$

- Full kinematics fully represented in terms of 2-component spinors:

$$\lambda(p) = |p+\rangle = |p\rangle \qquad \tilde{\lambda}(p) = |p-\rangle = |p]$$

- $\lambda, \tilde{\lambda}$ solutions to $\sigma \cdot p = 0$ and $\bar{\sigma} \cdot p = 0$

$$\sigma \cdot p = \begin{pmatrix} p^0 + p^3 & p_1 - ip_2 \\ p^0 + ip_2 & p^0 - p^3 \end{pmatrix} = \begin{pmatrix} p^+ & \bar{p}^\perp \\ p^\perp & p^- \end{pmatrix}$$

Spinor/Helicity Methods 2

$$\lambda(p) = \frac{\sqrt{p^+}}{\bar{p}^\perp} \begin{pmatrix} \bar{p}^\perp \\ p^- \end{pmatrix} \qquad \tilde{\lambda}(p) = \frac{\bar{p}^\perp}{\sqrt{p^+}} \begin{pmatrix} 1 \\ -\frac{p^+}{\bar{p}^\perp} \end{pmatrix}$$

Spinor products:

$$\langle pq \rangle = \lambda^\alpha(p) \varepsilon_{\alpha\beta} \lambda^\beta(q) = \frac{\sqrt{p^+}\sqrt{q^+}}{\bar{p}^\perp \bar{q}^\perp} \left(p^- \bar{q}^\perp - q^- \bar{p}^\perp \right)$$

$$[pq] = \tilde{\lambda}^{\dot{\alpha}}(p) \varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{\dot{\beta}}(q) = \frac{1}{\sqrt{p^+}\sqrt{q^+}} \left(p^+ \bar{q}^\perp - q^+ \bar{p}^\perp \right)$$

Spinor products are anti-symmetric functions

$$\langle pq \rangle [qp] = (p+q)^2 = 2p \cdot q$$

Spinor/Helicity Methods 3

Explicit representations for wave-functions and polarisation vectors:

$$u_+(p) = |p] \qquad u_-(p) = |p\rangle$$

$$\bar{u}_+(p) = [p| \qquad \bar{u}_-(p) = \langle p|$$

$$\epsilon_+^\mu(p,\xi) = \frac{\langle \xi|\gamma^\mu|p]}{\sqrt{2}\langle \xi p\rangle} \qquad \epsilon_-^\mu(p,\xi) = \frac{\langle p|\gamma^\mu|\xi]}{\sqrt{2}[p\xi]}$$

NB reversing momentum direction $\rightarrow$ helicity flip

- Free choice of the reference vectors $\xi \Rightarrow$ Gauge invariance
- Direct evaluation of amplitudes through spinor products

Example 1: $q\bar{q}gg$ Helicity Amplitudes

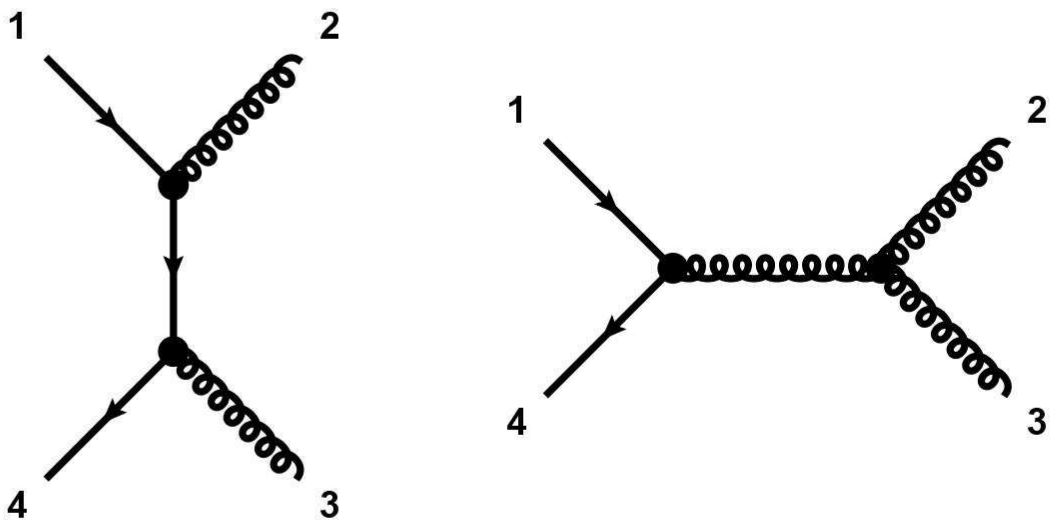


Diagram 1: All momenta considered incoming ($\sum_{i=1}^{4} p_i = 0$)

$$D_1^{+++-} = -i\bar{v}_-(4)\gamma_{\mu_3}\frac{\not{1}+\not{2}}{(1+2)^2}\gamma_{\mu_2}u_+(4)\epsilon_+^{\mu_2}(2,\xi_2)\epsilon_+^{\mu_3}(3,\xi_3)$$

$$= -i\frac{\langle 4|\gamma_{\mu_3}(1+2)\gamma_{\mu_2}|1]\langle\xi_2|\gamma^{\mu_2}|2]\langle\xi_3|\gamma^{\mu_3}|3]}{2\langle 12\rangle[21]\langle\xi_2 2\rangle\langle\xi_3 3\rangle}$$

$$= -i\frac{\langle 4\xi_3\rangle[3|1+2|\xi_2\rangle[21]}{\langle 12\rangle[21]\langle\xi_2 2\rangle\langle\xi_3 3\rangle} = i\frac{\langle 4\xi_3\rangle[34]\langle 4\xi_2\rangle}{\langle 12\rangle\langle\xi_2 2\rangle\langle\xi_3 3\rangle}$$

Example 1: $q\bar{q}gg$ Helicity Amplitudes

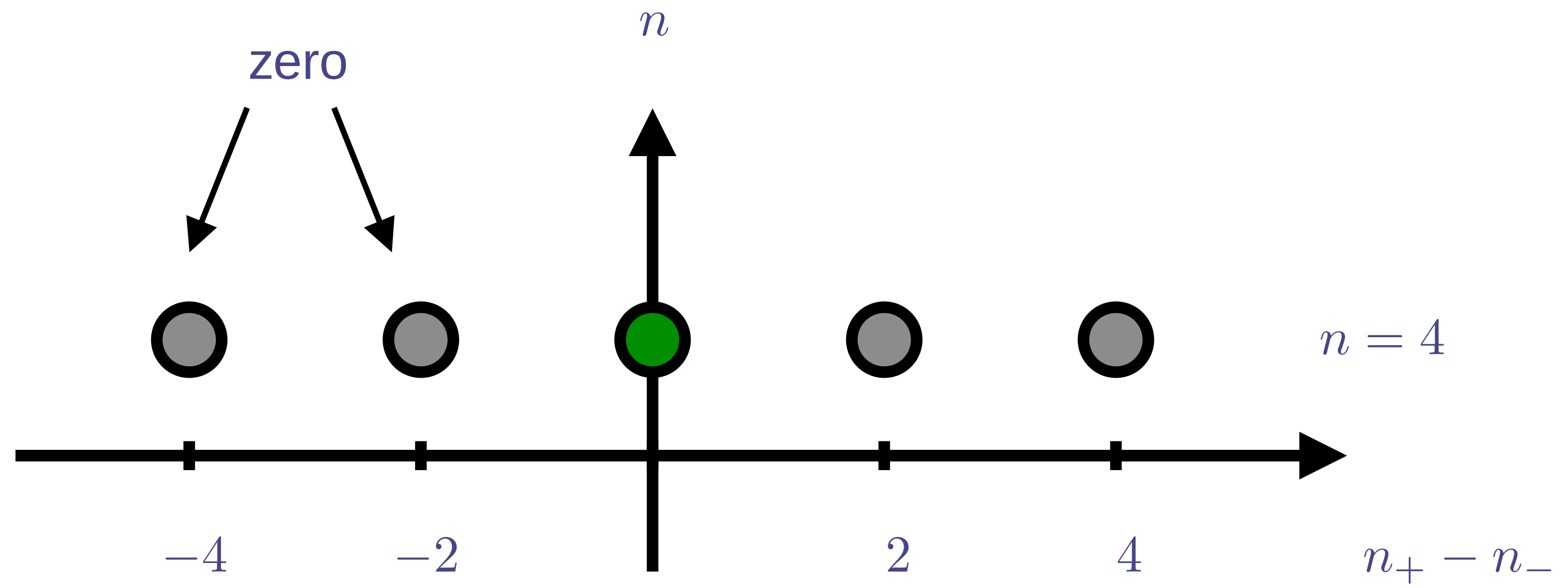
$$D_1^{+++-} = -i\frac{\langle 4\xi_3\rangle[34]\langle 4\xi_2\rangle}{\langle 12\rangle\langle \xi_2 2\rangle\langle \xi_3 3\rangle}$$

$$D_2^{+++-} = i\frac{(-\langle 43\rangle[31]\langle \xi_2\xi_3\rangle + \langle 4\xi_3\rangle[13]\langle 3\xi_2\rangle + \langle 4\xi_2\rangle[12]\langle 2\xi_3\rangle)}{\langle 23\rangle\langle \xi_2 2\rangle\langle \xi_3 3\rangle}$$

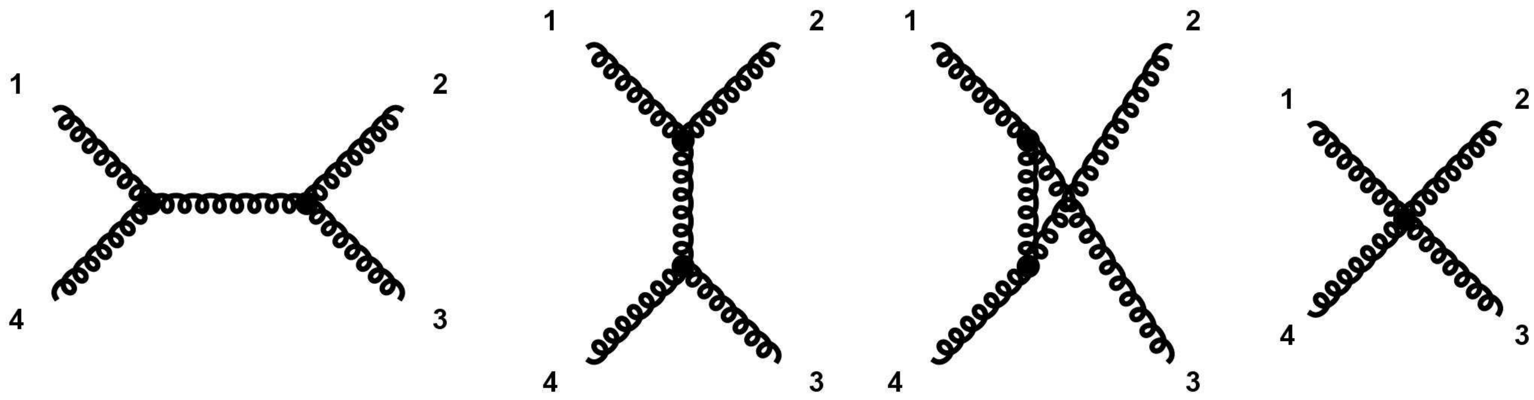
- Check independence on choice of ξ_2, ξ_3
- $\xi_2 = \xi_3 = p_4 \Rightarrow A_4(1_q^+, 2^+, 3^+, 4_{\bar{q}}^-) = 0$
- First non-zero configuration "Maximal Helicity Violating":

$$A_4(1_q^+, 2^+, 3^-, 4_{\bar{q}}^-) = i\frac{\langle 34\rangle^2\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 41\rangle}$$

Helicity Structures: $2 \to 2$ scattering



Example 2: $gggg$



- 6 colour structures $\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$
- 3 ordered diagrams:
  - 3 and 4 point interactions yield long expressions
- final result simple! same structure as $q\bar{q}gg$

$$A_4(1^\pm, 2^+, 3^+, 4^+) = 0$$

$$A_4(1^-, 2^-, 3^+, 4^+) = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

Example 2: $gggg$

Colour sum: $\mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}) = \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$

$$
\begin{aligned}
|\mathcal{A}_4(1^-,2^-,3^+,4^+)|^2 = 2N^2(N^2-1) \\
\times \left(|A_4(1^-,2^-,3^+,4^+)|^2 + |A_4(1^-,3^+,2^-,4^+)|^2 + |A_4(1^-,2^-,4^+,3^+)|^2\right) \\
+ \mathcal{O}\left(\frac{1}{N}\right)
\end{aligned}
$$

Sub-leading colour terms vanishes

- permutation sum kills self-interactions $\Rightarrow$ QED-like structure

Example 2: $gggg$

Square amplitudes $\rightarrow$ write back in terms of Lorentz invariants

$$|A_4(1^-,2^-,3^+,4^+)|^2 = \frac{\langle 12\rangle^3[12]^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle[23][34][41]} = \frac{s_{12}^3}{s_{23}s_{34}s_{41}}$$

$$|\mathcal{A}_4(1^-,2^-,3^+,4^+)|^2 = 2N^2(N^2-1)\frac{s^2(s^2+t^2+u^2)}{t^2u^2}$$

where $s_{ij} = (i+j)^2$ and $s_{12} = s, s_{23} = t, s_{13} = u$

- Benefit of helicity amplitudes:

$$(1+\gamma_5)(1-\gamma_5) = 0 \qquad (1)$$

- Helicity sum can be done incoherently

Example 2: $ggggg$

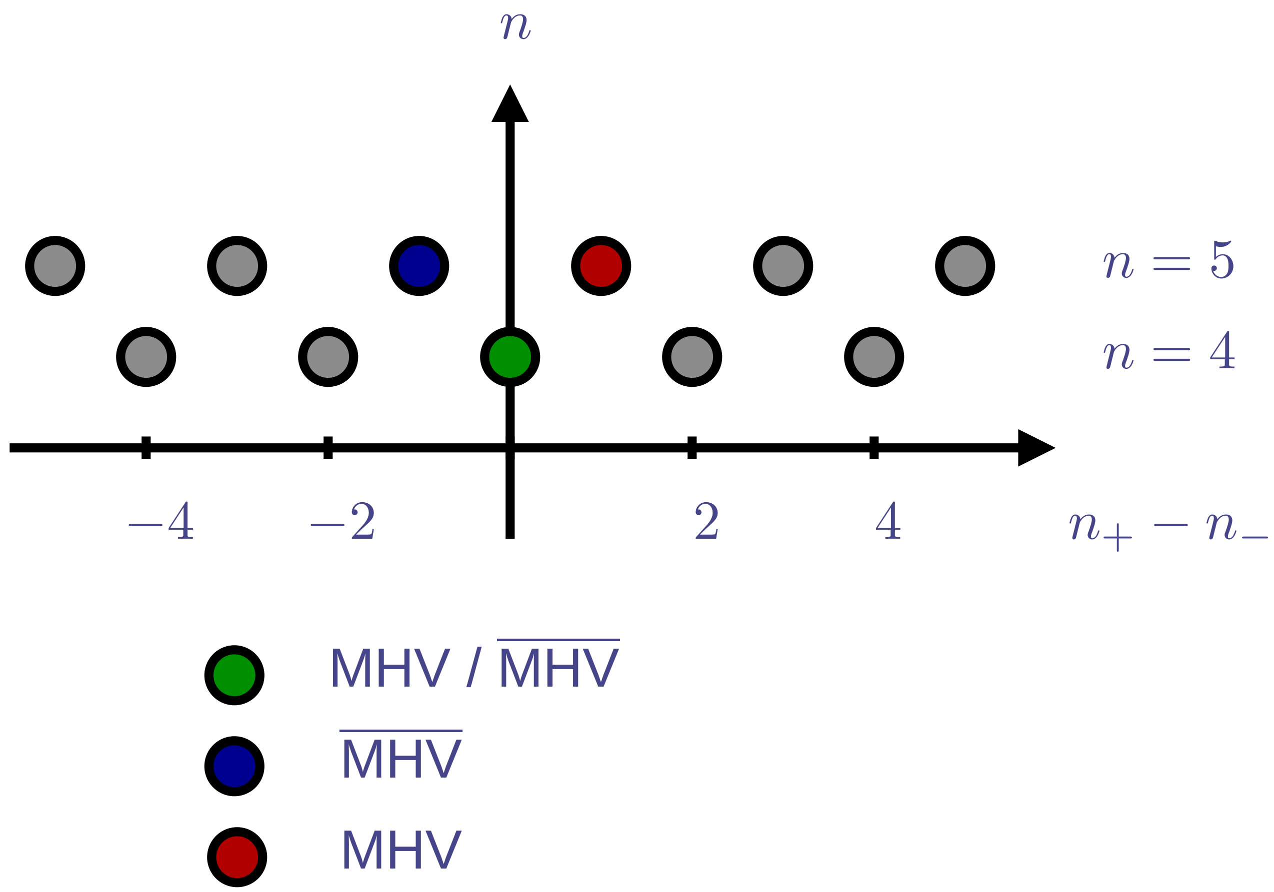
25 Feynman diagrams (10 ordered diagrams)

- large intermediate steps (lot's of self-interactions)
- large number of permutations in colour space $4! = 24$
- simple final results

Need efficient and automated methods

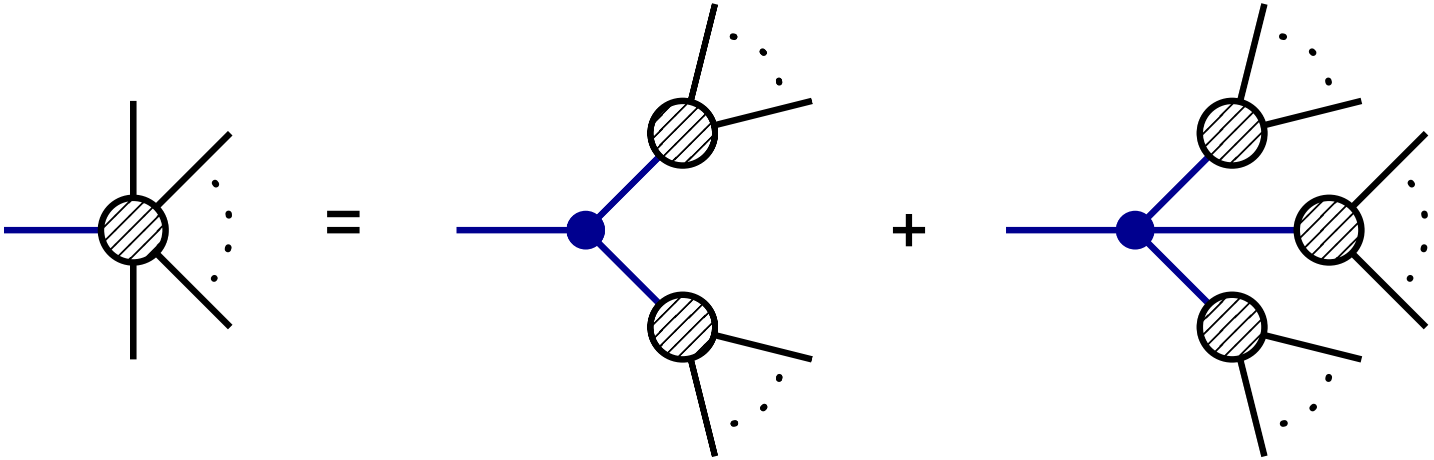
- long expressions mean long evaluation times
- high multiplicity amplitudes become unfeasible

Helicity Structures: $2 \to 3$ scattering



Off-Shell Recursive Techniques

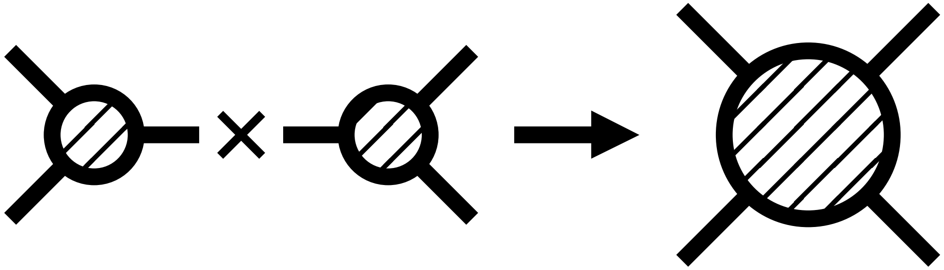
- Construct amplitudes from off-shell currents [Berends,Giele (1986)] [AlpGen,Helac]



- Extremely efficient numerical algorithm
- If you're clever you can also do Feynman diagrams! [MadGraph]

On-Shell Recursive Techniques

- On-shell amplitudes: only physical d.o.f $\Rightarrow$ simple expressions
- Build up tree amplitudes from simplest on-shell building blocks:



$$\text{MHV}_3 = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \qquad \overline{\text{MHV}}_3 = i\frac{[12]^3}{[23][31]}$$

On-Shell Recursive Techniques

- Problem: three-point amplitude vanishes on-shell

$$0 = p_3^2 = (p_1 + p_2)^2 = \langle 12 \rangle [21]$$

- Solution: move to complex momenta [Britto,Cachazo,Feng]

$$\langle 12 \rangle \neq [12]$$

Can find two complex solutions s.t. either $\langle 12 \rangle$ OR $[12]$ vanishes

BCFW Recursion

Principle: use complex analysis and factorisation to constrain amplitudes

- n-point amplitude: $A_n(1, 2, \ldots, n)$
- Complex continuation: $1 \to \widehat{1} \equiv 1(z)$, $2 \to \widehat{2} \equiv 2(z)$
- On-shell constraints: $\widehat{1}^2 = 0, \widehat{2}^2 = 0$ and $\widehat{1} + \widehat{2} + 3 = 0$
  Natural with Weyl spinors:

$$\widehat{1}^\mu = 1^\mu + \tfrac{z}{2}\langle 1|\gamma^\mu|2] \qquad \widehat{2}^\mu = 2^\mu - \tfrac{z}{2}\langle 1|\gamma^\mu|2]$$

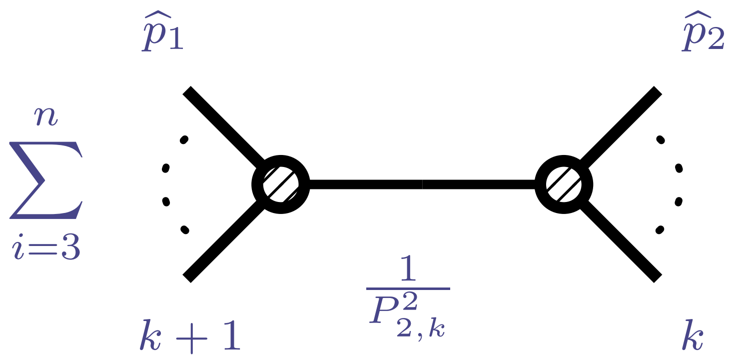
Construct the amplitude from partitions over residues in z

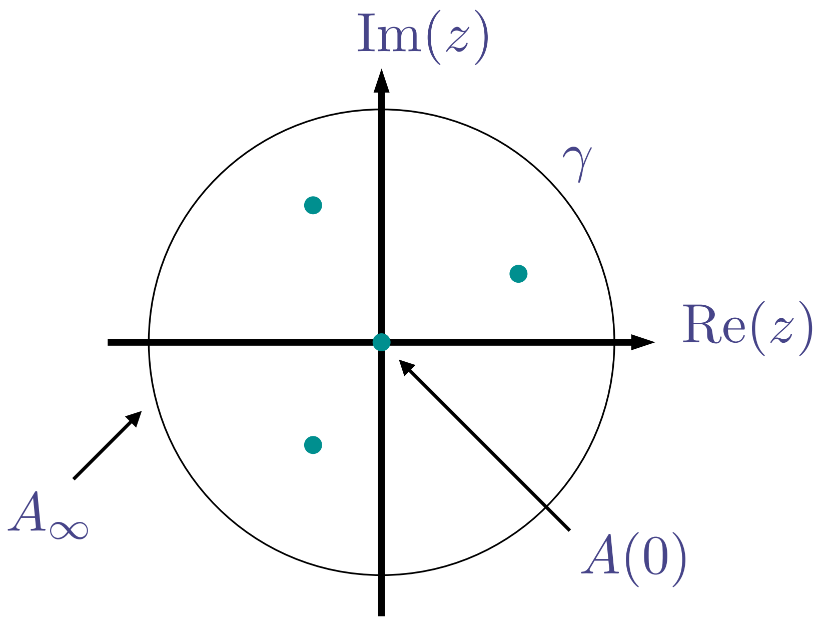
$$A_n(1, \ldots, n) = A_n(0) = \sum_{k=3}^{n-1} A_k(z_k) \frac{1}{P_k^2} A_{n-k+2}(z_k)$$

Proof of BCFW

Elegant proof from Cauchy's theorem [Britto,Cachazo,Feng,Witten]

$$0 = \oint dz \frac{A(z)}{z} = A(0) + \sum_k A_L(z_k) \frac{i}{P_k^2} A_R(z_k)$$





BCFW recursion: computing residues

- compute z_k s.t. propagator momentum goes on-shell $\widehat{P}(z_k)^2 = 0$

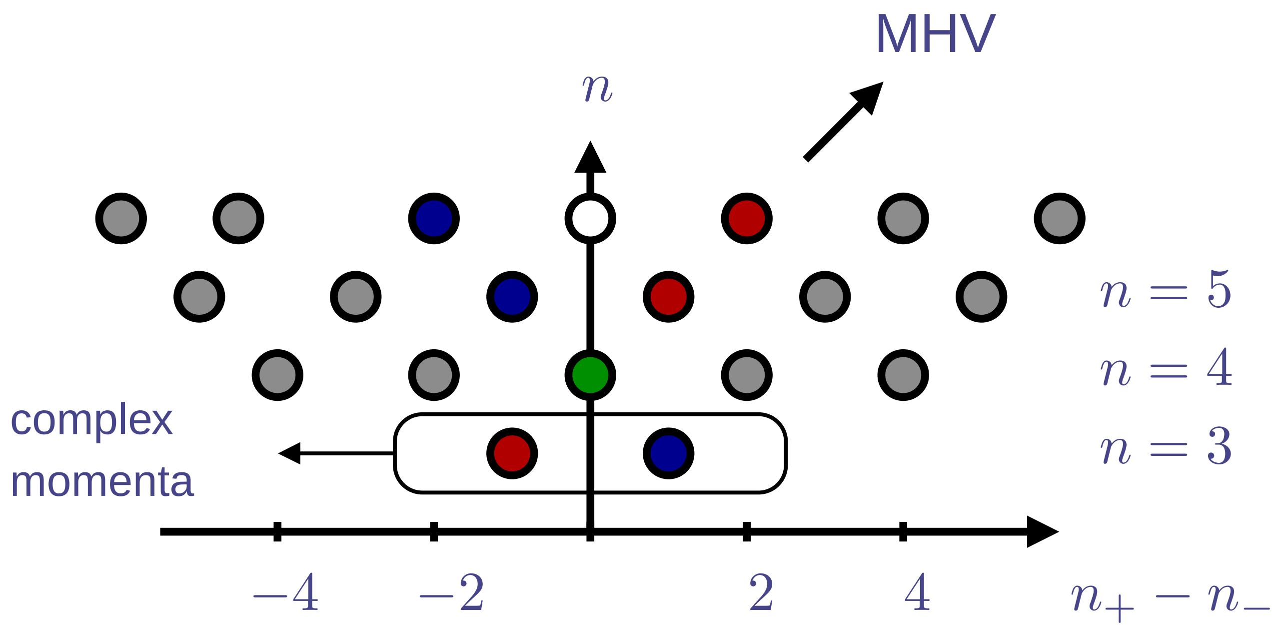
$$\widehat{P}(z_k)^\mu = P^\mu - \tfrac{z_k}{2}\langle 1|\gamma^\mu|2]$$
$$\Rightarrow z_k = \frac{P^2}{\langle 1|P|2]}$$

- Factorised amplitudes now defined on-shell, e.g.

$$A_R(-\widehat{P}(z_k), \{p_k\}, \widehat{2}(z_k))$$

- Sum over intermediate helicity states

Helicity Structures: n-point scattering



Parke-Taylor MHV formula [1986]

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 12 \rangle^3}{\prod_{\alpha=1}^{n} \langle \alpha \alpha + 1 \rangle}$$

Example 4: n-gluon MHV amplitude

BCFW allows for an extremely simple proof of the Parke-Taylor MHV formula:

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) =$$

$$\sum_{k=4}^{n} A_{n+3-k}((k+1)^+, \ldots, \widehat{2}^-, \widehat{P}^{\mp}_{3,k}) \frac{i}{P^2_{3,k}} A_{k-1}(-\widehat{P}^{\pm}_{3,k}, \widehat{3}^+, \ldots, k^+)$$

Use $\langle 2, 3]$ shift to avoid boundary contribution

- Only non-zero contribution : $k = 4$
- Spinor solution

$$|\widehat{2}\rangle = |2\rangle \qquad |\widehat{2}] = |2] + \frac{\langle 34\rangle}{\langle 24\rangle}|3]$$

$$|\widehat{3}\rangle = |3\rangle - \frac{\langle 34\rangle}{\langle 24\rangle}|2\rangle \qquad |\widehat{3}] = |3]$$

Example 4: n-gluon MHV amplitude

$$A_n(1^-,2^-,3^+,\ldots,n^+) =$$

$$A_{n-1}(1^-,\widehat{2}^-,\widehat{P}_{34}^+,5^+,\ldots,n^+)\frac{i}{P_{34}^2}A_3(-\widehat{P}_{34}^-,\widehat{3}^+,4^+)$$

- Already showed $n=4$ case, use inductive argument

$$A_n(1^-,2^-,3^+,\ldots,n^+) =$$

$$\frac{\langle 12\rangle^3}{\langle 2\widehat{P}_{34}\rangle\langle\widehat{P}_{34}5\rangle\prod_{\alpha=5}^{n}\langle\alpha\alpha+1\rangle}\frac{1}{\langle 34\rangle[43]}\frac{[34]^3}{[4(-\widehat{P}_{34})][(-\widehat{P}_{34})3]}$$

$$= i\frac{\langle 12\rangle^3}{\prod_{\alpha=2}^{n}\langle\alpha\alpha+1\rangle}$$

Further studies

- One-loop amplitudes get complicated more quickly
- On-shell and recursive techniques essential
- Modern method extremely powerful an general
- Look for future inclusion in NLO MC event generators
  - Real Radiation : Automated Dipole Subtraction
  - Amplitude generators : `BlackHat, Rocket, CutTools`
  - Monte-Carlo : `MC@NLO, POWHEG, Sherpa`