A slow review of the AGT correspondence

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Abstract

Starting with a gentle approach to the Alday–Gaiotto–Tachikawa (AGT) correspondence from its 6d origin, these notes provide a wide survey of the literature on numerous extensions of the correspondence. This is the writeup of the lectures given at the Winter School "YRISW 2020" to appear in a special issue of JPhysA.

Class S is a wide class of 4d $\mathcal{N}=2$ supersymmetric gauge theories (ranging from super-QCD to non-Lagrangian theories) obtained by twisted compactification of 6d $\mathcal{N}=(2,0)$ superconformal theories on a Riemann surface C. This 6d construction yields the Coulomb branch and Seiberg–Witten geometry of class S theories, geometrizes S-duality, and leads to the AGT correspondence, which states that many observables of class S theories are equal to 2d conformal field theory (CFT) correlators. For instance, the four-sphere partition function of 4d $\mathcal{N}=2$ SU(2) superconformal quiver theories is equal to a Liouville CFT correlator of primary operators.

Extensions of the AGT correspondence abound: asymptotically-free gauge theories and Argyres–Douglas (AD) theories correspond to irregular CFT operators, quivers with higher-rank gauge groups and non-Lagrangian tinkertoys such as T_N correspond to Toda CFT correlators, and nonlocal operators (Wilson–'t Hooft loops, surface operators, domain walls) correspond to Verlinde networks, degenerate primary operators, braiding and fusion kernels, and Riemann surfaces with boundaries.

Contents

1 Introduction and outline

Quantum field theories (QFTs) arise from many different constructions, be it Lagrangian descriptions, dimensional reduction or geometric engineering. The resulting building blocks can then be further deformed, coupled for instance by gauging symmetries, or reduced by decoupling a subsector. Theories living in different dimensions can also be fruitfully coupled together.

We explore some of these constructions in the world of 4d $\mathcal{N}=2$ supersymmetric theories, and specifically class S theories [1] obtained by suitable dimensional reduction of a 6d theory ("S" stands for "Six"). Class S includes the most commonly studied 4d $\mathcal{N}=2$ Lagrangian gauge theories (super-Yang–Mills (SYM), super-QCD (SQCD), quiver gauge theories, $\mathcal{N}=4$ SYM and its mass deformation) and non-Lagrangian ones such as AD theories [2], but also a plethora of previously unknown ones that have considerably broadened the set of known 4d $\mathcal{N}=2$ theories.

To construct a class S theory we start from a 6d $\mathcal{N}=(2,0)$ superconformal field theory (SCFT) denoted by $\mathcal{X}(\mathfrak{g})$, which is characterized by a simply-laced[1] Lie algebra $\mathfrak{g}$, for instance $\mathfrak{su}(N)$. We then reduce $\mathcal{X}(\mathfrak{g})$ on a Riemann surface C called the ultraviolet (UV) curve[2], while preserving 4d $\mathcal{N}=2$ supersymmetry thanks to a procedure called partial topological twist. The Riemann surface can have punctures (removed points, so that $C=\overline{C}\setminus\{z_1,\dots,z_n\}$ with $\overline{C}$ being compact) at which boundary conditions must be prescribed. Each choice of punctured Riemann surface, and data D_i describing the boundary condition at z_i, leads to one 4d $\mathcal{N}=2$ class S theory $\mathrm{T}(\mathfrak{g},C,D)$.

Due to their 6d origin, nonperturbative dynamics of class S theories are encoded in the geometry of C. For example the Seiberg–Witten (SW) curve [3, 4] of a theory, which determines the low-energy effective action in a given Coulomb branch vacuum, is a branched cover of C. Strikingly, this idea extends to many observables of the class S theory. The AGT correspondence [5] concerns the four-sphere (and ellipsoid) partition function:

$$Z_{S^4_b}(\mathrm{T}(\mathfrak{g},C,D)) = \langle V_{D_1}(z_1)\dots V_{D_n}(z_n)\rangle^{\mathrm{Toda}(\mathfrak{g})}_{\overline{C}} \tag{1.1}$$

where the right-hand side is a correlator of vertex operators in the Liouville CFT (for $\mathfrak{g}=\mathfrak{su}(2)$) or its generalization Toda CFT. The vertex operators are inserted at each puncture z_i and depend on the data D_i characterizing punctures.

The rest of the introduction summarizes this review, inevitably quickly: the reader should feel free to skip to the main text. Sections 2, 3, and 4 (summarized in subsection 1.1) describe the theories $\mathrm{T}(\mathfrak{g},C,D)$ and the data D_i. Sections 5 and 6 (summarized in subsection 1.2) explain how to define and compute both sides of (1.1), namely $Z_{S^4_b}$ and Liouville CFT correlators. Finally, sections 7, 8, 9, and 10 describe numerous extensions of the correspondence, with detailed pointers to the literature. In subsection 1.3 we present the preexisting reviews on topics related to AGT.

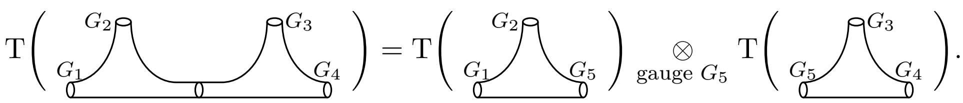
[1] A simple Lie algebra $\mathfrak{g}$ is simply-laced if all its roots have the same length. Such algebras have an ADE classification: concretely, $\mathfrak{g}$ is one of $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, or $\mathfrak{e}_8$.

[2] We often call C a curve, as it is a complex manifold of complex dimension 1. One should nevertheless remember that C is a two-dimensional Riemann surface.

1.1 Class S theories

In the main text we study the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ (section 2), its twisted dimensional reductions to class S theories (section 3), and Lagrangian descriptions of some of these 4d $\mathcal{N}=2$ theories (section 4). Here we only give some outcomes of these discussions. We often reduce to $\mathfrak{g}=\mathfrak{su}(N)$ for simplicity.

Building blocks for $\mathrm{T}(\mathfrak{g},C,D)$**.** A Riemann surface C of genus g with n punctures can be cut into $2g-2+n$ three-punctured spheres, also called trinions or pairs of pants[3] glued together by tubes that connect pairs of punctures. Such a description is often called a pants decomposition of C. Correspondingly, the general class S theory $\mathrm{T}(\mathfrak{g},C,D)$ can be decomposed into class S theories called *tinkertoys* [7] that correspond to each three-punctured sphere (tinkertoys range in complexity from free hypermultiplets to previously unknown non-Lagrangian isolated SCFTs). It turns out that each puncture is associated to a flavour symmetry, and connecting two punctures by a tube amounts to identifying the two associated flavour symmetries and gauging them using the same 4d $\mathcal{N}=2$ vector multiplet. For instance a four-punctured sphere can be split into two three-punctured spheres (for suitable groups $G_1,\dots,G_5$):



(1.2)

In simple cases where tinkertoys are collections of hypermultiplets, this results in gauge theories with an explicit Lagrangian made of hypermultiplets and vector multiplets. Thanks to the partial topological twist, the 4d theory does not depend on the metric of C [6] but only on the complex structure of C, which can be described by the "length" and "twist" of each tube. These two parameters control the complexified gauge coupling ($q=e^{2\pi i\tau}$ with $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$) that combines the Yang–Mills coupling g with the theta angle θ of the 4d vector multiplet corresponding to the tube. Weak coupling $g\to 0$ corresponds to a very long tube.

Of course, C can be decomposed in many ways into three-punctured spheres: correspondingly, $\mathrm{T}(\mathfrak{g},C,D)$ has many equivalent dual descriptions involving completely different sets of fields and gauge groups. The weak gauge coupling regime of these descriptions correspond to regimes where the complex structure on C is well-described by one pants decomposition where three-punctured spheres are joined by very long tubes. These regimes, which are different cusps of the space of complex structures on C, are continuously connected by varying the gauge couplings. In this way, gauge theories at strong coupling in one description may admit a different weakly-coupled description. This phenomenon [1] generalizes S-duality of the $\mathrm{SU}(2)$ $N_f=4$ theory and of $\mathcal{N}=4$ SYM. The 6d construction thus makes these S-dualities manifest through C.

[3] That number is zero or negative for the sphere with 0, 1 or 2 punctures and the torus with no punctures: these Riemann surfaces cannot be cut into three-punctured spheres, and the class S construction does not give a 4d theory, see [6].

In the 6d construction, the punctures at $z_i \in \overline{C}$ are codimension 2 defects that wrap the 4d spacetime on which the class S theory is defined. To preserve supersymmetry of the 4d theory the defects should be half-BPS, namely preserve half of the original supersymmetry. One must classify such defects, and then the tinkertoys corresponding to three-punctured spheres. Incidentally, the 6d theory also admits interesting half-BPS codimension 4 operators supported on 2d subspaces.

Coulomb branch and Seiberg–Witten curve. One way to get a handle on the theory $\mathrm{T}(\mathfrak{g}, C, D)$ is to describe its supersymmetric vacua, especially its Coulomb branch, and give the low-energy behaviour of the theory. This branch is spanned by Coulomb branch operators, namely local operators annihilated by all 4d antichiral supercharges.

Vacua of the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ are parametrized modulo gauge transformations by some (commuting, diagonalizable) adjoint-valued scalars Φ_I, where $I = 6, \dots, 10$ is an index for the $\mathfrak{so}(5)$ R-symmetry[4]. Alternatively they are parametrized by gauge-invariant polynomials (Casimirs), for instance $\mathrm{Tr}(\Phi_I \Phi_J)$. Coulomb branch vacua of the 4d theory are then determined by the values of these polynomials along the curve C. More precisely, tracking down how 4d $\mathcal{N} = 2$ antichiral supercharges embed into 6d $\mathcal{N} = (2,0)$, we find two restrictions: the Casimirs depend holomorphically on the coordinate $z \in C$, and among all the Φ_I only $\Phi_z \coloneqq \Phi_6 + i\Phi_7$ is non-zero. Because the partial topological twist mixes a subalgebra $\mathfrak{so}(2)$ of R-symmetry (under which Φ_z is charged) into the rotation group on C, $\Phi_z dz$ is tensorial (specifically a one-form) on C. Roughly speaking, then, the 4d Coulomb branch is parametrized by the adjoint-valued holomorphic one-form $\Phi_z dz$ on C modulo gauge transformations, and more invariantly by vacuum expectation values (VEVs) $\langle P_k(\Phi_z)\rangle dz^k$ of Casimir polynomials.

In the $\mathfrak{g} = \mathfrak{su}(N)$ case we repackage them as $\phi_k(z) = u_k(z) dz^k$, $k = 2, \dots, N$, defined in local coordinates $z \in C$ by expanding

$$\langle \det(x - \Phi_z)\rangle = x^N + \sum_{k=2}^{N} u_k(z) x^{N-k}. \tag{1.3}$$

It is then useful to consider the zeros of this determinant

$$\Sigma = \left\{ (z,x) \,\middle|\, x^N + \sum_{k=2}^{N} u_k(z) x^{N-k} = 0 \right\} \subset T^*C, \tag{1.4}$$

where z is a coordinate on C and x parametrizes the fiber of the cotangent bundle T^*C, the bundle of one-forms[5] on C. The complex curve Σ depends on the choice of vacuum (specified by $\phi_2, \dots, \phi_N$) and turns out to be the SW curve of $\mathrm{T}(\mathfrak{g}, C, D)$, presented as an N-fold (ramified) cover of C. It is equipped with a natural one-form $\lambda = x dz$, the SW differential.[6] From the SW curve and differential (Σ, λ) of $\mathrm{T}(\mathfrak{g}, C, D)$ in a given Coulomb branch vacuum one can derive the infrared effective action (the prepotential). Masses of BPS particles can also be extracted as integrals of λ along closed contours.

[4]In the M-theory construction of the 6d theory, $\mathfrak{so}(5)$ rotates coordinates $x^6, \dots, x^{10}$.

[5]This just means $x dz$ transforms as a tensor under changing the coordinate z on C.

[6]Equation (1.4) is often reformulated as $\lambda^N + \sum_{k=2}^{N} \phi_k(z) \lambda^{N-k} = 0$.

Tame punctures and tinkertoys. A puncture at $z_i \in \overline{C}$ is described in this language as a singularity of the gauge-invariants ϕ_k. An important example is the *full tame puncture* which imposes a first order pole $\Phi_z \sim m_i(z-z_i)^{-1}dz + O(1)$ at z_i, up to conjugation, where the residue $m_i \in \mathfrak{g}_\mathbb{C}$ is a suitably generic element of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. This mass[7] parameter m_i can be understood as a constant value for the background vector multiplet scalar that couples to the flavour symmetry $\mathfrak{g}$ corresponding to the puncture at z_i. In gauge-invariant terms this first order pole translates to

$$\langle P_k(\Phi_z)\rangle = \frac{P_k(m_i)}{(z-z_i)^k} + \dots, \tag{1.5}$$

or equivalently $\phi_k \propto dz^k/(z-z_i)^k + \dots$ with a leading-order coefficient determined from m_i, using (1.3) in the $\mathfrak{su}(N)$ case. It turns out that this type of singularity occurs generically when C gets pinched and split into two in the limit where a tube becomes infinitely thin.

The main building block of class S theories is thus *the tinkertoy* $T_\mathfrak{g}$, namely the theory associated to a sphere with three full tame punctures. A frequent notation is $T_N \coloneqq T_{\mathfrak{su}(N)}$. By matching SW curves and SW differentials of $\mathrm{T}(\mathfrak{su}(2), C, D)$ theories to previously known theories such as SU(2) $N_f = 4$ SQCD, one checks that T_2 is simply a collection of 4 free hypermultiplets [1]. In general, however, the theory $T_\mathfrak{g}$ is a non-Lagrangian theory, with (at least) one flavour symmetry $\mathfrak{g}$ for each puncture. For instance, $T_{\mathfrak{su}(3)}$ is the Minahan-Nemeschansky SCFT with flavour symmetry $\mathfrak{e}_6 \supset \mathfrak{su}(3)^3$.

There are more general *tame punctures*, defined as points where one imposes a first order pole of Φ_z with a residue m that may be non-generic. For $\mathfrak{su}(N)$ they are characterized by the pattern of equal eigenvalues of m, encoded as a partition of N, and they lead to lower-order poles for the ϕ_k. The partition for a full tame puncture is $N = 1+1+\dots+1$, also denoted by $[1^N]$; it carries $\mathfrak{su}(N)$ flavour symmetry (broken explicitly by the mass m). At the other extreme, the puncture corresponding to the partition $[N]$ (all eigenvalues equal, hence vanishing) is a trivial absence of puncture since it is a pole with zero residue. The next "smallest" puncture, called a *simple tame puncture* corresponds to the partition $[N-1,1]$ so $m = \mathrm{diag}(m_1,\dots,m_1,-(N-1)m_1)$; it carries $\mathfrak{u}(1)$ flavour symmetry, enhanced to $\mathfrak{su}(2)$ for $N=2$ since in that case the simple and full punctures are identical. It appears in the class S description of SU(N) $N_f = 2N$ SQCD, as depicted in Figure 1.

While the gauge algebra carried by each tube is $\mathfrak{g}$ when punctures are full tame punctures, more general tame punctures may lead to smaller gauge algebras. For example, $\mathfrak{su}(N)$ class S includes *linear quiver gauge theories* with gauge group $\prod_i \mathrm{SU}(N_i)$ (with $N_i \leq N$), one hypermultiplet in each bifundamental representation $N_i \otimes \overline{N_{i+1}}$, and $M_i \leq 2N_i - N_{i-1} - N_{i+1}$ hypermultiplets[8] in fundamental representations N_i of

[7] For $\mathfrak{su}(N)$, the N eigenvalues of m_i give residues of λ at each of the N points of Σ projecting to z_i. Integrating λ to compute masses of BPS particles picks up such residues, which are thus mass parameters.

[8] When this bound is saturated the gauge coupling of that group does not run. When $M_i < 2N_i - N_{i-1} - N_{i+1}$ we get an asymptotically free gauge theory, which can be realized in class S using wild punctures. When the bound is violated instead, the theory is only an effective theory and does not have a class S construction.

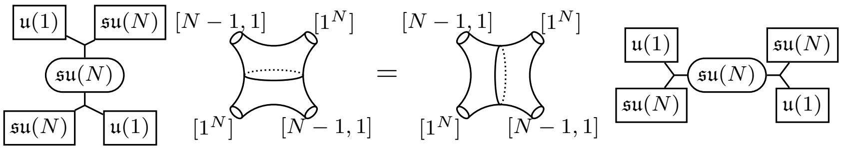


Figure 1: The $\mathfrak{su}(N)$ class S theory corresponding to a sphere with two full tame punctures (labelled $[1^N]$, flavour symmetry $\mathfrak{su}(N)$) and two simple tame punctures (labelled $[N-1,1]$, symmetry $\mathfrak{u}(1)$). We depict two pants decompositions constructed from spheres with one simple and two full punctures, whose corresponding tinkertoy is a collection of hypermultiplets. The decompositions lead to two S-dual Lagrangian descriptions of the theory as $\mathrm{SU}(N)$ SQCD with $N_f = 2N$. The third pants decomposition turns out to involve non-Lagrangian tinkertoys (for $N > 2$).

each $\mathrm{SU}(N_i)$. This is summarized in the quiver diagram

$$\begin{array}{ccccccc} \boxed{M_1} & & \boxed{M_2} & & & & \boxed{M_p} \\ | & & | & & & & | \\ \mathrm{SU}(N_1) & - & \mathrm{SU}(N_2) & - & \cdots & - & \mathrm{SU}(N_p) \end{array} \tag{1.6}$$

1.2 Basic AGT correspondence

We summarize here two sections that build up to the full AGT correspondence. First, section 5 describes how the (squashed) sphere partition function $Z_{S_b^4}$ of quiver gauge theories is computed using supersymmetric localization, and especially the issue of instanton counting. Then, section 6 explains basic aspects of Liouville CFT and gives the precise statement of the AGT correspondence for $\mathfrak{g} = \mathfrak{su}(2)$ generalized quivers.

Supersymmetric localization. In section 5 we explain how to place class S theories on the (squashed) four-sphere $S_b^4 := \{y_5^2 + b^2(y_1^2 + y_2^2) + b^{-2}(y_3^2 + y_4^2) = r^2\} \subset \mathbb{R}^5$ supersymmetrically, and how to evaluate the partition function on this ellipsoid using supersymmetric localization [8, 9]. This path integral technique applies to each 4d $\mathcal{N} = 2$ Lagrangian description of $\mathrm{T}(\mathfrak{g}, C, D)$—if such a description exists.[9] Supersymmetric localization can reduce the infinite-dimensional path integral down to a finite-dimensional integral over supersymmetric configurations of the hypermultiplets and vector multiplets. One finds configurations labeled by the constant value of a vector multiplet scalar a that can be gauge-fixed to lie in the Cartan subalgebras of the gauge algebras. These configurations are additionally dressed by point-like instantons at one pole ($y_5 = r$) and anti-instantons at the other pole ($y_5 = -r$) of S_b^4. The partition function then reads

$$Z_{S_b^4}(q, \bar{q}) = \int da\, Z_{\mathrm{cl}}(a, q, \bar{q})\, Z_{\text{one-loop}}(a) Z_{\mathrm{inst}}(a, q) Z_{\mathrm{inst}}(a, \bar{q}), \tag{1.7}$$

[9]Factorization properties of $Z_{S_b^4}$ that we find upon cutting the Riemann surface also hold for non-Lagrangian class S theories. They are obtained by applying supersymmetric localization to the vector multiplets only, and not to the tinkertoys.

where we omit the dependence on $\mathfrak{g}$ and data D at the punctures but write explicitly the dependence on complex structure parameters q of the curve C. Here, Z_{cl} comes from the classical action of supersymmetric configurations; it depends non-holomorphically on the complex gauge couplings q, but factorizes as $Z_{\text{cl}}(a,q,\bar{q}) = Z_{\text{cl}'}(a,q)Z_{\text{cl}'}(a,\bar{q})$. Quadratic fluctuations around these configurations yield $Z_{\text{one-loop}}(a)$, a straightforward product of special functions that is completely independent of the shape of C. Finally, (anti-)instantons at each pole bring a factor of Z_{inst} that depends (anti-)holomorphically on gauge couplings q. This factor $Z_{\text{inst}}(a,q) = \sum_{k\geq 0} q^k Z_{\text{inst},k}(a) = 1 + O(q^1)$ is Nekrasov's instanton partition function with parameters $\epsilon_1 = b/r$, $\epsilon_2 = 1/(rb)$ [10, 11], computable in favourable cases.

The main difficulty is to compute each k-instanton contribution $Z_{\text{inst},k}(a)$, which is an integral over the k-instanton moduli space. This space is finite-dimensional but very singular, and its singularities are understood best for unitary gauge groups. For linear quivers of unitary groups, which are obtained from (1.6) by replacing all $\mathrm{SU}(N_i)$ gauge groups by $\mathrm{U}(N_i)$, the Nekrasov partition function can be determined by equivariant localization or through IIA brane constructions. The instanton partition function of the SU theories (1.6) that we care about can then be derived by an appropriate decoupling of the U(1) factors, which divides $Z_{\text{inst}}(a,q)$ by simple factors such as powers of $(1-q)$ [5]. Various other methods have been devised, but there is as of yet no complete first principles derivation of Z_{inst} for general class S theories, and even when restricting to $\mathfrak{g} = \mathfrak{su}(2)$ with tame punctures.[10]

S-dual Lagrangian descriptions of the same theory, obtained by different pants decompositions of C, should have the same partition function if S-duality is to hold. The equality of explicit integral expressions (1.7) is extremely challenging to prove, even for the SU(2) $N_f = 4$ theory. In fact the easiest way I know is to derive the AGT correspondence in that case (e.g., [12]) and then rely on modularity properties on the 2d CFT side [13–15].

Liouville CFT correlators and basic AGT correspondence. In section 6 we move on to the other side of the correspondence for $\mathfrak{g} = \mathfrak{su}(2)$, namely Liouville CFT correlators. Liouville CFT depends on a "background charge" $Q = b + 1/b \geq 2$ (the central charge is $c = 1 + 6Q^2 \geq 25$), which translates on the 4d side to a deformation parameter of S^4 into the ellipsoid S^4_b. As in any 2d CFT, local operators organize into conformal families constructed by acting with the Virasoro algebra on primary operators. In the Liouville CFT these primaries are the vertex operators V_α, labeled by a continuous parameter $\alpha = Q/2 + iP$ with $P \in \mathbb{R}/\mathbb{Z}_2$ (called momentum), and they have equal holomorphic and antiholomorphic dimension $h(\alpha) = \alpha(Q-\alpha) = Q^2/4 + P^2$. In the $\mathfrak{su}(2)$ case the data D_i for each tame puncture reduces to specifying a mass $m_i \in \mathbb{R}/\mathbb{Z}_2$, naturally identified with a Liouville momentum (up to the sphere's radius r): the AGT correspondence then states

$$Z_{S^4}(\mathrm{T}(\mathfrak{su}(2), C, m)) = \langle V_{Q/2+irm_1}(z_1) \dots V_{Q/2+irm_n}(z_n)\rangle_C^{\text{Liouville}}. \tag{1.8}$$

[10] I thank Jaewon Song for clarifications on this point.

As in any 2d CFT, n-point functions of Virasoro primary operators on the Riemann surface $\overline{C}$ have a useful expression for each pants decomposition of the punctured Riemann surface C. The idea is to insert a complete set of states along each cut in the decomposition, then use Virasoro symmetry to rewrite all resulting three-point functions in terms of those of primaries. Schematically this gives

$$\langle V_{\mu_1}(z_1,\bar{z}_1)\dots V_{\mu_n}(z_n,\bar{z}_n)\rangle_{\overline{C}}^{\text{Liouville}} = \int d\alpha\, C(\mu,\alpha)\mathcal{F}(\mu,\alpha,q)\mathcal{F}(\mu,\alpha,\bar{q}). \tag{1.9}$$

Here we integrate over all internal momenta α labelling the conformal family in each inserted complete set of states. The factor $C(\mu,\alpha)$ is a combination of structure constants of Liouville CFT. The other two factors are conformal blocks, which are purely about representation theory of the Virasoro algebra, and which depend (anti-)holomorphically on the complex structure parameters q of C, including (cross-ratios of) z_i.

Both sides of the AGT correspondence admit the same kind of expressions (1.7) and (1.9) for each pants decomposition of C, with one integration variable a or α for each tube, and a factorization of the dependence on q into holomorphic and antiholomorphic. In fact these expressions match factor by factor: $Z_{\text{one-loop}}(m,a) = C(\mu,\alpha)$ and $Z_{\text{cl}'}(a,q)Z_{\text{inst}}(m,a,q) = \mathcal{F}(\mu,\alpha,q)$. An additional entry in the dictionary is that ϕ_2 on the 4d side corresponds to the holomorphic stress-tensor $T(z)$ on the Liouville side in the classical limit $r\to\infty$: the leading term in an operator product expansion (OPE) with $T(z)$ matches $r^2\phi_2(z)$,

$$T(z)V_\mu(0) = \frac{h(\mu)V_\mu(0)}{z^2} + \cdots \underset{r\to\infty}{\simeq} \frac{r^2m^2}{z^2}V_\mu(0) + \cdots \simeq r^2\phi_2(z)V_\mu(0) + \dots. \tag{1.10}$$

We end section 6 by outlining a rather technical derivation of how Liouville CFT appears upon reducing the 6d theory on S^4 [16].

Extensions of the AGT correspondence. The AGT correspondence is generalized in two ways in section 7. First, AD theories are described by replacing tame punctures by wild punctures, which replaces primary vertex operators by irregular ones on the CFT side. Second, $\mathfrak{su}(2)$ is replaced by $\mathfrak{g} = \mathfrak{su}(N)$: hypermultiplets are then replaced by non-Lagrangian building blocks T_N.

In section 8 we investigate how to include in the AGT correspondence various gauge theory operators (local operators, Wilson–'t Hooft loops, ...). On the CFT side we encounter Verlinde loops, degenerate vertex operators, fusion and braiding kernels, and Riemann surfaces with boundaries. The dictionary is summarized in Table 1.

We discuss some offshoots of the AGT correspondence in section 9. Some class S theories, especially linear quiver gauge theories, can be realized as reductions of 5d $\mathcal{N}=1$ theories. Instanton partition functions have direct analogues in 5d as certain q-deformations of the 4d results. This leads to a q-deformed AGT correspondence [101] equating these 5d instanton partition functions to chiral correlators ("conformal blocks") of q-deformed Virasoro or W_N algebras. The S^5 partition function involves *three* instanton partition functions, and its proper translation to non-chiral correlators of a complete q-Toda theory is still under investigation [102].

Table 1: AGT correspondence for extended operators, organized according to the codimension of the 6d operator or orbifold that yields them, and further sorted by increasing dimension on the 4d side. Most entries are hyperlinked to the main text.

		Operator in class S theory	Liouville/Toda CFT	References
Codimension 4	0d	Coulomb branch operator	Integrated current	[17]
Codimension 4	0d	Orbifold $\mathbb{C}^2/\mathbb{Z}_M$	Change CFT to coset	[18–35]
Codimension 4	1d	Dyonic loop: Wilson loop/'t Hooft loop	Degenerate Verlinde loop: around a tube/transverse	$\mathfrak{su}(2)$ [36–41], $\mathfrak{g}$ [42–49]
Codimension 4	2d	Vortex string operator	Degenerate vertex operator	[37, 50–64]
Codimension 2	2d	Gukov–Witten surface defect or orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$	Change CFT by Drinfeld–Sokolov reduction	[65–78]
Codimension 2	3d	Symmetry-breaking wall	Verlinde loop	[41]
Codimension 2	3d	S-duality domain wall	Modular kernel	[79–83]
Codimension 2	3d	Boundary	Boundary CFT	[84, 85]
Codimension 2	4d	Coupling to a tinkertoy	Vertex operator	[5, 7, 86–100]

Placing the 6d theory onto other product spaces $M \times C$ (with some twist) leads to interesting relations between theories on M and on C: the index/qYang–Mills (YM) correspondence [103], the 3d/3d correspondence [104], the 2d/4d correspondence [105].

We end in section 10 with a quick outline of many topics omitted in this review, such as matrix models, topological strings, quantum integrable systems, etc.

1.3 Earlier reviews

There have been many good reviews related to the AGT correspondence, including in several PhD theses. I particularly recommend Tachikawa's very clear collection of reviews [106–109].

- **6d $(2,0)$ SCFTs.** These theories, and more generally 6d $(1,0)$ SCFT, are reviewed in [110] from an F-theory perspective. For codimension 2 defects, which are central in the AGT correspondence, see [111].

- **4d $\mathcal{N}=2$ and Seiberg–Witten.** While there are nice introductions from the late 1990's [112, 113] to the SW solution of 4d $\mathcal{N}=2$ theories, I recommend more modern explanations such as Martone's notes in this school [114], and the well-known review "for pedestrians" [106] which covers a lot of ground, including how AD theories arise from limits of SQCD. The book [115] discusses many modern relations between 4d $\mathcal{N}=2$ theories and other topics. The review [108] is focussed on the very important non-Lagrangian 4d $\mathcal{N}=2$ theory T_N.

- **Localization and instanton counting.** Supersymmetric localization is reviewed in the book [116], and in particular the squashed four-sphere partition function in [117]. Its expression involves Nekrasov's instanton partition function, for which a good starting point is [107], followed by [118] which discusses all gauge groups, subtleties regarding the U(1) factor, and the choice of renormalization scheme.
- **Toda CFT and W-algebras.** Liouville CFT is reviewed in [119, 120] among many others, and it is worth reading [121] for some subtleties. There are no recent reviews on Toda CFT or on W-algebras. For W-algebras see the old [122, 123] (and possibly [124, 125]) or the discussion of truncations of $W_{1+\infty}$ in [126]. For Toda CFT perhaps the early article [127] or my thesis [128][11].
- **AGT for physicists.** See [109] (or perhaps [129], in Japanese) for a brief review, and the longer [130] ranging from SW basics to AD theories arising from degenerations of SQCD. The matrix model approach to AGT is reviewed in [131, 132].
- **AGT for mathematicians.** Possible starting points for mathematicians include the introductory seminar notes [133], a "pseudo-mathematical pseudo-review" [134, 135], incomplete (nevertheless 200 pages long) lecture notes [136], a review that focuses on moduli spaces of flat connections [137] and one discussing instanton counting on asymptotically locally Euclidean (ALE) spaces [138]. There are also notes on mathematical applications of the 6d $(2,0)$ SCFT to geometric representation theory, symplectic duality, knot homology, and Hitchin systems [139].
- **Generalizations of AGT.** These include the 3d/3d correspondence reviewed in [140] and the AGT relation between 5d $\mathcal{N} = 1$ gauge theories and q-Toda correlators in [141].

Given these numerous reviews, writing yet another set of notes is perhaps futile, but hopefully the rather different approach taken here, starting from the 6d theory, is the right one for some readers. I apologize for omitting many directions from this review, listed in the conclusion section 10, especially the deep links to topological strings, matrix models, quantization of integrable models underlying SW geometry, a broader discussion of the BPS/CFT correspondence, etc. Some of these may be covered in a later version, upon request.

2 6d $(2,0)$ SCFT of ADE type

Superconformal algebras exist in dimensions up to 6, and there is by now ample evidence for the existence of 6d $\mathcal{N} = (2,0)$ (maximally supersymmetric) SCFTs $\mathcal{X}(\mathfrak{g})$, labelled

[11] Better reference very welcome: only the A_{N-1} case is considered there, and only full, simple, and degenerate punctures rather than general tame punctured labeled by partitions of N.

by a Lie algebra $\mathfrak{g}$ that is simply-laced[12,13]. Nobody knows how to actually define $\mathcal{X}(\mathfrak{g})$ directly in a QFT language, for instance through a Lagrangian formulation. It is instead obtained as a decoupling limit of certain string theory or M-theory brane setups. Despite its stringy construction, the theory is expected to be a bona-fide local QFT, for instance having a local conserved stress-tensor. These constructions entail three important properties which we explain below:

- $\mathcal{X}(\mathfrak{g})$ has vacua on which the infra-red (IR) description is an abelian 6d $(2,0)$ theory valued in the Cartan algebra of $\mathfrak{g}$ modulo the Weyl group;
- $\mathcal{X}(\mathfrak{g})$ is a UV-completion of 5d $\mathcal{N}=2$ SYM in the sense that SYM with gauge algebra $\mathfrak{g}$ and gauge coupling $g_{5\text{d}}$ gives an IR description of $\mathcal{X}(\mathfrak{g})$ compactified on a circle of radius $g_{5\text{d}}^2$;
- $\mathcal{X}(\mathfrak{g})$ admits half-BPS defects supported on codimension 2 subspaces and others on codimension 4 subspaces.

The first two properties are compatible because both 5d $\mathcal{N}=2$ SYM on its Coulomb branch, and the abelian 6d theory on a circle, are described by 5d abelian vector multiplets in the Cartan of $\mathfrak{g}$. The last property is compatible as well, as the defects have rather explicit descriptions when one moves along the Coulomb branch or when one places the theory on a circle. The existence of $\mathcal{X}(\mathfrak{g})$ with these properties is confirmed by many consistency checks involving better-understood theories. A major set of consistency checks is the AGT correspondence obtained by placing these theories on the product $M_4 \times C_2$ of a 4d and a 2d manifolds.

In this section we describe the symmetry algebra $\mathfrak{osp}(8^*|4)$ (subsection 2.1), properties of self-dual two-form gauge fields (subsection 2.2), string/M-theory constructions (subsection 2.3) and extended operators (subsection 2.4) of $\mathcal{X}(\mathfrak{g})$.

2.1 Superconformal algebras

Superconformal algebras in dimensions $d > 2$ have been classified by Nahm [143] under certain conditions. Their even (bosonic) part consists of the conformal algebra $\mathfrak{so}(2,d)$ (in Lorentzian signature) and an R-symmetry algebra, and their odd (fermionic) part consists of supercharges that must transform in the spinor representation of $\mathfrak{so}(2,d)$, and such that translations are realized as anticommutators of supercharges.

The classification is in Table 2. In dimensions $d = 3,4,6$ the conformal algebra coincides with the expected $\mathfrak{so}(2,d)$ thanks to accidental isomorphisms[14] $\mathfrak{so}(2,3) = \mathfrak{sp}(4,\mathbb{R})$

[12] As a reminder, simply-laced Lie algebras are $\mathfrak{a}_{N-1} = \mathfrak{su}(N)$, $\mathfrak{so}(2N) = \mathfrak{d}_N$, and the three exceptional algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ (in each case the subscript is the rank). This ADE classification has several beautiful avatars in theoretical physics but we will not get to explore them in this review.

[13] While different constructions of $\mathcal{X}(\mathfrak{g})$ give the same condition that $\mathfrak{g}$ is simply-laced, including some field theoretic arguments [142], it has not been proven that $\mathcal{X}(\mathfrak{g})$ exhaust all 6d $\mathcal{N}=(2,0)$ SCFTs. The situation is the same in 4d $\mathcal{N}=4$ SCFT: there might possibly be such theories other than $\mathcal{N}=4$ SYM theories.

[14] There are no such accidental isomorphisms for $d > 6$, which more or less explains the lack of higher-dimensional superconformal algebras.

Table 2: Nahm classification of superconformal algebras in Lorentzian signature. Here we list the superconformal algebras in each dimension, the two bosonic factors (conformal algebra and R-symmetry algebra), and the representations (of these bosonic factors) in which Poincaré and conformal supercharges Q and S transform.

	Superalgebra	Conformal	R-symmetry	$Q\&S$
3d $\mathcal{N} \leq 8$	$\mathfrak{osp}(\mathcal{N}\|4)$	$\mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{so}(\mathcal{N})$	$(4,\mathcal{N})$
4d $\mathcal{N} \leq 3$	$\mathfrak{su}(2,2\|\mathcal{N})$	$\mathfrak{su}(2,2)$	$\mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1)$	$(4,\overline{\mathcal{N}}) \oplus (\overline{4},\mathcal{N})$
4d $\mathcal{N} = 4$	$\mathfrak{psu}(2,2\|4)$	$\mathfrak{su}(2,2)$	$\mathfrak{su}(4)$	$(4,\overline{4}) \oplus (\overline{4},4)$
5d $\mathcal{N} = 1$	$\mathfrak{f}^2(4)$	$\mathfrak{so}(2,5)$	$\mathfrak{su}(2)$	$(8,2)$
6d $\mathcal{N} \leq 2$	$\mathfrak{osp}(8^*\|2\mathcal{N})$	$\mathfrak{so}^*(8)$	$\mathfrak{usp}(2\mathcal{N})$	$(8,2\mathcal{N})$

and $\mathfrak{so}(2,4) = \mathfrak{su}(2,2)$ and $\mathfrak{so}(2,6) = \mathfrak{so}^*(8)$. In each case, the spinor representation of $\mathfrak{so}(2,d)$ is the fundamental (vector) representation of the other group. It is known that SCFTs with more than 16 Poincaré supercharges do not exist for $d \geq 4$ (and are free for $d = 3$) [144], and this leads to the bounds on $\mathcal{N}$ given in the table.

For the 6d case of interest to us, minimal spinor representation of the Lorentz algebra $\mathfrak{so}(2,6)$ are chiral, and the superconformal algebras contain $\mathcal{N} = 1$ or 2 such chiral spinors (technically, symplectic Majorana–Weyl spinors) with the same chirality. These algebras are thus called 6d $\mathcal{N} = (1,0)$ and 6d $\mathcal{N} = (2,0)$ superconformal algebras. There is no 6d $\mathcal{N} = (1,1)$ superconformal algebra. We are interested in the largest superconformal algebra of all: the 6d $(2,0)$ algebra $\mathfrak{osp}(8^*|4)$.

Supercharges of this algebra transform in the $(\mathbf{8}_s, \mathbf{4})$ representation[15] of the conformal and R-symmetry algebras $\mathfrak{so}(6,2) \times \mathfrak{so}(5)_{\mathrm{R}}$, with a reality condition. Decomposing this into representations of the Lorentz algebra $\mathfrak{so}(1,5)$ gives $(\mathbf{4},\mathbf{4}) \oplus (\mathbf{4},\mathbf{4})$, with a symplectic reality condition. One set $(\mathbf{4},\mathbf{4})$ consists of Poincaré supercharges and the other of superconformal transformations.

2.2 Self-dual forms

The 6d $\mathcal{N} = (2,0)$ SCFT $\mathcal{X}(\mathfrak{g})$ is roughly speaking a theory of self-dual two-forms gauge fields for a gauge Lie algebra $\mathfrak{g}$ among $\mathfrak{a}_{N-1}, \mathfrak{d}_N, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, as we explain next.

Abelian self-dual forms. In even dimension d there exists an interesting notion of (anti)[16] self-dual k-form for $k = d/2 - 1$: a k-form B with components $B_{\alpha_1\dots\alpha_k}$ (antisymmetric in $\alpha_1,\dots,\alpha_k$) such that the field strength $H = dB$ is mapped to a

[15] We denote irreps (irreducible representations) of a simple Lie algebra by their dimension in bold face. When ambiguities arise there are standard decorations to distinguish them, such as overlines for conjugating the representation, or primes when there are several irreps of the same dimension and they are not related by conjugation. A peculiar example is $\mathfrak{so}(8)$ and other real forms thereof like $\mathfrak{so}(p,8-p)$ as they have three dimension 8 irreps: the defining representation of $\mathfrak{so}(8)$ called 8_v, and two conjugate spinor representations 8_s and 8_c.

[16] Self-dual and anti-self-dual cases differ by a sign, and we shall just write "self-dual" for simplicity.

multiple of itself by the Hodge star, that is,

$$H_{\alpha_0\alpha_1\dots\alpha_k} := (k+1)!\,\partial_{[\alpha_0} B_{\alpha_1\dots\alpha_k]} = \pm i^{d/2+s} \epsilon_{\alpha_0\dots\alpha_k\beta_0\dots\beta_k} \partial^{[\beta_0} B^{\beta_1\dots\beta_k]}. \tag{2.1}$$

Here indices within square brackets are antisymmetrized and the power of $i = \sqrt{-1}$ involves $s = 0$ for Euclidean and $s = 1$ for Lorentzian signature. The self-duality condition regards the field strength hence is invariant under gauge transformations $B \to B + d\Lambda$ for any k-form Λ: explicitly this adds $k!\,\partial_{[\alpha_1}\Lambda_{\alpha_2\dots\alpha_k]}$ to the component $B_{\alpha_1\dots\alpha_k}$ of the k-form gauge field B.

From (2.1) we see that real self-dual k-forms exist only if $d/2 + s$ is even. In 2d this happens in Lorentzian signature, and it corresponds to a real scalar field propagating only in one lightlike direction. (In the Euclidean case it is a complex chiral boson depending on one holomorphic coordinate.) In 4d with Euclidean signature, (2.1) defines self-dual gauge field configurations, also called instantons, which play a crucial role on the 4d side of the AGT correspondence. (In the Lorentzian case they are complex saddle-points.) In 6d with Lorentzian signature we get a real self-dual two-form gauge field $B_{\alpha\beta}$.

We care about 6d $(2,0)$ supersymmetry, in which case the multiplet containing $B_{\alpha\beta}$ consists of B, spinors λ, and scalars Φ that transform respectively as the singlet, the 4-dimensional, and the 5-dimensional representations of R-symmetry $\mathfrak{usp}(4) = \mathfrak{so}(5)$.

Compactifying on a circle. Let us place this 6d $(2,0)$ abelian theory of (B, λ, Φ) on a circle and decompose into Kaluza–Klein (KK) modes. As determined in the following exercise, the five scalars Φ_I remain scalars, the spinors λ as well, and the self-dual two-form gauge field B becomes a usual gauge field A in 5d. Altogether this gives abelian 5d $\mathcal{N} = 2$ SYM.

We review dimensional reduction in Exercise 2.1 below. An important aspect for the reduction from $\mathcal{X}(\mathfrak{g})$ to 5d is that 5d SYM has instanton particles, namely gauge field configurations with non-trivial topological number $\int \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} d^4x$ on each spatial slice. These excitations of the gauge field A play the role of the tower of KK modes: their mass (proportional to) $1/g_{5\mathrm{d}}^2$ is correctly identified with the mass $1/R$ of KK modes.

Exercise 2.1. *1. Consider a D-dimensional scalar field φ, with Lagrangian $\mathcal{L}(\varphi) = \partial_\alpha\varphi\partial^\alpha\varphi - V(\varphi)$ (you can take $V = 0$ for simplicity). Consider it on a d-dimensional Minkowski space times a $(D-d)$-dimensional torus of radius R (you can take $D - d = 1$ for simplicity). Write a Fourier decomposition of φ along the circle direction and rewrite the action of φ as an action for these components. In the limit $R \to 0$ notice that all Fourier modes become infinitely massive except the zero mode.*

2. Repeat the exercise for an abelian vector field A_α ($\alpha = 0, \dots, D-1$) with Lagrangian $F_{\alpha\beta}F^{\alpha\beta}$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Check that the dimensionally-reduced theory has both a vector field A_μ ($\mu = 0, \dots, d-1$) and $D - d$ scalar fields. These can be gauge-invariantly understood for *finite R as Wilson loops of A_α around coordinate circles of the torus. How do D-dimensional gauge transformations act?*

3. Repeat the exercise for a two-form $B_{\alpha\beta}$ reduced from 6d to 5d. This results in a two-form $B_{\mu\nu}$ and a one-form A_μ. By imposing the self-duality condition on $B_{\alpha\beta}$ find that $B_{\mu\nu}$ can be reconstructed (up to gauge transformations) from A_μ.

Nonabelian theory. Recall the Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0$ in 4d. It generalizes to $dH = ddB = 0$. For a self-dual form this implies the free equations of motion $d \star dB = 0$, namely $\partial^\mu H_{\mu\nu\dots} = 0$. How can we add interactions? In 4d, the equation $F_{\mu\nu} = \mp\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ defining instantons makes sense even for the field strength of non-abelian gauge fields, $F = dA + A \wedge A$, explicitly $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The non-abelian version of the Bianchi identity is $\epsilon^{\lambda\mu\nu\rho}D_\mu F_{\nu\rho} = 0$. When the gauge field configuration is self-dual this implies the standard Yang–Mills equations of motion $D_\mu F^{\mu\lambda} = 0$. In contrast, for other values $k \neq 1$ there is no obvious non-abelian generalization of the relation $H = dB$, hence no obvious way to introduce interactions. Instead, we use two stringy constructions.

2.3 Brane construction of 6d theories

Two string theory constructions teach us about the description of $\mathcal{X}(\mathfrak{g})$ upon moving on the tensor branch[17], and about how $\mathcal{X}(\mathfrak{g})$ on a circle is equivalent to 5d $\mathcal{N} = 2$ SYM.

M-theory fivebranes. The first construction applies to the A-type case: $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of a stack of N coincident M5 branes in M-theory, with the decoupled center of mass degrees of freedom removed.

M-theory is a 11 dimensional theory (Lorentzian signature) with 32 supersymmetries (one Majorana spinor). It is related by various dualities to better-known string theories and supergravity. For our purposes, the most interesting aspect is that M-theory on a circle times a 10-dimensional spacetime is equivalent to IIA strings on that spacetime. The aim of this review is not to discuss the intricate web of dualities relating M-theory to IIA and other string theories, so we are quite schematic.

A standard comment on terminology: p branes are $(p+1)$ dimensional objects, with p space and 1 time directions, so for instance the M5 brane is 6-dimensional and has Lorentzian signature, as we wanted. M-theory has two such half-BPS objects: the M5 brane and the M2 brane. Stacks of flat[18] parallel branes of the same type preserve

[17] The 6d $\mathcal{N} = (2,0)$ tensor multiplet splits into a 6d $\mathcal{N} = (1,0)$ tensor multiplet and a hypermultiplet. The tensor branch and Higgs branch are vacua where scalar fields in tensor or hyper multiplets acquire a VEV (with $(2,0)$ supersymmetry the two branches combine). The tensor branch is sometimes called Coulomb branch because it reduces to Coulomb branches in 5d and 4d. In 6d $\mathcal{N} = (1,0)$ theories one also has vector multiplets but they contain no scalars so there is no corresponding branch.

[18] Here we work in flat space; the backreaction of branes on the geometry does not invalidate our statements.

the same half of supersymmetry (see Exercise 2.2), and there is no energy cost to moving the branes while keeping them flat and parallel. While the world-volume theory of a stack of N Dp branes has been known for a long time to be maximally supersymmetric SYM in $p+1$ dimensions (see the review [145]), the world-volume theory of stacks of branes in M-theory has proven more difficult to pin down.

- The world-volume theory of a stack of coincident M2 branes is now known[19] to be the Aharony–Bergman–Jafferis–Maldacena (ABJM) Chern–Simons matter theory, an SCFT with an explicit 3d $\mathcal{N}=2$ Lagrangian description, whose supersymmetry enhances to the expected 3d $\mathcal{N}=8$ superconformal algebra $\mathfrak{osp}(8|4)$ preserved by the branes (see the review [146]). The R-symmetry $\mathfrak{so}(8)$ rotates the 11-dimensional space around the M2 branes. Its holographic dual is $\mathrm{AdS}_4 \times S^7$. That is all we will say in this review.

- The world-volume theory of a stack of N coincident M5 branes is what we call $\mathcal{X}(\mathfrak{su}(N))$, a 6d $(2,0)$ SCFT with no Lagrangian description.[20] More precisely, this would give $\mathfrak{u}(N)$, but the $\mathfrak{u}(1)$ center of mass of the branes decouples. The R-symmetry $\mathfrak{so}(5)$ rotates space around the M5 branes. The holographic dual is $\mathrm{AdS}_7 \times S^4$, which has the expected symmetry algebra $\mathfrak{osp}(8^*|4)$, differing only from the 3d case by some signs in the 7d and 4d parts.

Consider now $\mathcal{X}(\mathfrak{su}(N))$ on a circle (times five-dimensional Minkowski space). M-theory on a circle is equivalent to IIA string theory, and M5 branes wrapping the circle become D4 branes. Thus, $\mathcal{X}(\mathfrak{su}(N))$ on a circle is equivalent to the world-volume theory of N D4 branes, which is 5d $\mathcal{N}=2$ SYM, as announced at the start of this section 2.

We move on to describing the vacua of $\mathcal{X}(\mathfrak{g})$ from its M-theory construction. Supersymmetric vacua are parametrized by the positions of the N M5 branes in the 5 transverse directions, modulo relabelling of the branes since they are indistinguishable. The vacua are thus $(\mathbb{R}^N)^5/S_N$. At any generic vacuum, all degrees of freedom are massive (with mass proportional to the separation between the branes), except fluctuations around each individual brane, which are known to be described by one 6d abelian theory of (B,λ,Φ) for each brane. The scalar fields Φ_I, $I=6,\dots,10$, describe fluctuations of each of the N M5 branes in the transverse directions.

IIB strings. The M-theory construction gives a lot of insight on $\mathcal{X}(\mathfrak{g})$ for A-type $\mathfrak{g}$, and can be extended to D-type by orbifolding, but it cannot realize the exceptional cases $\mathfrak{e}_6,\mathfrak{e}_7,\mathfrak{e}_8$. For this a dual IIB description is needed.

[19] Depending on one's point of view, most words "known" in this review should be replaced by "conjectured". Ultimately, since the path integral has not been properly defined in most cases of interest to physicists, almost all non-perturbative QFT results are conjectural. One can think about how much "evidence" there is for one result or another. Results that are consistent with many others should then serve as a guide to determine if a given mathematical definition of the theories is acceptable.

[20] Instead of $\mathfrak{g}=\mathfrak{a}_{N-1}=\mathfrak{su}(N)$ one can realize $\mathfrak{g}=\mathfrak{d}_N=\mathfrak{so}(2N)$ by including an O5 orbifold plane on top of the M5 branes.

The second construction, which we will not use much, is to place IIB string theory on Minkowski space $\mathbb{R}^{1,5}$ times a quotient $\mathbb{C}^2/\Gamma$ by a finite subgroup $\Gamma \subset \mathrm{SU}(2)$, and "dimensionally reducing" $\mathbb{C}^2/\Gamma$ (more precisely, one uses a geometric setup where that part is compact to make dimensional reduction meaningful). Such subgroups are classified by ADE Lie algebras $\mathfrak{g}$. For instance, the A_{N-1} case is $\Gamma = \mathbb{Z}_N$ acting as $(z, w) \mapsto (e^{2\pi i/N} z, e^{-2\pi i/N} w)$ on coordinates of $\mathbb{C}^2$.

Moving along the vacuum moduli space of the 6d theory corresponds to blowing up $\mathbb{C}^2/\Gamma$ into ALE space, namely resolving the singularity at the origin of $\mathbb{C}^2/\Gamma$ into a collection of $r = \operatorname{rank}\mathfrak{g}$ finite-size two-cycles. The sizes of these two-cycles become r scalar fields Φ of the 6d theory on $\mathbb{R}^{1,5}$, in the Cartan subalgebra of $\mathfrak{g}$. In fact their VEV parametrizes the vacua of $\mathcal{X}(\mathfrak{g})$. In any vacuum, the IR degrees of freedom are: these scalar fields Φ, the two-form B obtained by integrating the chiral four-form of IIB string theory around each of the two-cycles, and some spinors. We end up as wanted with the 6d $(2,0)$ theory of an abelian self-dual two-form gauge field multiplet (B, λ, Φ) in the Cartan subalgebra of $\mathfrak{g}$.

In this description only $SO(4)$ R-symmetry is manifest, and the reduction to 5d $\mathcal{N} = 2$ SYM is also nontrivial to see.

2.4 Codimension 2 and 4 defects

We return to the M-theory construction of $\mathcal{X}(\mathfrak{su}(N))$ and consider intersecting brane configurations with branes placed along the following directions inside $\mathbb{R}^{1,10}$.

M5	0	1	2	3	4	5	.	.	.	.	.	$\to$ 6d $(2,0)$ theory
M5'	0	1	2	3	.	.	6	7	.	.	.	$\to$ codimension 2 defect
M2	0	1	.	.	.	.	.	.	.	.	10	$\to$ codimension 4 defect

Each column is a direction in $\mathbb{R}^{1,10}$; a dot indicate that a stack of branes is localized at a given value of that coordinate, and a number indicates that the brane extends along the corresponding direction. For instance the M5 branes are at given values of $x^6, x^7, x^8, x^9, x^{10}$ and extend in all other coordinates. The prime on M5' branes just helps us distinguish them from the M5 branes on which $\mathcal{X}(\mathfrak{g})$ lives. Each additional stack of branes in this table breaks half of supersymmetry (see Exercise 2.2). There can be several stacks of the same kind of branes parallel to each other, in which case they don't break further supersymmetry.

The M2 branes extend only in one direction transverse to the M5 branes. In this direction x^{10} they can either be infinite, or semi-infinite ending on one M5 brane, or finite stretching between two M5 branes. Either way, from the point of view of $\mathcal{X}(\mathfrak{g})$, stacks of M2 branes insert a half-BPS codimension 4 operator, namely an operator supported on a two-dimensional slice of the 6d theory.

The way it is written here, it would seem the M5 and M5' branes intersect in codimension 2. In truth they turn out to merge into a smooth complex manifold that asymptotes at large distances to the configuration we wrote. For this to happen, the x^6, x^7 positions of the M5 branes should grow to infinity as x^4, x^5 get closer to the positions of

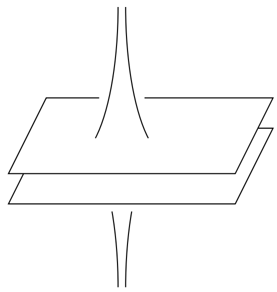


Figure 2: Configuration of a pair of M5 branes spanning the x^4, x^5 directions (depicted horizontally) in the presence of an M5' brane at a point in the x^4, x^5 plane. The M5 and M5' branes merge into a complex manifold. The x^6, x^7 positions (depicted vertically) diverge at one point in the x^4, x^5 plane. We depicted the situation after decoupling the center of mass modes, which is why the branes diverge symmetrically.

M5' branes, as depicted in Figure 2. We discuss this later in more concrete situations. From the point of view of $\mathcal{X}(\mathfrak{g})$, at large distance, the intersection with M5' branes has an effective description as a four-dimensional (codimension 2) half-BPS operator.

As we explore the AGT correspondence in this review we learn various properties of these defects, and especially the data that describes them. We find that:

- Codimension 2 operators are labeled by partitions of N specifying the way in which the N M5 branes cluster into different groups as they go to infinity in the x^6, x^7 directions, and additional continuous data describing scales in these directions.
- Codimension 4 operators are labeled by representations of $\mathrm{SU}(N)$. We recall that to each representation is associated a Young diagram, such that $\square$ is the fundamental N-dimensional representation, $\square\square$ is the symmetric representation, etc. The total number of boxes is the number of M2 branes necessary to describe the operator in M-theory. Roughly speaking, the number of boxes in each row of the Young diagram indicates how many M2 branes can end on the same M5 brane.

Exercise 2.2. *A flat M5 brane along directions $x^0, x^1, \ldots, x^5$ preserves supersymmetries with $\Gamma^{012345}\epsilon = \epsilon$ while a flat M2 brane along directions x^0, x^1, x^{10} preserves supersymmetries with $\Gamma^{23456789}\epsilon = \epsilon$. Check that the brane configurations above are such that each additional stack of branes breaks half of supersymmetry. (Hint: check that Γ^{01}, Γ^{23}, Γ^{45} etc. commute with each other.) What other relative orientations of the stacks of branes preserve half of the supersymmetry?*

3 Class S theories from 6d

Our next task is to dimensionally reduce the 6d theory $\mathcal{X}(\mathfrak{g})$ on a Riemann surface C_2. We explain in subsection 3.1 a partial topological twist such that the reduced theory has 4d $\mathcal{N}=2$ supersymmetry. The Coulomb branch and SW curves giving the IR physics are worked out in subsection 3.2 and subsection 3.3. We then explain in subsection 3.4 how the 4d theory decomposes into building blocks called tinkertoys.

3.1 Partial topological twist

Our aim is to place the 6d $(2,0)$ theory on $\mathbb{R}^4 \times C_2$, where C_2 is an arbitrary punctured Riemann surface. Doing this too naively would not preserve any symmetry beyond the Poincaré symmetry of $\mathbb{R}^4$. We explain a procedure, the *partial topological twist*, that allows 4d $\mathcal{N}=2$ supersymmetry to be preserved regardless of C_2.

Generalities on topological twist. First we comment on the topological twist of supersymmetric theories in general terms from several point of views.

When placing a field theory on a curved background, the metric $g_{\mu\nu}$ acts as a source for the stress tensor $T^{\mu\nu}$. For a supersymmetric field theory, $T^{\mu\nu}$ typically belongs to a multiplet together with supersymmetry currents S^μ_α and R-symmetry currents J^μ. These can also be coupled to sources ψ^α_μ and A_μ. The partial topological twist consists of setting $S^\mu_\alpha = 0$ and choosing A_μ equal to the spin connection derived from $g_{\mu\nu}$. Schematically, at linearized order around some background values of g, ψ, A, when these fields are changed the Lagrangian varies by

$$\delta\mathcal{L} = T\delta g + J\delta A = T\delta g - J\partial(\delta g) \simeq (T + \partial J)\delta g. \tag{3.1}$$

In the second step we used our choice that A is related to derivatives of the metric, and in the second step we integrate by parts.

In this way the topological twist amounts to redefining the stress-tensor from T to $T_{\text{twist}} = T + \partial J$ before placing the theory on a non-trivial background metric. The twist mixes the stress-tensor $T^{\mu\nu}$ with the R-symmetry current J^μ, but it is good to remember that it does not affect any observables of the theory in flat space, only what we call the stress-tensor. Through the change of stress-tensor it changes how the theory is put on curved spaces.

One job of the stress-tensor is to keep track of Poincaré symmetries: $T^{\mu\nu}$ is the conserved current of translation symmetries, while $x^{[\mu}T^{\nu]\rho}$ is the conserved current of rotations. Since the twist shifts T by a total derivatives it is simply an improvement transformation of the translation symmetry current, and it does not change the corresponding conserved charge, the momentum operator. In contrast, it has a non-trivial effect on what we call rotations: twisted rotation acts by a rotation plus an R-symmetry transformation.

What happens to supercharges? They typically transform as spinors under the original rotations and under R-symmetry transformations. Under the new rotations embedded diagonally the supercharges typically split into a scalar supercharge Q and a vector. Upon placing the theory on a curved manifold using the twisted stress-tensor, the supersymmetry Q is preserved. The next step is typically to restrict to operators in the Q-cohomology, and show that their correlators are described by a topological quantum field theory (TQFT). We will not need this in our case.

Partial topological twist of 6d theories. The *partial* topological twist we use consists of only mixing *some* of the R-symmetries into *some* of the rotation symmetries. To define the specific twist we use, consider rotations $\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{\text{old}}$ preserving separately the two factors of a product $\mathbb{R}^{1,3} \times \mathbb{R}^2$, and consider the block-diagonal subalgebra $\mathfrak{so}(2)_{\mathrm{R}} \times \mathfrak{so}(3)_{\mathrm{R}} \subset \mathfrak{so}(5)_{\mathrm{R}}$ of R-symmetry. We define twisted rotations to be embedded diagonally into $\mathfrak{so}(2)_{\text{old}} \times \mathfrak{so}(2)_{\mathrm{R}}$, namely we treat the following symmetries as our (twisted) Lorentz and R-symmetries:

$$\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{\text{twist}} \times \mathfrak{so}(3)_{\mathrm{R}}. \tag{3.2}$$

This is done by changing the stress-tensor to

$$T^{\mu\nu}_{\text{twist}} = T^{\mu\nu}_{\text{old}} + \frac{1}{4}(\epsilon^{\mu\rho}\partial_\rho J^\nu_{12} + \epsilon^{\nu\rho}\partial_\rho J^\mu_{12}), \tag{3.3}$$

where J_{12} is the R-symmetry rotation generator of $\mathfrak{so}(2)_{\mathrm{R}}$ and $\epsilon^{\mu\nu} = \delta^\mu_4\delta^\nu_5 - \delta^\nu_5\delta^\mu_4$ is the Levi-Civita tensor on the $\mathbb{R}^2$ factor.

> **Exercise 3.1.** *Check that* (3.3) *shift the* x^4, x^5 *rotation current* $x^{[4}T^{5]\mu}$ *by* J_{12} *up to total derivatives (an improvement term), so that the twisted rotation is a combination of rotation and R-symmetry.*

Let us track supersymmetries as we twist and then compactify. Under the $\mathfrak{so}(1,5)$ rotations of $\mathbb{R}^{1,5}$ and $\mathfrak{so}(5)_{\mathrm{R}}$ R-symmetry, the Poincaré supersymmetries transform in the spinor representation of each, denoted $(\mathbf{4},\mathbf{4})$, with a symplectic reality condition that we hide for simplicity. Each 6d Weyl spinor, namely each representation $\mathbf{4}$ of $\mathfrak{so}(1,5)$ decomposes into a pair of 4d Weyl spinors of opposite chirality $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ under $\mathfrak{so}(1,3)$, and these spinors have opposite charges $1/2$ and $-1/2$ under $\mathfrak{so}(2)_{\text{old}}$. Each spinor $\mathbf{4}$ of $\mathfrak{so}(5)$ decomposes into two $\mathbf{2}$ of $\mathfrak{so}(3)_{\mathrm{R}}$ with $\mathfrak{so}(2)_{\mathrm{R}}$ charges $\pm 1/2$. Altogether we denote this as follows, with subscripts denoting charges under the two $\mathfrak{so}(2)$ algebras:

$$\begin{aligned}(\mathbf{4},\mathbf{4}) &= \Big((\mathbf{2},\mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\Big) \otimes \Big(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}\Big) \\ &= (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},-\frac{1}{2}}.\end{aligned} \tag{3.4}$$

By construction the charge under $\mathfrak{so}(2)_{\text{twist}}$ is the sum of those under $\mathfrak{so}(2)_{\text{old}}$ and $\mathfrak{so}(2)_{\mathrm{R}}$. Thus, under the $\mathfrak{so}(1,3) \times \mathfrak{so}(3)_{\mathrm{R}} \times \mathfrak{so}(2)_{\text{twist}}$ symmetry of $\mathbb{R}^{1,5}$ that we are concentrating on, Poincaré supercharges transform as

$$(\mathbf{2},\mathbf{1};\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{1};\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_{-1}. \tag{3.5}$$

We denote them respectively as

$$Q_z^{\alpha A}, Q^{\alpha A}, \bar{Q}^{\dot{\alpha} A}, \bar{Q}_{\bar{z}}^{\dot{\alpha} A}, \tag{3.6}$$

where $\alpha, \dot{\alpha}, A$, ranging from 1 to 2, are indices for spinors of $\mathfrak{so}(1,3)$ of the two chiralities and spinors of $\mathfrak{so}(3)_{\mathrm{R}}$, respectively, while z is a complex coordinate on the $\mathbb{R}^2$ factor that keeps track of $\mathfrak{so}(2)_{\text{twist}}$ charges ± 1 of the first and last supercharges $Q_z, \bar{Q}_{\bar{z}}$.

The middle two supercharges $Q, \bar{Q}$ are scalars under $\mathfrak{so}(2)_{\text{twist}}$ rotations, so that deforming the metric on $\mathbb{R}^2$ to any curved metric preserves these supercharges. Altogether, upon compactifying on $\mathbb{R}^{1,3} \times C$ with the partial topological twist we obtain a system that preserves $\mathfrak{iso}(1,3)$ symmetry, supercharges $Q^{\alpha A}$ and $\bar{Q}^{\dot{\alpha} A}$, and the $\mathfrak{so}(3)_{\mathrm{R}} = \mathfrak{su}(2)$ R-symmetry. Together these form the 4d $\mathcal{N} = 2$ Poincaré supersymmetry algebra.

In the limit where C has zero size, we thus obtain a 4d $\mathcal{N} = 2$ theory, generically.[21] Twisting (3.3) does not preserve the tracelessness of T, so even though the original 6d rotation symmetry extends to conformal symmetry, this is not the case of the twisted rotation symmetry. In the zero area limit, 4d conformal symmetry can be restored and we get an SCFT unless data at punctures of C carry an intrinsic scale.

3.2 Coulomb branch

The Coulomb branch of a 4d $\mathcal{N} = 2$ theory is described by giving a VEV to Coulomb branch operators, namely (gauge-invariant) operators of the 4d theory that are annihilated by all antichiral Poincaré supercharges $\bar{Q}^{\dot{\alpha} A}$. Let us identify these operators starting from the 6d theory $\mathcal{X}(\mathfrak{g})$, following roughly [147, section 3].

Importantly, the resulting Coulomb branch $\mathcal{B}$ obtained in (3.13) *only depends on the complex structure of* C, not on its metric. This lets us deform the Riemann surface in various ways to understand the resulting 4d theory, and it underlies the appearance of 2d CFT objects on C in the AGT correspondence.

Coulomb branch operators. The vacua of $\mathcal{X}(\mathfrak{g})$ are parametrized by the VEV of scalar fields Φ_I, $I = 6, \ldots, 10$, in the Cartan subalgebra of $\mathfrak{g}$ (modulo the Weyl group). The low-energy theory in a given vacuum is described by these fields as well as spinors λ and a self-dual two-form B. Under the $\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{\text{old}} \times \mathfrak{so}(2)_{\mathrm{R}} \times \mathfrak{so}(3)_{\mathrm{R}}$ symmetry algebra of interest to us just before the twist, these fields transform as

$$\begin{aligned} &\Phi_z \coloneqq \Phi_6 + i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,1}, \qquad \Phi_8, \Phi_9, \Phi_{10} \in (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0}, \\ &\Phi_{\bar{z}} \coloneqq \Phi_6 - i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,-1}, \qquad \lambda \in (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\pm\frac{1}{2}} \end{aligned} \tag{3.7}$$

On the other hand the supercharges $\bar{Q}^{\dot{\alpha} A}$ transform as $(\mathbf{1},\mathbf{2},\mathbf{2})_{-1/2,1/2}$. We deduce that

$$\bar{Q}^{\dot{\alpha} A} \Phi_z = 0 \tag{3.8}$$

[21] The system at finite area of C has a certain moduli space of vacua, and in the scaling limit where the area is sent to zero one must specify around which vacuum to expand. If C has "enough" handles or punctures, then its Higgs branch has a maximally symmetric point around which it is natural to expand, and the 4d $\mathcal{N} = 2$ limit is well-defined. If C is a sphere with "too few punctures" or is a torus without punctures, there is no maximally symmetric point and the situation is more subtle, as explained in [6].

because no component of λ has the appropriate $\mathfrak{so}(2)_{\mathrm{R}}$ charge $3/2$.

Really, we should be working with the corresponding gauge-invariant operators, such as traces $\mathrm{Tr}(\Phi_z^j)$ in classical cases $\mathfrak{su}(N)$ and $\mathfrak{so}(2N)$. These are the Casimirs of $\mathfrak{g}$, polynomials $P_k(\Phi_z)$ of degrees d_k for $k=1,\dots,\operatorname{rank}\mathfrak{g}$. Concretely, for classical gauge groups these gauge-invariant operators annihilated by $\bar{Q}^{\dot{\alpha}A}$ are

$$\begin{aligned}
&\mathrm{Tr}(\Phi_z^j), \quad j=2,3,4,\dots,N && && \text{for } \mathfrak{su}(N),\\
&\mathrm{Tr}(\Phi_z^j), \quad j=2,4,6,\dots,2N-2, \text{ and} && \mathrm{Pfaff}(\Phi_z) && \text{for } \mathfrak{so}(2N).
\end{aligned} \tag{3.9}$$

(We recall that the Pfaffian is a square root of the determinant.) For reference, the degrees of Casimirs of $\mathfrak{su}(N)$ are $2,3,\dots,N$; of $\mathfrak{so}(2N)$ are $2,4,6,\dots,2N-2$ and N; of $\mathfrak{e}_6$ are $2,5,6,8,9,12$; of $\mathfrak{e}_7$ are $2,6,8,10,12,14,18$; of $\mathfrak{e}_8$ are $2,8,12,14,18,20,24,30$.

It is often convenient to replace Φ_z by $\Phi_z dz$ to soak up the z index and obtain a tensor. Then we work with the order j differentials $\mathrm{Tr}(\Phi_z^j)dz^j$ on the holomorphic curve (aka Riemann surface) C. A somewhat different basis is more practical: for instance for $\mathfrak{su}(N)$ one expands

$$\det(X-\Phi_z dz) = X^N - \sum_{j=2}^{N} \mathcal{O}_j \, X^{N-j}. \tag{3.10}$$

Exercise 3.2. *Check that* $\mathcal{O}_2 = \mathrm{Tr}(\Phi_z^2/2)dz^2$, $\mathcal{O}_3 = \mathrm{Tr}(\Phi_z^3/3)dz^3$, *and perhaps check that* $\mathcal{O}_4 = \mathrm{Tr}(\Phi_z^4/4)dz^4 - \mathcal{O}_2^2/2$. *Why is there no* $\mathcal{O}_1$*?*

Coulomb branch. What VEV can we give $\mathcal{O}_j$? Denote it by[22]

$$\phi_j \coloneqq \langle \mathcal{O}_j \rangle. \tag{3.11}$$

It should be constant along $\mathbb{R}^{1,3}$ to avoid breaking Poincaré symmetry. Next we use the anticommutator $\{\bar{Q}^{\dot{\alpha}A}, \bar{Q}_{\bar{z}}^{\dot{\beta}B}\} \sim \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\partial_{\bar{z}}$ to deduce that ϕ_j must depend holomorphically on z:

$$\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\partial_{\bar{z}}\phi_j \sim \left\langle \bar{Q}^{\dot{\alpha}A}(\bar{Q}_{\bar{z}}^{\dot{\beta}B}\mathcal{O}_j)\right\rangle + \left\langle \bar{Q}_{\bar{z}}^{\dot{\beta}B}(\bar{Q}^{\dot{\alpha}A}\mathcal{O}_j)\right\rangle = 0. \tag{3.12}$$

The first term vanishes because the twisted compactification preserves the supercharge $\bar{Q}$. The second term vanishes by construction of $\mathcal{O}_j$.

The Coulomb branch of the 4d $\mathcal{N}=2$ theory is thus parametrized by degree d_k differentials ϕ_{d_k} on C for $k=1,\dots,\operatorname{rank}\mathfrak{g}$. In symbols,

$$\mathcal{B} = \bigoplus_{k=1}^{r} H^0(C, K^{\otimes d_k}), \qquad \phi_j \in H^0(C, K^{\otimes j}), \tag{3.13}$$

where K is the canonical bundle on the curve C and $H^0(C,\mathcal{L})$ is the vector space of sections of the line bundle $\mathcal{L}$ on C. Starting in section 4 we explain, for concrete choices

[22] The notation is slightly ill-defined in the case of $\mathfrak{so}(4K)$ because there are then two Casimirs of the same degree $2K$, leading to two order $2K$ differentials: ϕ_{2K} defined from traces of powers of Φ_z, and $\tilde{\phi}_{2K} = \langle \mathrm{Pfaff}(\Phi_z)\rangle \, dz^{2K}$.

of C giving usual 4d $\mathcal{N}=2$ gauge theories, how to relate the parametrization (3.13) to the usual description in terms of scalars in 4d $\mathcal{N}=2$ vector multiplets.

We would like to say intuitively that the 4d Coulomb branch is parametrized by the "VEV" of the adjoint-valued holomorphic one-form $\Phi_z dz$, a putative element of $H^0(C, K \otimes \mathfrak{g})$, modulo gauge transformations. Of course, VEVs of non-gauge-invariant operators don't make sense (or are automatically zero, depending on your point of view) so talking about them is an abuse of language. Nevertheless in our case there is a construction of the so-called Hitchin field (or Higgs field), a holomorphic one-form $\varphi = \varphi_z dz$ with component $\varphi_z \in \mathfrak{g}$, whose Casimirs give $\mathrm{Tr}(\Phi_z^j) dz^j$ in the $\mathfrak{su}(N)$ case and likewise in other cases.[23] For convenience we occasionally use φ rather than the gauge-invariants ϕ_j in some explanations.

Comment on the IR behaviour. The low-energy limit of the 6d theory in a generic vacuum is given by an abelian 6d $(2,0)$ theory valued in the vacuum moduli space. Likewise, in a Coulomb branch vacuum described by a given choice of differentials ϕ_j in (3.13), the effective description of the 4d $\mathcal{N}=2$ theory includes massless scalar fields describing fluctuations of the ϕ_j. Together with similar dimensional reductions of $B_{\mu\nu}$ and λ, these scalar fields form 4d $\mathcal{N}=2$ abelian vector multiplets.

How many? The scalar fields have the Coulomb branch $\mathcal{B}$ as their target, so we should expect an infrared description as a 4d $\mathcal{N}=2$ gauge theory with gauge group $\mathrm{U}(1)^{\dim_{\mathbb{C}} \mathcal{B}}$. Additionally, at particular points on the Coulomb branch there are massless hypermultiplets charged under this gauge group. There are even more singular points on the Coulomb branch where the low-energy dynamics are not abelian.

3.3 Seiberg–Witten curve

Seiberg–Witten curve. In the $\mathfrak{su}(N) = \mathfrak{a}_{N-1}$ case we can repackage the data of ϕ_k in a geometric way in terms of the SW curve Σ and SW differential λ defined next.

Consider the total space T^*C of the canonical line bundle over C: in a local coordinate z on C it has points (z,x) where $x \in \mathbb{C}$ describes a one-form $x dz$. There is a natural projection $T^*C \twoheadrightarrow C$ that "forgets" the coordinate x. There is a natural injection $C \hookrightarrow T^*C$, the "zero section", that maps $z \in C$ to $(z, x=0)$. We define the (complex) curve $\Sigma \subset T^*C$ as the locus (z,x) such that

$$\langle \det(x - \Phi_z) \rangle = \det(x - \varphi_z) = x^N - \sum_{j=2}^{N} u_j(z) x^{N-j} = 0 \tag{3.14}$$

where we used the construction (3.10) of $\mathcal{O}_j$ and wrote $\phi_j = \langle \mathcal{O}_j \rangle = u_j(z) dz^j$ for each exponent j of $\mathfrak{g}$. Note that (3.14) is consistent with transformation properties of x and of the u_j since each term is (the sole component of an) N-form. At generic points $z \in C$ this

[23] The story is quite a bit longer: one compactifies the 4d theory further on S^1. Coulomb branch vacua of the 3d theory are described by (A, φ) solutions of the Hitchin system $0 = F + [\varphi, \bar{\varphi}] = \bar{\partial}_A \varphi = \partial_A \bar{\varphi}$ on C, and the resulting Coulomb branch (the Hitchin moduli space) $\mathcal{M}$ projects onto the Coulomb branch $\mathcal{B}$ of the 4d theory by mapping (A, φ) to Casimirs of φ.

equation (3.14) has N solutions, which locally gives an N sheeted cover of C. Generically, at certain isolated points on C two sheets intersect with a branch point of order 2. We have constructed in this way an N-sheeted ramified cover Σ of C.

As we will see in concrete examples, Σ turns out to be the SW curve of the 4d $\mathcal{N}=2$ theory in the given Coulomb branch vacuum, and the SW differential is the holomorphic one-form λ defined as $\lambda = xdz$ in coordinates (z,x) of T^*C. The fact that our (Σ,λ) matches the usual one is easier to see for concrete theories later on, but we can give some intuition. Besides indirectly giving the prepotential for the low-energy $\mathrm{U}(1)^{\dim_{\mathbb{C}}\mathcal{B}}$ vector multiplets, one of the jobs of the SW curves is to calculate the central charge of particles (which puts a BPS lower bound on their mass) in terms of their electric, magnetic, and flavour charges: it should be obtained by integrating λ along closed contours in Σ. Let us confirm this from the M-theory perspective in the A-type case.

M-theory perspective on SW curve. We recall that $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of N M5 branes (with the decoupled center of mass modes removed). The R-symmetry is then realized geometrically as transverse rotations. The topological twist corresponds to tieing the 2d rotations with 2d transverse rotations, and one finds that the full geometrical set-up corresponding to $\mathcal{X}(\mathfrak{su}(N))$ partially twisted on $\mathbb{R}^{1,3}\times C$ is to consider M-theory on $\mathbb{R}^{1,3}\times T^*C\times\mathbb{R}^3$ and to place M5 branes along $\mathbb{R}^{1,3}\times C$, the zero section.[24]

Moving onto the Coulomb branch corresponds to shifting the M5 branes away from each other along the fibers of T^*C. Since the branes are indistinguishable they generically reconnect into an N-sheeted ramified cover $\Sigma\subset T^*C$ of C. Supersymmetry requires it to be holomorphic and we thus reproduce the above classification of Coulomb branch vacua. We emphasize that the UV curve C characterizes the theory, while the IR curve (or SW curve) Σ depends on (and characterizes) the given Coulomb branch vacuum.

Excitations of the brane system include massless fluctuations along the Coulomb branch of course, but also very interesting massive excitations coming from M2 branes ending on the M5 branes. Consider a two-dimensional surface $D\subset T^*C$ whose boundary lies in the SW curve, $\partial D\subset\Sigma$, and let us place an M2 brane along $D\times\mathbb{R}$ where $\mathbb{R}$ is the time direction in 4d Minkowski space. From the 4d point of view this describes a particle sitting still as time passes. Its mass m is simply the area of D,

$$m=\int_D |dzdx| \geq \Big|\int_D dzdx\Big| = \Big|\int_D d(xdz)\Big| = \Big|\int_{\partial D}\lambda\Big|. \tag{3.15}$$

This reproduces the BPS lower bound expected from the SW curve and differential (Σ,λ). In fact, realizing the SW curve Σ as a fibration over C gives slightly finer control of the BPS spectrum than just knowing Σ (and λ). Indeed, some closed curves on Σ are not the boundary of any two-dimensional $D\subset T^*C$, so that the M-theory setup "knows" that no BPS state with these charges exist, while the data of (Σ,λ) only would not know it.

[24] More generally, T^*C can be replaced by a four-dimensional hyper-Kähler manifold and C by a holomorphic cycle inside Q.

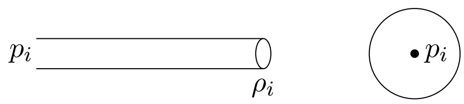
These M-theory considerations suggest that we found the right notion of SW curve and differential for class S theories. But we have yet to explain any concrete description of the 4d theories, rather than only their IR behaviour on the Coulomb branch. We turn to this next.

3.4 Tubes and tinkertoys

So far we only discussed the low-energy effective description of $\mathrm{T}(\mathfrak{g}, C, D)$ on its Coulomb branch. We now study how the class S theory can be described without moving along its Coulomb branch. Our guide to find such a description is that it should reproduce the aforementioned IR physics (it also reproduces some protected observables), and that different descriptions we find should be (exactly) dual to each other. Recall that the partial twist ensures that 4d physics we are interested in only depends on the complex structure of the Riemann surface C on which we compactify. We can thus pick any metric compatible with this complex structure.

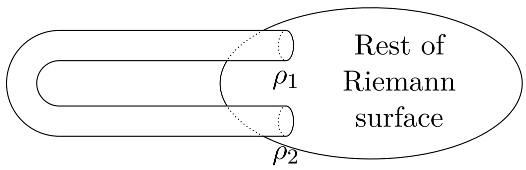
Gluing. Consider two punctures $p_1, p_2 \in \overline{C}$ of the same (or of different) punctured Riemann surface $C = \overline{C} \setminus \{p_i\}$ and consider disks around p_1 and p_2. As far as the complex structure is concerned, these punctured disks are the same as semi-infinite cylinders thanks to the exponential map (expressed here in coordinates centered at p_i)

$$\exp\colon (-\infty, \rho_i] \times (\mathbb{R}/(2\pi\mathbb{Z})) \xrightarrow{\sim} \{z \mid |z| \leq e^{\rho_i}\} \setminus \{0\}.$$



(3.16)

We can glue two such semi-infinite cylinders by cutting their infinite end off at some finite distance and identifying the cutoff points on the left side of the following diagram:



(3.17)

In terms of complex coordinates w and z around p_1 and p_2 respectively (with p_1 at $w = 0$ and p_2 at $z = 0$), the identification is

$$zw = q \tag{3.18}$$

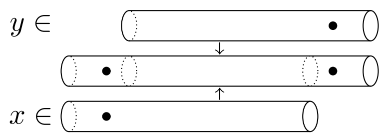
for some parameter q. The modulus $|q|$ gives the aspect ratio (length over circumference) $(-\log |q|)/2\pi$ of the tube, while the phase of q indicates how the cylinders are rotated before gluing. The coordinates w, z are only locally defined so $|q|$ cannot be too big: the tube can be arbitrarily long/thin but not short as the description otherwise breaks down.

Exercise 3.3 (On punctured spheres)**.** *Choose a coordinate w on the complex projective plane $\mathbb{CP}^1$ (the two-sphere), where $w \in \mathbb{C} \cup \{\infty\}$.*

1. For $n \geq 3$ arbitrary distinct points $w_j \in \mathbb{C} \cup \{\infty\}$, $j = 1, \dots, n$, define a new coordinate $z(w) := \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$. Check that $w \mapsto z$ is bijective on $\mathbb{CP}^1$ so that the definition gives a good coordinate on $\mathbb{CP}^1$. Check that w_1, w_2, w_3 are mapped to $0, 1, \infty$. The coordinate $z(w_j)$ for $j > 3$ is called cross-ratio of w_1, w_2, w_3, w_j.

2. In the four-punctured sphere, how does the cross-ratio q change when w_1, w_2, w_3, w_4 are permuted?

3. Construct the four-punctured sphere $\mathbb{CP}^1 \setminus \{0, q, 1, \infty\}$ by gluing two three-punctured spheres $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. (Hint: let x, y be coordinates on the two three-punctured spheres; identify $qx = y$ for some region $1 < |x| < 1/|q|$.) Generalize to the n-punctured sphere.



Vector multiplets. Despite how it is drawn in (3.17), the cylinder connecting the two punctures is flat and of constant circumference $2\pi L_5$ (for some metric). We know that the 6d theory $\mathcal{X}(\mathfrak{g})$ reduced on a circle gives 5d $\mathcal{N} = 2$ SYM with gauge algebra $\mathfrak{g}$ and coupling $g_{5\mathrm{d}}^2 \simeq L_5$. We should thus expect that part of the system obtained by reducing $\mathcal{X}(\mathfrak{g})$ on the glued surface (3.17) is 5d $\mathcal{N} = 2$ SYM on an interval of length $L_4 \sim (-\log|q|)L_5$. In the limit where C shrinks to a point, the 4d Lagrangian ought to have a term

$$\frac{1}{g_{5\mathrm{d}}^2} \int_I \mathrm{Tr}(F^2) = \frac{1}{g_{4\mathrm{d}}^2} \mathrm{Tr}(F^2), \qquad \frac{1}{g_{4\mathrm{d}}^2} = \frac{L_4}{g_{5\mathrm{d}}^2} \simeq \frac{L_4}{L_5} = -\log|q|. \tag{3.19}$$

What about the phase of q, which implements a rotation along the circle direction? The instanton current of a 5d gauge field is defined as

$$J_\mu^{\mathrm{inst}} = \epsilon_{\mu\nu\rho\sigma\tau} \mathrm{Tr}(F^{\nu\rho} F^{\sigma\tau}). \tag{3.20}$$

As we mentioned earlier, in the reduction from $\mathcal{X}(\mathfrak{g})$ to 5d $\mathcal{N} = 2$ SYM the KK (Kaluza–Klein) modes correspond to instanton particles of the 5d theory, namely the KK number is equal to the charge under J^{inst}. Let us denote the directions along and around the cylinder as x^4, x^5 and think of x^4 as Euclidean time. A rotation in x^5 measures KK number and is thus implemented as the 4d "spatial" integral of the "time" component J_4^{inst}. Twisting the cylinder by an angle $\theta = \mathrm{Im} \log|q|$ thus contributes a term $\theta \, \mathrm{Tr}(F \wedge F)$ to the 4d Lagrangian when we reduce C to a point.

Altogether we expect that a long cylinder as in (3.17) should yield a 4d $\mathcal{N}=2$ vector multiplet with complexified gauge coupling τ roughly given by $\log q$:

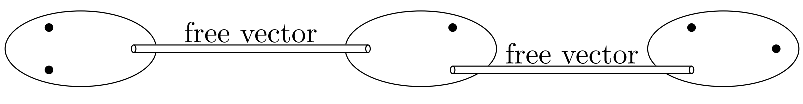
$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \qquad q \sim e^{2\pi i \tau}. \tag{3.21}$$

The relation is made more precise later in concrete geometries.

Pants decomposition and S-duality. Vector multiplets must gauge flavour symmetries of some matter sector, and our next task is to understand where that matter comes from. For this, the key is to send gauge couplings to zero, because in this limit the vector multiplet decouples and leaves behind the matter sector with its flavour symmetries.

> **Exercise 3.4.** *As a toy model, consider a scalar field ϕ transforming in some representation of a group G, and gauge the symmetry G using a gauge field A. We denote by $D = d + A$ the covariant derivative and $F = dA + A \wedge A$, and ignore numerical factors. By introducing a field $\tilde{A} = g^{-1}A$ with canonically normalized kinetic term, show how*
>
> $$\mathcal{L} = \frac{1}{g^2}\operatorname{Tr}(F^2) + |D\phi|^2 \xrightarrow{g\to 0} \operatorname{Tr}(d\tilde{A})^2 + |d\phi|^2. \tag{3.22}$$
>
> *Note that in the limit the flavour symmetry G of ϕ is not gauged any longer. The original gauge theory can be then restored (up to the free gauge field $\tilde{A}$) by gauging this flavour symmetry with a new gauge field. Check the same decoupling happens for spinors ($\bar{\psi}\gamma^\mu D_\mu \psi$).*

Any punctured Riemann surface C with genus g and n punctures, except for (g,n) among $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, can be decomposed into three-punctured spheres (pairs of pants) glued as described above. Such a decomposition is called a pants decomposition. For each pants decomposition of C there is a corresponding cusp in the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces with genus g and n punctures. At this cusp, C is described by three-punctured spheres joined by infinitely thin tubes. Each such tube yields an infinitely weakly coupled vector multiplet in the 4d theory, so that in this limit we can expect 6d fields "localized" on each pair of pants to decouple from each other since the 4d vector multiplets joining them become free:



(3.23)

As in the toy model, the symmetries gauged by the vector multiplet are restored as flavour symmetries in the zero coupling limit.

The picture that emerges is as follows. The building blocks of $\mathrm{T}(\mathfrak{g}, C, D)$ are class S theories called tinkertoys associated to three-punctured spheres. These (4d $\mathcal{N}=2$) tinkertoys have flavour symmetries associated to each puncture, which we study carefully later. For each tube, consider the flavour symmetry groups F_1 and F_2 associated to the two punctures that it connects, and gauge a suitable diagonal subgroup $G \subset F_1 \times F_2$ using a 4d $\mathcal{N} = 2$ vector multiplet. This yields $\mathrm{T}(\mathfrak{g}, C, D)$. This description of $\mathrm{T}(\mathfrak{g}, C, D)$ for each pants decomposition of C can be written schematically as

$$\mathrm{T}(\mathfrak{g}, C, D) = \Bigl(\prod_{\text{pants}} \mathrm{T}(\mathfrak{g}, \text{sphere} \setminus 3\text{pt})\Bigr) \Big/ \Bigl(\prod_{\text{tubes}} \text{gauge group}\Bigr). \tag{3.24}$$

A large part of the work in understanding the AGT correspondence is to classify tinkertoys obtained from three punctured spheres with different types of punctures. Their flavour symmetry can be rather intricate, which is why we cannot make the gauge groups more explicit in (3.24) in such generality.

When all punctures are so-called full tame punctures (explained later), all building blocks are the same tinkertoy $T_{\mathfrak{g}}$. This theory is an isolated[25] SCFT with (at least) $\mathfrak{g}^3$ flavour symmetry associated to its three punctures. For $\mathfrak{g} = \mathfrak{su}(2)$ it consists of four free hypermultiplets, while for other $\mathfrak{g}$ it has no 4d $\mathcal{N} = 2$ Lagrangian description.

Of course, a given Riemann surface has many inequivalent decompositions into pairs of pants. Each one leads to a description of $\mathrm{T}(\mathfrak{g}, C, D)$ as a weakly coupled gauge theory in one corner of parameter space. At strong coupling (short tubes) another description may be weakly coupled hence more useful. In concrete cases this reproduces known 4d $\mathcal{N} = 2$ S-dualities. Here is an exercise to get an intuition about pants decompositions.

Exercise 3.5 (Combinatorics of pants decompositions)**.** *1. Given a surface $C_{g,n}$ with genus g and n punctures, check that all pants decompositions use the same number of three-punctured spheres.*

2. Draw the three topologically different[26] pants decompositions of a four-punctured sphere (assuming punctures are distinguishable). How many pants decompositions does an n-punctured sphere have? Does a once-punctured torus have a finite number of pants decompositions?

3. Return to point 3 of Exercise 3.3 and construct the sphere with $n = 4$ (or 5) punctures by gluing three-punctured spheres in all possible ways.

4. We don't need to degenerate the Riemann surface completely down to pairs of pants: as soon as C involves one long tube the theory $\mathrm{T}(\mathfrak{g}, C, D)$ can be written in terms of a weakly coupled vector multiplet gauging flavour symmetries of a "smaller" class S theory. What Riemann surface (genus, punctures) underlies the latter theory? There are two cases.

[25] An SCFT with a certain amount of supersymmetry is *isolated* if it does not have any exactly marginal deformation (such as gauge couplings in 4d).

[26] Two pants decompositions are the same in this sense if the closed curves cutting the surface into pieces with three boundaries can be deformed into each other without (self-)intersection or crossing punctures.

4 Lagrangians for class S theories

After discussing the tame punctures that arise when pinching tubes, we argue in subsection 4.1 that $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with three tame punctures yields 4 free hypermultiplets, with a flavour symmetry (larger than) $\mathrm{SU}(2)^3$. In subsection 4.2 we glue two such building blocks to learn how $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with four tame punctures reproduces known aspects of 4d $\mathcal{N}=2$ SQCD with gauge group SU(2) and $N_f=4$ flavours. This is the conventional starting point of AGT reviews: one usually studies S-duality of SU(2) SQCD [4] and of quivers gauge theories [148], before explaining the unifying 6d point of view [1]. We extend the discussion in subsection 4.3 to generalized SU(2) quivers arising from $\mathcal{X}(\mathfrak{su}(2))$ on arbitrary punctured Riemann surfaces. In subsection 4.4 we realize as class S theories some $\mathrm{SU}(N)$ linear quiver gauge theories including $\mathrm{SU}(N)$ SQCD with $N_f=2N$ flavours. This teaches us that there are several types of tame punctures hence several types of codimension 4 operators in the 6d theory.

4.1 Trifundamental tinkertoy

We discuss tame punctures; for $\mathfrak{su}(2)$ there is only one type. We then consider $\mathcal{X}(\mathfrak{su}(2))$ on a sphere $\mathbb{CP}^1$ with three tame punctures at $0,1,\infty$ and we argue that the resulting tinkertoy $T_2=T_{\mathfrak{su}(2)}$, which is the main building block of $\mathfrak{su}(2)$ class S theories, is a collection of four free hypermultiplets. There is no first principle derivation of that fact, but we will see many checks of it, especially correct postdictions of Coulomb branch and SW curves of many gauge theories, as well as consistency with string theory dualities.

Tame punctures. We describe punctures in terms of their effect on the Hitchin field $\varphi(z)$ parametrizing the Coulomb branch, or gauge-invariantly in terms of the higher-order differentials ϕ_{d_k}, $k=1,\dots,\operatorname{rank}\mathfrak{g}$.

Punctures can arise from pinching a thin tube. In a complex coordinate $w\in\mathbb{R}\times S^1$ describing this tube, the ϕ_{d_k} often tend to constants (times dw^{d_k}) inside the thin tube. Cutting the cylinder (the opposite of what we did in (3.17)) and applying the exponential map (3.16) $z=e^w$, we generically expect

$$\phi_{d_k}\simeq\frac{dz^{d_k}}{z^{d_k}}+\dots \tag{4.1}$$

with some coefficients, in a local coordinate z in which the puncture is at $z=0$.

This motivates us to work with *tame punctures*, namely points where $\varphi(z)dz$ has a first order pole with a prescribed residue, of course up to gauge conjugation: the prescribed residue translates generically to a prescribed leading coefficient in (4.1) —a *full* tame puncture. We study other punctures later in section 7: tame punctures in which ϕ_{d_k} have lower-order poles instead of (4.1), and irregular punctures defined as having higher-order poles.

For now we focus on $\mathfrak{su}(2)$: there is then a single type of tame puncture.

This case has a single Casimir, the quadratic differential $\phi_2=\frac{1}{2}\operatorname{Tr}(\varphi^2)dz^2$. We impose the residue of the Hitchin field φ up to conjugation (which we denote $\sim$): for non-zero $m \in \mathbb{C}$,

$$\varphi(z) \sim \Big(\frac{\operatorname{diag}(m,-m)}{z-z_i} + O(1)\Big)dz \implies \phi_2(z) = \Big(\frac{m^2}{(z-z_i)^2} + O\Big(\frac{1}{z-z_i}\Big)\Big)dz^2. \quad (4.2)$$

We call $m \neq 0$ the *mass parameter* of the puncture for the following reason. The sheets of Σ defined in (3.14) behave as $x_\pm(z) = \pm m/(z-z_i) + O(1)$, and integrating the SW differential λ around z_i on one of the two sheets picks up the residue $\pm m$. This means m appears as a contribution to the central charge hence to masses of BPS particles.

Naively, taking the $m \to 0$ limit in the $\varphi(z)$ asymptotics changes z_i into a regular point. In the ϕ_2 equation however, the puncture remains as a first order pole. This is explained from the $\varphi(z)$ point of view by conjugating the diagonal matrix $\operatorname{diag}(m,-m)$ before taking the $m \to 0$ limit:

$$\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \sim \begin{pmatrix} m & 1 \\ 0 & -m \end{pmatrix} \xrightarrow{m\to 0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (4.3)$$

Indeed, we find a consistent *massless* tame puncture

$$\varphi(z) \sim \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{z-z_i} + O(1)\right)dz \implies \phi_2(z) = O\Big(\frac{1}{z-z_i}\Big)dz^2 \quad (4.4)$$

where the pole has a free coefficient. Interestingly, the sheets of Σ defined by $x^2dz^2 = \phi_2$ admit a branch point at such a puncture.

> **Exercise 4.1.** *1. For any $\alpha \in \mathbb{C}$ find an invertible matrix $g \in \mathrm{SL}(2,\mathbb{C})$ such that $g^{-1}\operatorname{diag}(m,-m)g = \left(\begin{smallmatrix} m & \alpha \\ 0 & -m \end{smallmatrix}\right)$.*
>
> *2. Check that all $\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$, $\alpha \neq 0$, are conjugate to each other.*
>
> *3. How does the coefficient of $1/(z-z_i)$ arise in (4.2) and (4.4) from components of the $(z-z_i)^0$ term in the expansion of φ?*

We interpret (4.2) as follows: the massless puncture (4.4) carries SU(2) flavour symmetry, and turning on a constant scalar $\phi_{\text{background}} = m$ in a background vector multiplet coupled to that symmetry changes the puncture to the massive one (4.2).

Three-punctured sphere and symmetries. Now consider T_2, the result of placing $\mathcal{X}(\mathfrak{su}(2))$ on a sphere $\mathbb{CP}^1$ with three tame punctures. What do we know for sure about T_2?

First, it should have at least $\mathrm{SU}(2)^3$ flavour symmetry, one SU(2) per puncture. We learn in the next exercise that 4 free hypermultiplets indeed have an USp(8) flavour symmetry, which contains $\mathrm{SU}(2) \times \mathrm{Spin}(4) = \mathrm{SU}(2)^3$. The USp(8) flavour symmetry is in this context an emergent symmetry that is only present in the limit where C shrinks to a point; it is not a symmetry of the 6d setup.

Exercise 4.2. *1. Check that k free scalar fields carry $\mathrm{O}(k)$ flavour symmetry. Check that p free hypermultiplets contain $4p$ free scalars hence have $\mathrm{O}(4p)$ symmetries (and more from spinors). Out of these, check a $\mathrm{USp}(2p)$ subgroup commutes with $\mathrm{SU}(2)_\mathrm{R}$ R-symmetry. (Hint: as an intermediate step, the $\mathrm{U}(2p)$ subgroup commutes with $J_3 \in \mathrm{SU}(2)_\mathrm{R}$.)*

2. Now gauge an $\mathrm{SU}(2) = \mathrm{USp}(2)$ flavour symmetry embedded diagonally into $\mathrm{USp}(2)^p \subset \mathrm{USp}(2p)$. The gauged $\mathfrak{su}(2)$ times $\mathfrak{su}(2)_\mathrm{R}$ combine into $\mathfrak{so}(4)$. Check that the $4p$ scalars organize as p copies of the fundamental representation of $\mathfrak{so}(4)$. Deduce that the remaining flavour symmetry is[27] $\mathrm{Spin}(p)$.

The trifundamental half-hypermultiplet. All three $\mathrm{SU}(2)$ symmetries of the four hypermultiplets can be made manifest, at the cost of hiding $\mathcal{N}=2$ supersymmetry. Split each hypermultiplet into a pair of $\mathcal{N}=1$ chiral multiplets $(q,\tilde{q})$. The four hypermultiplets split into eight $\mathcal{N}=1$ chiral multiplets q_{aiu} where a,i,u (ranging from 1 to 2) are indices for the three independent $\mathrm{SU}(2)$. To reconstruct the hypermultiplets as $(q,\tilde{q})$ simply introduce the notation

$$\tilde{q}^{aiu} = \epsilon^{ab}\epsilon^{ij}\epsilon^{uv} q_{bjv}. \tag{4.5}$$

The hypermultiplets are thus in a trifundamental representation of $\mathrm{SU}(2)^3$ with a reality property (4.5) that halves the number of components. This set of matter fields is called a half-hypermultiplet.

If background vector multiplet scalars (i.e., masses m_1, m_2, m_3) are turned on for the three $\mathrm{SU}(2)^3$, then the underlying 8 chiral multiplets have complex masses $\pm m_1 \pm m_2 \pm m_3$ for all choices of signs. In particular one of the 4 hypermultiplets becomes massless when $m_2 = \pm m_1 \pm m_3$. This is important later.

Seiberg–Witten curve of T_2. We denote by m_1, m_2, m_3 the mass parameters of punctures at $0, 1, \infty$ in the sense of (4.2) or (4.4).[28] The Coulomb branch (if any) of the 4d theory is parametrized by holomorphic quadratic differential $\phi_2(z)$ that have second order poles (4.2) or first-order in the massless case (4.4) at each of the punctures, and no other pole. The puncture at infinity translates to a condition as $z \to \infty$:

$$\phi_2(z) = \left(\frac{m_3^2}{z^2} + O\left(\frac{1}{z^3}\right)\right) dz^2. \tag{4.6}$$

We recall Liouville's theorem regarding entire functions (holomorphic functions on $\mathbb{C}$ with no pole): if an entire function f is bounded as $|f(z)| < K z^p$ for some constant K and exponent p then f is a polynomial of degree at most p.

[27] Discrete factors may be wrong?

[28] We don't know at this stage that they are the same parameters as in the last paragraph about the trifundamental half-hypermultiplet.

Exercise 4.3. *Find a quadratic differential $\phi_2(z) = u_2(z)dz^2$ that has the prescribed second order poles at $0, 1, \infty$ and no other singularity and show it is unique. (Hint: write it as $u_2(z) = f(z)/(z^2(z-1)^2)$, change variables to $w = 1/z$ to polynomially bound f at infinity and use Liouville's theorem to bound the degree of f, then compare with the prescribed asymptotics to fix coefficients.)*

The Coulomb branch is thus a single point, which is consistent with the lack of vector multiplet in our description of T_2 as free hypermultiplets. Explicitly,

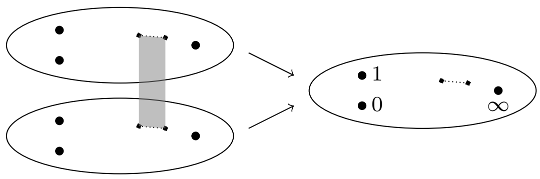
$$\phi_2(z) = u_2(z)dz^2, \qquad u_2(z) = \frac{-m_1^2}{z^2(z-1)} + \frac{m_2^2}{z(z-1)^2} + \frac{m_3^2}{z(z-1)}. \tag{4.7}$$

Let us find the IR description of T_2 at this unique Coulomb branch vacuum. As we commented on page 23, the low-energy theory is generically a 4d $\mathcal{N} = 2$ abelian gauge theory with the vector multiplet scalars living in the Coulomb branch $\mathcal{B}$. Here there is no Coulomb branch hence no gauge fields in the IR. There may be hypermultiplets: for this we have to study the SW curve Σ defined by $x^2 = u_2$ and the SW differential $\lambda = xdz$. The integral of λ over closed cycles tells us about masses of BPS states.

The curve Σ is a ramified double cover of $\mathbb{CP}^1$. How many branch points does it have? Branch points are where the two sheets $x = \pm\sqrt{u_2}$ rejoin, namely where $u_2 = 0$. This happens at the (generically) two roots of the quadratic polynomial

$$z^2(z-1)^2 u_2(z) = -(z-1)m_1^2 + zm_2^2 + z(z-1)m_3^2. \tag{4.8}$$

Altogether, Σ wraps the sphere twice, with a single branch cut. It is thus topologically a sphere. The three punctures at $0, 1, \infty \in \mathbb{CP}^1$ become six point on Σ where the SW differential λ blows up:

$\Sigma:$  $:C$ (4.9)

BPS spectrum. Contour integrals of λ give integer[29] linear combinations of residues of $\lambda = xdz = \pm\sqrt{u_2}\,dz$ at the poles $z = 0, 1, \infty$. By construction these residues are $\pm m_1, \pm m_2, \pm m_3$, so we find that masses (or rather central charges) of BPS states take the form $Z = f_1 m_1 + f_2 m_2 + f_3 m_3$ for $f_1, f_2, f_3 \in \mathbb{Z}$. On the other hand, the trifundamental half-hypermultiplet only has BPS states with integer linear combinations of $\pm m_1 \pm m_2 \pm m_3$: this imposes the further restriction that $f_1 = f_2 = f_3 \mod 2$. Does the tinkertoy T_2 also have that restriction?

In subsection 3.3 we learned that M-theory instructs us to only integrate λ over contours γ in $\Sigma \subset T^*C$ that can be written as the boundary $\gamma = \partial D$ of some two-dimensional surface $D \subset T^*C$.

[29]We ignore the factor of $2\pi i$ in the residue theorem.

Exercise 4.4. *1. First choose D to be a small circle around one of the punctures, times the interval connecting the two sheets of Σ. Its boundary is a pair of circles picking up twice the same residue m_i (from different sheets). Deduce that $2m_1, 2m_2, 2m_3$ and all their integer linear combinations are in the spectrum.*

2. Next, choose D such that ∂D is a contour from one branch point to the other (on one sheet) and back via the other sheet. Note that the contour ∂D can be deformed to a contour staying on one sheet and surrounding the cut. Deduce that the integral of λ is one of the combinations $\pm m_1 \pm m_2 \pm m_3$ (three poles are on each side of the contour) and conclude that the BPS spectrum of T_2 contains all $Z = f_1 m_1 + f_2 m_2 + f_3 m_3$ with $f_1 = f_2 = f_3 \mod 2$.

3. (Mathematical.) For any $D \subset T^*C$ with boundary $\partial D \subset \Sigma$, consider the projection $\pi\colon T^*C \to C$ and deduce that $\pi(\partial D) = \partial(\pi(D))$ cannot surround a pole. Deduce that the BPS spectrum of T_2 is exactly that of the trifundamental half-hypermultiplet.

Generically, all of these BPS particles are massive so that the low-energy theory is empty. An interesting case is the limit $m_2 \to \pm(m_1 \pm m_3)$ where one of the four hypermultiplets in the trifundamental half-hypermultiplet becomes massless. Then the SW curve degenerates because the two branch points collide: indeed, u_2 has a double root (4.8)

$$z^2(z-1)^2 u_2(z) = (m_1 \pm z m_3)^2. \tag{4.10}$$

The contour we considered in point 2 of the above exercise shrinks to zero size while λ itself remains finite, so the integral is indeed zero, consistent with the vanishing mass. We will run this kind of easy consistency checks for the more complicated theories.

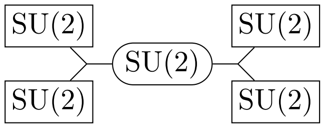
4.2 4d $\mathcal{N} = 2$ SU(2) $N_f = 4$ SQCD

After so many generalities we are ready to study the Coulomb branch, SW curve and S-dualities of our first interesting concrete theory: $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with four tame punctures.

Identifying the 4d theory (spoilers in the title above). We place the four punctures at $z_1 = 0$, $z_2 = q$, $z_3 = 1$, $z_4 = \infty$ on the two-sphere $\mathbb{CP}^1$. The three degeneration limits $q \to 0, 1, \infty$ of the four-punctured sphere correspond to all ways of clustering the punctures pairwise. Since the three limits are identical up to permuting the punctures we concentrate on $q \to 0$. In this limit, we expect on general grounds that the 4d theory consists of two tinkertoys $T_{\mathfrak{su}(2)}$ and one SU(2) vector multiplet gauging an SU(2) flavour symmetry of each tinkertoy. After this gauging each tinkertoy should still carry at least

$\mathfrak{su}(2) \times \mathfrak{su}(2)$ flavour symmetries associated to its two remaining punctures. We can depict this as a (generalized) quiver making all symmetries explicit:

$$\mathrm{T}(\mathfrak{su}(2), \mathbb{CP}^1 \setminus \{0, q, 1, \infty\}) = \tag{4.11}$$



Here the round node denotes a gauge group while square nodes denote flavour symmetries. Each junction $\succ$ represents our favorite tinkertoy T_2, the trifundamental half-hypermultiplet, i.e., four hypermultiplets transforming as two doublet representations of the SU(2) gauge group.

We thus get two flavours from the left junction and two flavours from the right junction, hence the theory is SU(2) SQCD with $N_f = 4$ flavours. While each tinkertoy in (4.11) has Spin(4) flavour symmetry after gauging SU(2), the full theory has $N_f = 4$ doublets of SU(2) on an equal footing hence has a larger Spin(8) flavour symmetry. This symmetry of the 4d theory only emerges in the limit where C shrinks to a point.

Coulomb branch. We denote by m_1, m_2, m_3, m_4 the mass parameters at each of these punctures in the sense of (4.2) or (4.4). Coulomb branch vacua of the 4d theory are parametrized by holomorphic quadratic differential $\phi_2(z)$ that have second order poles (4.2) or first-order in the massless case (4.4) at each of the punctures, and no other pole. The puncture at infinity translates to a condition as $z \to \infty$:

$$\phi_2(z) = \left(\frac{m_4^2}{z^2} + O\left(\frac{1}{z^3}\right)\right) dz^2. \tag{4.12}$$

We parametrize the possible $\phi_2(z)$ in the next exercise, starting with the massless case $m_1 = m_2 = m_3 = m_4 = 0$ for which ϕ_2 has first-order poles.

> **Exercise 4.5.** *1. Find all quadratic differentials $\phi_2(z) = u_2(z) dz^2$ that have first order poles at $0, q, 1, \infty$ and no other. (Hint: after writing $u_2(z) = f(z)/(z(z-q)(z-1))$, change variables to $w = 1/z$ to deduce a polynomial bound on $f(z)$, then use Liouville's theorem mentioned above.)*
>
> *2. Find one quadratic differential ϕ_2 that has leading behaviour $m_i^2/(z-z_i)^2$ for $i = 1, 2, 3$ and m_4^2/z^2 at infinity as per (4.12). Combining with the massless case deduce all such quadratic differentials.*

We find a one-dimensional Coulomb branch $\mathcal{B} = \mathbb{C}$ with vacua[30]

$$\phi_2 = u_2 dz^2, \qquad u_2(z) = \frac{\frac{q}{z} m_1^2 + \frac{q(q-1)}{z-q} m_2^2 + \frac{z-q}{z-1} m_3^2 + z m_4^2 - u}{z(z-q)(z-1)} \tag{4.13}$$

labeled by $u \in \mathcal{B} = \mathbb{C}$. A zero-th order check that we did not go astray is that we got the correct dimension for the Coulomb branch of SU(2) SQCD with $N_f = 4$ flavours.

[30] The variable u parametrizing the Coulomb branch can be freely redefined, hence you may have gotten a slightly different expression in Exercise 4.5.

Degeneration limit. As $q \to 0$, the surface degenerates, and we should obtain in a suitable sense two disconnected three-punctured spheres. For $|q|, |z| \ll 1$, (4.13) behaves as

$$\phi_2(z) \simeq \frac{\frac{-q}{z} m_1^2 + \frac{q}{z-q} m_2^2 + u}{z(z-q)} dz^2, \tag{4.14}$$

which is precisely the quadratic differential on a sphere with three tame punctures of masses squared m_1^2, m_2^2, and u. Likewise, for $|q| \ll |z|, 1$ (4.13) behaves as the quadratic differential on a three-punctured sphere with masses squared u, m_3^2 and m_4^2. This is consistent with how we introduced tame punctures in subsection 4.1.

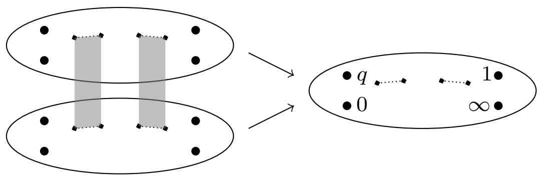
Since masses are background values of vector multiplet scalars, we learn from (4.14) the identification

$$u = \frac{1}{2} \left\langle \operatorname{Tr} \phi^2 \right\rangle \tag{4.15}$$

in the weakly-coupled limit $|q| \ll 1$, where ϕ is the (dynamical) vector multiplet scalar corresponding to the SU(2) gauge group. In other words u is the usual parametrization of the Coulomb branch of SQCD.

Seiberg–Witten curve. We now return to general q. The SW curve and differential are defined by $\Sigma = \{(x,z) \in T^*\mathbb{CP}^1 \mid x^2 = u_2(z)\}$ and $\lambda = x dz$.

The curve Σ is a ramified double cover of $\mathbb{CP}^1$. How many branch points does it have? Branch points are where the two sheets $x = \pm\sqrt{u_2}$ rejoin, namely where $u_2 = 0$. This happens at the (generically) four roots of the polynomial $z^2(z-q)^2(z-1)^2 u_2(z)$, which is quartic. Altogether, Σ wraps the sphere twice, with four branch points joined by two branch cuts. It is thus topologically a torus. In addition to these branch cuts we have four punctures at $0, q, 1, \infty \in \mathbb{CP}^1$, hence eight point on Σ where the SW differential λ blows up:

Σ: 

$: C$ (4.16)

> **Exercise 4.6.** *1. By changing coordinates as $x = \tilde{x}/z + m_2/(z-q) + m_3/(z-1)$, rewrite the curve $x^2 = u_2$ in a form that only has simple poles at $z = 0, q, 1$. Show that $\tilde{\lambda} \coloneqq \tilde{x} dz/z$ differs from the SW differential $\lambda = x dz$ by a u-independent term whose contour integrals (residues) are linear combinations of masses. Recall the BPS mass formula $\oint \lambda = na + ma_D + f_i m_i$ and check what changing λ to $\tilde{\lambda}$ amounts to a redefinition of flavour charges. Up to simple changes of coordinates perhaps[31] match with more conventional expressions of the SW curve and differential of SU(2) $N_f = 4$ SQCD given in [114]. The match confirms that we correctly identified the tinkertoy $T_{\mathfrak{su}(2)}$.*

[31] I have not checked yet what form Martone uses in his notes.

Singularities on the Coulomb branch. As we described on page 23, the low-energy theory is generically a 4d $\mathcal{N}=2$ abelian gauge theory with the vector multiplet scalars living in the Coulomb branch $\mathcal{B}$. For generic values of u and of masses, we thus get a U(1) vector multiplet, but at special values of the parameters some branch points may collide, which leads to interesting low-energy behaviours. We already saw that near (4.10) in our study of the tinkertoy: we found particular values of the masses where a pair of branch points of the SW curve collide. This collision made a certain contour shrink to zero size, hence lead to a massless BPS particle which remains present in the IR theory. For SQCD such collisions of branch points enrich the IR theory by adding one or more massless hypermultiplets charged under the low-energy U(1).

> **Exercise 4.7** (On the discriminant)**.** *The discriminant of a degree d polynomial $P(z)=p_d\prod_{a=1}^d(z-z_a)$ is $\Delta_P=p_d^{2d-2}\prod_{a<b}(z_a-z_b)^2$. It vanishes by construction exactly when $P(z)$ has double roots. It is known that Δ_P can be expressed as a polynomial of degree $2d-2$ in the coefficients p_j of $P(z)=\sum_{j=0}^d p_j z^j$. Check this for quadratic polynomials.*

Our question is to find when $P(z)=z^2(z-q)^2(z-1)^2u_2(z)$, which is a quartic polynomial given explicitly in (4.13), has double roots (hence when two branch points collide). The discriminant Δ_P is then of degree 6 in the coefficients of P. Since P depends linearly on u we find that Δ_P is of degree 6 in u (the leading coefficient turns out to be nonzero). We should thus expect 6 singularities on the Coulomb branch.

Four of these can be seen concretely in the weak coupling limit. Then ϕ_2 is roughly given by the quadratic differential on each pair of pants, connected by a long tube where ϕ_2 is suitably constant, see (4.14). Each three-punctured sphere has two zeros of ϕ_2, hence one branch cut of the SW curve. Consider the pair of pants with masses squared m_1^2, m_2^2, u, for definiteness. Its branch cut shrinks to zero size whenever any combination $\pm m_1 \pm m_2 \pm \sqrt{u}$ of the mass parameters vanishes. We thus find four of the six singular points of the Coulomb branch:

$$u=(m_1\pm m_2)^2+O(q) \quad \text{and} \quad u=(m_3\pm m_4)^2+O(q). \tag{4.17}$$

The remaining two points are not so easy to determine from the explicit quadratic differential (4.13) of the class S theory, partly because they correspond to the collision of branch points that sit in different pair of pants in our decomposition above. A tedious series expansion shows that at[32]

$$u=\pm 2\big(q(m_2^2-m_1^2)(m_3^2-m_4^2)\big)^{1/2}+O(q) \tag{4.18}$$

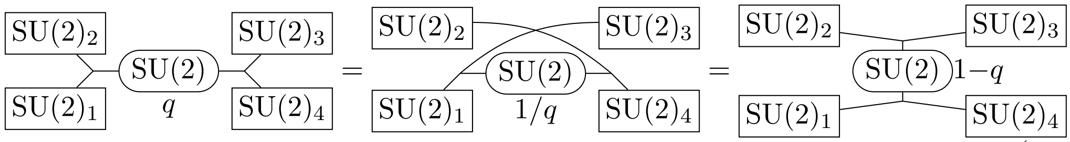
two branch points collide at $z=\mp 2\big(q(m_2^2-m_1^2)/(m_3^2-m_4^2)\big)^{1/2}+O(q)$, with the sign being correlated to that of u.

[32] It would be nice to understand the formulae better from our 6d construction.

From the point of view of SQCD with $N_f = 4$ flavours, what happens is as follows. The four doublet hypermultiplets have mass parameters $m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4$, so when the "VEV" of the vector multiplet ϕ matches one of these we get a massless hypermultiplet; its charge is $+1$ or -1 under the low-energy U(1) because that is how the diagonal $\mathrm{U}(1) \subset \mathrm{SU}(2)$ acts on a doublet. At low energies $|u| \ll |m_i|, |m_i \pm m_j|$, all hypermultiplets are massive and can be integrated out, leaving behind pure SU(2) SYM, whose Coulomb branch is known to have two singular points at $u = \pm 2\Lambda$, the monopole and dyon points. Incidentally we learn that the dynamically generated scale is at (4.18) divided by 2. The main takeaway for our purposes is that the 6d perspective reproduces all the expected physics of SQCD.

By tuning more than just u we can get more than two branch points to collide, hence more than one set of fields to become massless. Such limits can lead in the IR to non-trivial SCFT including the AD theory, which we return to in section 7. The limits are also interesting on the 2d side.

S-duality. The four-punctured sphere $\mathbb{CP}^1 \setminus \{0, q, 1, \infty\}$ has three pants decompositions hence three Lagrangian descriptions. The descriptions are identical except for permutations of masses m_1, m_2, m_3, m_4 and changing $q \to 1/q$ or $q \to 1 - q$. This is S-duality of SQCD [4]. In the notations of (4.11),



(4.19)

While these equalities are manifest in the 6d perspective they hide deep non-perturbative physics, as they are equalities between QFTs involving completely different elementary gauge fields and matter fields (the gauge field A_μ in some description is unrelated to A_μ in another).

4.3 Generalized SU(2) quivers

We have given all the ingredients to determine $\mathfrak{su}(2)$ class S theories arising from $\mathcal{X}(\mathfrak{su}(2))$ on an arbitrary punctured Riemann surface C with tame punctures.[33] This subsection will thus consist essentially of exercises.

Five-punctured sphere. We consider here $C = \mathbb{CP}^1 \setminus \{0, z_1, z_2, 1, \infty\}$; note that we shifted indices of punctures z_i a bit compared to our earlier conventions.

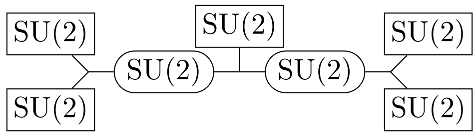
> **Exercise 4.8.** *Write C as the gluing of three pairs of pants with gluing parameters z_1/z_2 and z_2 following Exercise 3.3.*

[33] We exclude the sphere with no puncture, one puncture (a plane), or two punctures (a cylinder), and the torus without punctures, as they are pathological.

Exercise 4.9. *For each pants decomposition of C find a Lagrangian description with gauge group* $\mathrm{SU}(2)^2$ *and twelve hypermultiplets. (Hint: see* (4.20)*.) In what representations of the* $\mathrm{SU}(2)^2$ *gauge group do they transform? What flavour symmetries do these representations carry?*

Contrarily to spheres with $n = 3, 4$ punctures, the $\mathrm{SU}(2)^5$ flavour symmetry manifest from 6d does not enhance in the 4d theory (manifestly, at least).

$$\mathrm{T}(\mathfrak{su}(2), \mathbb{CP}^1 \setminus \{0, z_1, z_2, 1, \infty\}) = \tag{4.20}$$



Exercise 4.10. *1. Each* $\mathrm{SU}(2)$ *gauge group is coupled to four doublet hypermultiplets. When the other gauge group is weakly coupled the theory is thus SQCD coupled to further matter by a weakly coupled gauge field. "Apply" S-duality to this SQCD theory and check that the resulting description is the description one would have written for some pair of pants of the five-punctured sphere.*

2. Check that elementary S-dualities (4.19) *applied to different gauge nodes do not commute so that the S-duality group(oid) of the* $\mathrm{SU}(2)^2$ *gauge theory is not the product of S-duality groups of two SQCD theories.*

Punctured sphere. Next we consider $\mathbb{CP}^1 \setminus \{z_0, \dots, z_{n-1}\}$ with n punctures, with $z_0 = 0$, $z_{n-2} = 1$, $z_{n-1} = \infty$. Denote by $m_0, m_1, \dots, m_{n-1}$ the mass parameters of the punctures.

Exercise 4.11. *1. By using Liouville's theorem as in Exercise 4.3 and Exercise 4.5, find all quadratic differentials $\phi_2(z)$ that have the prescribed second order poles at punctures. Deduce that the Coulomb branch is $\mathcal{B} = \mathbb{C}^{n-3}$.*

2. Write a $\mathrm{SU}(2)^{n-3}$ *linear quiver description of the theory that is weakly coupled in the regime $|z_1| \ll |z_2| \ll \cdots \ll |z_{n-3}| \ll 1$.*

2. Expand $\phi_2(z)$ in this regime for z in an annulus $|z_{i-1}| \ll |z| \ll |z_{i+1}|$ $(i = 1, \dots, n-2)$. Check that ϕ_2 reduces to the differential of T_2 on each of these pair of pants building blocks. Check that $\mathcal{B} = \mathbb{C}^{n-3}$ can be parametrized by the parameters u_i, $i = 1, \dots, n-3$ of punctures in these pants. Identify $u_i = \frac{1}{2} \operatorname{Tr} \phi_i^2$ where ϕ_i is the vector multiplet scalar of the i-th vector multiplet.

3. Check that starting at $n = 6$ pants decompositions can be topologically distinct beyond just the permutation of punctures.

Punctured torus.

> **Exercise 4.12.** *1. The once-punctured torus is obtained by gluing two punctures of the same pair of pants. Write the theory as an* $\mathrm{SU}(2)$ *gauge theory and note that there is a decoupled gauge singlet in addition to the adjoint hypermultiplet. We henceforth ignore such gauge singlets.*[34]
>
> *2. Write the theory associated to an* n*-punctured torus as a circular* $\mathrm{SU}(2)^n$ *quiver with a bifundamental hypermultiplet for each pair of neighboring groups. The weak gauge coupling regime corresponds to a long torus with well-separated punctures.*
>
> *3. If you know enough about elliptic functions determine all quadratic differentials with prescribed second order poles at the punctures. Expand them in the weak gauge coupling limit as in Exercise 4.11.*

4.4 Linear quiver $\mathfrak{su}(N)$ theories

We move on to $\mathfrak{su}(N)$ class S theories, specifically a particular subclass that is ad-hoc from the 6d perspective but leads to linear quiver gauge theories in 4d, as can be understood using brane constructions. [35]

SQCD. Let us try and realize $\mathrm{SU}(N)$ SQCD with $N_f = 2N$ flavours (the number of flavours needed for a vanishing beta function) as a class S theory. Its flavour symmetry is $\mathfrak{u}(N_f) = \mathfrak{u}(2N)$ (enhanced to $\mathfrak{so}(8)$ when $N = 2$). In analogy to the $N = 2$ case we split the $2N$ flavours as two groups of N where each group should come from some three-punctured sphere. The flavour symmetry of each group is $\mathfrak{u}(N) = \mathfrak{u}(1) \times \mathfrak{su}(N)$, so that this split makes $\mathfrak{su}(N)^2 \times \mathfrak{u}(1)^2$ flavour symmetry manifest. In analogy to the $N = 2$ case we associate each of the four factors to one puncture and write an analogue of (4.11):[36]

$$\mathrm{T}(\mathfrak{su}(N), \mathbb{CP}^1 \setminus 4\mathrm{pt}, \text{suitable data}) = \begin{array}{ccccc} \boxed{\mathrm{U}(1)} & & & & \boxed{\mathrm{U}(1)} \\ & > \!\!- & (\mathrm{SU}(N)) & -\!\! < & \\ \boxed{\mathrm{SU}(N)} & & & & \boxed{\mathrm{SU}(N)} \end{array} \tag{4.21}$$

A major difference in the $N = 2$ case is that the $\mathfrak{u}(N) = \mathfrak{u}(2)$ symmetry enhances to $\mathfrak{so}(4)$, namely the $\mathfrak{u}(1)$ factor enhances to $\mathfrak{su}(2)$. For $N > 2$ we have to deal with the presence of different kinds of punctures. We delay the full story to subsection 7.2. For now we shall be content with using two types of tame punctures: full punctures that carry $\mathfrak{su}(N)$ flavour symmetry and simple punctures that carry $\mathfrak{u}(1)$.

---

[34] It is not immediately clear to me how these work out when considering different Lagrangian descriptions of n-punctured tori or of higher genus surfaces.

[35] Section to be completed in a later version of these notes.

[36] As in various other places in this review there are inaccuracies about the global structure of groups.

Free hypermultiplets. The left and right sides of the quiver (4.21) consist of N^2 hypermultiplets that each have $\mathfrak{u}(1) \times \mathfrak{su}(N)^2$ flavour symmetry (actually more before gauging), of which one $\mathfrak{su}(N)$ factor is gauged. This collection of N^2 free hypermultiplets is the tinkertoy associated to a sphere with two full and one simple puncture.

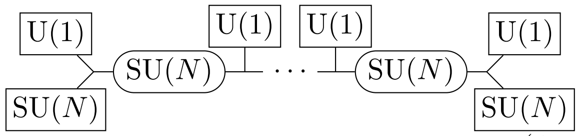
Following the general ideas from the $\mathfrak{su}(2)$ case the full punctures are described as a boundary condition like (4.2):

$$\varphi(z) \sim \Big(\frac{m_i}{z-z_i} + O(1)\Big)dz \implies \phi_k(z) = \bigg(\frac{(-1)^{k+1}\sigma_k(m_i)}{(z-z_i)^k} + O\Big(\frac{1}{(z-z_i)^{k-1}}\Big)\bigg)dz^2, \tag{4.22}$$

where $\sigma_k(m_i)$ is defined by the expansion $\det(X - m_i) = X^N + \sum_{k\geq 2}(-X)^k \sigma_k(m_i)$.

Linear quiver gauge theory. Starting with collections of N^2 free hypermultiplets, identifying pairs of $\mathfrak{su}(N)$ symmetries, and gauging them using vector multiplets, we find

$$\mathrm{T}\big(\mathfrak{su}(N), \mathbb{CP}^1 \setminus \{0, \underline{z_1}, \dots, \underline{z_{n-2}}, \infty\}\big) =$$

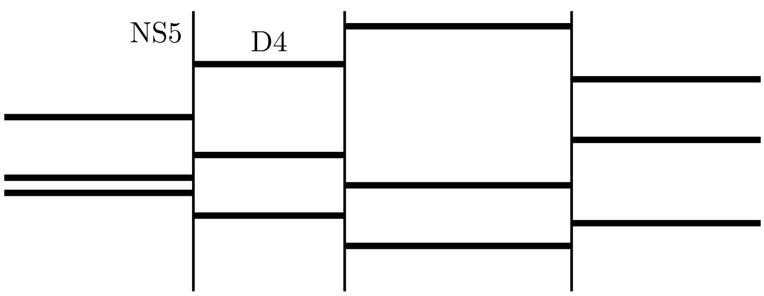


(4.23)

where we have underlined the simple punctures (so that only 0 and ∞ are full punctures). This is a linear quiver gauge theory. The free hypermultiplet tinkertoy is only enough to describe some of the degeneration limits of C: we only learned so far how to describe the theory if every pair of pants involves at least one simple puncture.

M-theory construction. We know that the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of a stack of N coincident M5 branes (minus the center of mass). The Riemann surface in (4.23) can be taken to be a cylinder, with some punctures. Then the brane setup can be described by N M5 branes wrapping the cylinder, with the insertion of transverse M5' branes at $n-2$ points on the cylinder.

Now M-theory on a circle is IIA string theory, M5 branes wrapping the circle become D4 branes, while M5' branes at points on the circle become NS5 branes. This gives a well-known brane set-up [149] with N D4 brane segments stretching between each pair of neighboring NS5 branes:



(4.24)

The world-volume description of this is known to be the linear quiver gauge theory (4.23). Mass parameters of the two SU(N) flavour symmetries are positions (vertically in the figure) of the semi-infinite D4 branes on either end. Mass parameters of all U(1) flavour symmetries are distances between centers of masses of each collection of N D4 branes. The remaining vertical positions are dynamical and appear on the gauge theory side as Coulomb branch parameters.

The SW curve and differential of the linear quiver can be extracted from this construction and coincides with what we find from the 6d perspective.

5 Localization for 4d quivers

Up to this point we have been working with 4d $\mathcal{N}=2$ class S theories in Minkowski space. We now turn[37] to Euclidean signature. Our aim in this section and the next is to explain both sides of the AGT relation (1.1) for the case $\mathfrak{g}=\mathfrak{su}(2)$ with tame punctures:

$$Z_{S^4}(\mathrm{T}(\mathfrak{su}(2),C,m)) = \langle V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\rangle_C^{\text{Liouville}}. \tag{5.1}$$

The AGT correspondence concerns the sphere (and squashed sphere) partition function. We explain how 4d $\mathcal{N}=2$ Lagrangian theories are placed on this curved background geometry in subsection 5.1, and in subsection 5.2 how the infinite-dimensional path integral is reduced to a finite-dimensional one (a matrix model) by supersymmetric localization in the Lagrangian case. The resulting expression is built from Nekrasov instanton partition functions, which we explore in subsection 5.3. Supersymmetric localization implies that some factorization properties remain true even for non-Lagrangian theories, see subsection 5.4.

5.1 Theories on an ellipsoid

Round sphere. To place an SCFT on a round sphere S^4 is in principle straightforward: just apply a conformal transformation to the flat space theory since the sphere is conformally flat. The 4d $\mathcal{N}=2$ superconformal algebra on S^4 is then the same as on $\mathbb{R}^4$, namely $\mathfrak{su}^*(4|2)$, whose bosonic part is the conformal algebra $\mathfrak{su}^*(4)=\mathfrak{so}(5,1)$ times the R-symmetry algebras $\mathfrak{u}(1)$ and $\mathfrak{su}^*(2)=\mathfrak{su}(2)$.

The class S theories we study (for tame punctures) are mass deformations of SCFTs. They can thus be placed on S^4 by conformally mapping the SCFT to the sphere, then turning on masses as background values for vector multiplet scalars coupled to the various flavour symmetries. Mass terms turn out to break half of supersymmetry, break the conformal algebra to the rotation algebra $\mathfrak{so}(5)=\mathfrak{usp}(4)$, and the R-symmetry to $\mathfrak{so}(2)=\mathfrak{so}^*(2)$. Altogether one can work out that massive theories preserve the supersymmetry subalgebra

$$\mathfrak{osp}^*(2|4)\subset\mathfrak{su}^*(2|4). \tag{5.2}$$

Note that this differs quite a bit from the Poincaré algebra preserved by massive theories on $\mathbb{R}^4$: for instance spatial isometries of $\mathbb{R}^4$ are $\mathfrak{iso}(4)=\mathbb{R}^4\rtimes\mathfrak{so}(4)$.

[37] We shall ignore possible difficulties with Wick rotation.

The AGT correspondence involves an ellipsoid (often called squashed sphere) S_b^4 and not only S^4. The squashed sphere is not conformally flat, and defining theories on this background requires technology that we now explain.

Conformal Killing vectors. We are interested in QFTs on rigid curved spaces (no dynamical gravity). Placing a Poincaré-invariant QFT on a curved space is done by coupling the theory to gravity and freezing the value of the metric[38]. As the next exercise shows, the resulting curved-space theory preserves some space-time (Poincaré) symmetries provided the metric admits *Killing vectors* Y_μ, defined by the Killing vector equation $\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$. More generally if the flat-space QFT is conformal, spatial symmetries are given by *conformal Killing vectors*

$$\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = \frac{2}{d} g_{\mu\nu} \nabla_\rho Y^\rho. \tag{5.3}$$

Exercise 5.1. *1. A Poincaré-invariant local QFT has a conserved stress-tensor $T^{\mu\nu}$ that is symmetric. Check that the current $Y_\mu T^{\mu\nu}$ is conserved if Y is a Killing vector.*

2. If the flat-space QFT is conformally invariant, $T^{\mu\nu}$ is traceless as well. Check that $Y_\mu T^{\mu\nu}$ is conserved if Y is a conformal Killing vector. Explain the factor $2/d$ in (5.3) *by taking the trace of the equation.*

Conformal Killing spinors. Consider now a supersymmetric theory. This means that there are conserved supersymmetry currents G^μ_α and $\tilde{G}^{\dot\alpha\mu}$, where μ is a vector index of the SO(4) rotation group, and $\alpha = 1, 2$ and $\dot\alpha = 1, 2$ are spinor indices with both chiralities. Leaving the spinor index of G^μ implicit, the conservation equation reads

$$D_\mu G^\mu := \nabla_\mu G^\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_a \gamma_b G^\mu = 0 \tag{5.4}$$

where ∇ is the Levi–Civita connection of the given metric, ω its spin connection, a, b are vielbein indices, and γ are Dirac matrices.

To get a usual conserved translation current from the conserved stress-tensor in flat space one contracts $T^{\mu\nu}$ with a constant translation vector a_μ. Likewise here we have usual currents $\xi G^\mu = \xi^\alpha G^\mu_\alpha$ and $\tilde\xi \tilde G^\mu = \tilde\xi_{\dot\alpha} G^{\dot\alpha\mu}$ for constant[39] spinors ξ. In curved space we can check that ξG^μ is conserved provided ξ is a Killing spinor:

$$D_\mu \xi := \left(\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_a \gamma_b \right) \xi = 0. \tag{5.5}$$

(Note that we put ∂ instead of ∇ because ξ has no vector index.)

[38] To be precise, if the theory has spinors one must additionally give a spin structure rather than only the metric (for instance giving a vielbein is enough).

[39] Here we use a common abuse of language: talking about constant spinors requires a choice of vielbein, for which we choose the standard Cartesian one on flat space.

Just as a theory is conformally invariant if $x_\mu T^{\mu\nu}$ is conserved, a theory is superconformally invariant if $x^\nu \gamma_\nu G^\mu$ is conserved in the same sense as (5.4). When put on curved space, the theory now has super(conformal) symmetries if the spacetime admits a conformal Killing spinor

$$D_\mu \xi = \frac{1}{d} \gamma_\mu \gamma^\nu D_\nu \xi. \tag{5.6}$$

Exercise 5.2. *Check that* (5.6) *indeed leads to a conserved current ξG if the theory is superconformal.*

Generalized Killing spinors. To define the (partial) topological twist in subsection 3.1, we have used a generalization that is available if the flat-space theory has an R-symmetry, which lead to a conserved current J^μ. When placing the QFT on curved space we can turn on a background gauge field V_μ coupled to J^μ in addition to the metric $g_{\mu\nu}$ that is coupled to $T^{\mu\nu}$ (we typically don't turn on fermionic backgrounds coupled to supersymmetry currents).

In such a background, the Killing spinor equation (5.5) generalizes by including the R-symmetry gauge field:

$$D_\mu \xi := \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b + i V_\mu \right) \xi = 0. \tag{5.7}$$

Here $V_\mu \xi$ should be suitably weighted by the R-charge under the given R-symmetry, as is standard for covariant derivatives in the presence of a gauge field. The conformal Killing spinor equation generalizes in the same way to (5.6) with the new D_μ. The background gauge field V can make it possible to preserve some supersymmetries even if the curved manifold of interest does not have any (conformal) Killing spinors.

Squashed sphere. The supergravity background found in [9] to place 4d $\mathcal{N} = 2$ theories on S_b^4 is rather complicated and gives non-zero values to most bosonic fields in the supergravity multiplet. We point to the review [117] for actual expressions. For our purposes we only need two aspects. The metric is the one induced from that of Euclidean $\mathbb{R}^5$ in the embedding

$$S_b^4 := \{ y_5^2 + b^2 (y_1^2 + y_2^2) + b^{-2} (y_3^2 + y_4^2) = r^2 \} \subset \mathbb{R}^5. \tag{5.8}$$

Parts of 4d $\mathcal{N} = 2$ supersymmetry remains: $\mathrm{U}(1)^2$ rotations in the y_1, y_2 and y_3, y_4 planes, and a supercharge Q such that

$$Q^2 = \frac{b}{r} (M_{12} - \tfrac{1}{2} J_3^{\mathrm{R}}) + \frac{1}{br} (M_{34} - \tfrac{1}{2} J_3^{\mathrm{R}}) \tag{5.9}$$

where J_3^{R} is the Cartan generator of $\mathfrak{su}(2)_{\mathrm{R}}$.

5.2 Supersymmetric localization

Many supersymmetric observables can be determined using the supersymmetric localization technique. The idea of localization is several decades old when applied to scalar supercharges of topologically twisted field theories. It received a new life since Pestun's calculation in 2007 [8] of the sphere partition function of 4d $\mathcal{N}=2$ theories, and of Wilson loop expectation values. In the following decade the technique was sucessfully applied to many dimensions (from 1d to 7d and even continuous dimensions) and geometries (such as spheres S^d, products $S^{d-1}\times S^1$, hemispheres and other spaces with boundaries), as summarized in the 2016 review volume [116]. We introduce here the technique and present in subsection 5.4 a lesser-known variant that explains various factorization properties.

Supersymmetric localization. Our goal is to compute a path integral

$$\langle\mathcal{O}\rangle = \int [D\phi] e^{-S}\mathcal{O} \tag{5.10}$$

that is invariant under some supercharge Q (we denote collectively all the fields as ϕ). This means that the action and path integration measure are Q-invariant ($QS=0$ and $\int[D\phi]Q(\text{anything})=0$) and that $Q\mathcal{O}=0$. Note that this is a path integral technique so we need Q to be a symmetry *off-shell*, not only on-shell.

Supersymmetric localization can often be used to reduce the calculation to an integral over Q-invariant field configurations. The rough intuition is that the integrand is invariant along orbits of Q in the space of field configurations, so the integral on each non-trivial orbit is the Grassmann integral of a constant, which gives zero.

The technique relies on deforming the action in the path integral in a way that does not change $\mathcal{O}$, and that suppresses contributions from most configurations, thus reducing the path integral down to a smaller space of configurations. Concretely, one needs some functional of the fields with three properties: it is Q-exact (namely of the form QV), Q-closed (so V is invariant under the bosonic symmetry Q^2), and has nonnegative bosonic part on the path integration contour we wish to consider. A typical choice is roughly speaking a sum over all fermions of the theory (collectively denoted as ψ) of the form

$$V = \sum_{\text{all spinors }\psi} \psi\overline{Q\psi} \implies (QV)_{\text{bosonic}} = \sum_{\psi} |Q\psi|^2 \tag{5.11}$$

for a suitable definition of $\overline{Q\psi}$.

Once such a term is chosen, we deform the action by tQV for $t\in[0,+\infty)$ and notice that the observable is unaffected since

$$\begin{aligned} \langle\mathcal{O}\rangle_t &= \int [D\phi] e^{-S-tQV}\mathcal{O} \\ \implies \partial_t\langle\mathcal{O}\rangle_t &= -\int [D\phi] e^{-S-tQV}\mathcal{O}\, QV = -\int [D\phi] Q\big(e^{-S-tQV}\mathcal{O}\, V\big) = 0. \end{aligned} \tag{5.12}$$

Here we used that $Q\mathcal{O}=0=Q(S+tQV)$ to write the integrand as Q of something.

The observable is t-independent, so we can take the limit $t \to \infty$, in which limit the saddle-point approximation becomes exact. In addition, any saddle with $(QV)_{\text{bosonic}} > 0$ is infinitely suppressed. Since we assumed $QV \geq 0$, in this limit we are left with an integral over field configurations with $QV = 0$ and the Gaussian integral of quadratic fluctuations around it:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_0 = \lim_{t \to \infty} \langle \mathcal{O} \rangle_t = \int_{QV=0} [D\phi] e^{-S[\phi]} Z_{\text{one-loop}}[\phi] \mathcal{O}[\phi]. \tag{5.13}$$

Here we wrote schematically $\int_{QV=0}$, but this may also involve discrete sums if the space of zeros of QV is disconnected. Here $Z_{\text{one-loop}}[\phi]$ is the result of a Gaussian integral of $\exp(-tQV)$ around a field configuration ϕ that is a zero of QV.

Let us summarize the steps in doing supersymmetric localization on some manifold M.

- Choose a supergravity background on M with at least one generalized conformal Killing spinor ξ, so that the theory on M has at least one supersymmetry Q.
- Find a fermionic functional V that is Q^2-invariant and has $(QV)_{\text{bosonic}} \geq 0$ on the path integral contour.
- Find zeros of $(QV)_{\text{bosonic}}$, which will be the resulting integration locus, often finite-dimensional.
- Expand $(QV)_{\text{bosonic}}$ to quadratic order around these zeros and compute the one-loop determinant (Gaussian integrals) $Z_{\text{one-loop}}$. This is often the most difficult step.
- Study the resulting integral (5.13) and extract physical predictions such as dualities, information about correlators, etc.

Saddle-points on ellipsoid. We now apply localization to 4d $\mathcal{N} = 2$ theories on the squashed sphere S^4_b. We take the standard deformation term (5.11) where the sum ranges over quarks ψ (hypermultiplet spinors) and gauginos λ (vector multiplet spinors). The resulting QV is pretty similar to the 4d $\mathcal{N} = 2$ action of these multiplets, and we only mention what is needed to determine the space $(QV)_{\text{bosonic}} = 0$.

Supersymmetric localization relies on the existence of a supercharge Q that is an *off-shell* symmetry. This requires the addition of some auxiliary fields K to the 4d $\mathcal{N} = 2$ theory. For now we focus on the sphere [8], restoring the squashing only in the final expressions [9].

For the hypermultiplet we have

$$(QV)_{\text{hyper,bosonic}} = |Dq|^2 + |D\tilde{q}|^2 + \cdots + \frac{R}{6}|q|^2 + \frac{R}{6}|\tilde{q}|^2 + |K_q|^2. \tag{5.14}$$

Here, R is the Ricci scalar: this term arises upon conformally mapping $|Dq|^2$ from flat space to the sphere. The "…" are a sum of squares, so the zero locus has the whole hypermultiplet set to zero:

$$q = \tilde{q} = K_q = 0 \tag{5.15}$$

and fermions as well since they are Grassmann variables.

For the vector multiplet we have similar terms with $q, \tilde{q}$ replaced by the vector multiplet scalar ϕ, but we also have terms like $|F_{\mu\nu}|^2$ and terms due to the supergravity background. Eventually (the bosonic part of) the deformation term can be massaged to a sum of squares of the form

$$\begin{aligned}(QV)_{\text{vector,bosonic}} = \frac{r-x^0}{2r}\left(F^-_{\mu\nu} + w^-_{\mu\nu}\operatorname{Re}\phi\right)^2 + \frac{r+x^0}{2r}\left(F^+_{\mu\nu} + w^+_{\mu\nu}\operatorname{Re}\phi\right)^2 \\ + |D\phi|^2 + [\phi,\phi^\dagger]^2 + |K_{\phi,i} + w_i \operatorname{Im}\phi|^2.\end{aligned} \tag{5.16}$$

Here F^- and F^+ are the (anti-)self-dual parts of the gauge field strength, $K_{\phi,i}$, $i = 1,2,3$ are auxiliary fields (a triplet of $\mathfrak{su}(2)_{\mathrm{R}}$), and $w^\pm_{\mu\nu}$ and w_i are determined by the supergravity background.

Let us find zeros of (5.16). Each term must vanish, so in particular ϕ is covariantly constant ($D\phi = 0$). Away from the poles $x^0 = \pm r$, we have $F^\pm_{\mu\nu} = -w^\pm_{\mu\nu}\operatorname{Re}\phi$ so $F_{\mu\nu} = -w_{\mu\nu}\operatorname{Re}\phi$ where $w = w^+ + w^-$. Then the Bianchi identities imply

$$0 = D^\mu F_{\mu\nu} = D^\mu(-w_{\mu\nu}\operatorname{Re}\phi) = -(\partial^\mu w_{\mu\nu})\operatorname{Re}\phi - w_{\mu\nu}\operatorname{Re}(\underbrace{D^\mu\phi}_{=0}). \tag{5.17}$$

In the specific supergravity background we have here, $\partial^\mu w_{\mu\nu} \neq 0$, so we learn that $\operatorname{Re}\phi = 0$, hence $F_{\mu\nu} = 0$. Thus, in a suitable gauge,

$$A_\mu = 0, \quad \phi = ia, \quad K_{\phi,i} = -w_i a \qquad \text{away from poles,} \tag{5.18}$$

where $a \in \mathfrak{g}$ is a (real) constant.

At the poles $x^0 = \pm r$, on the other hand, we only have one of the two equations $F^\pm_{\mu\nu} = -w^\pm_{\mu\nu}\operatorname{Re}\phi$, while the other part of $F_{\mu\nu}$ is unconstrained. This suggests to include point-like instanton configurations at the poles:

$$\text{instantons } (F^+ = 0) \text{ at } x^0 = r; \quad \text{anti-instantons } (F^- = 0) \text{ at } x^0 = -r. \tag{5.19}$$

Let us concentrate on the North pole $x^0 = r$. Instanton configurations are insensitive to matter, so that the instanton moduli space $\mathcal{M}$ is a product, over simple gauge group factors, of a moduli space of instantons for each gauge group. This, in turn, splits as a union of infinitely many connected components, labeled by the instanton number $k = \#\int \operatorname{Tr}(F\wedge F) \in \mathbb{Z}_{\geq 0}$ (for some calculable constant $\#$), with one instanton number per gauge group.

Result. Saddle-point configurations defined by (5.15), (5.18), (5.19) are thus characterized by a choice, for each gauge group, of a Coulomb branch parameter a and point-like (anti-)instanton configurations at the poles. For any such saddle-point we compute the classical action

$$S_{\text{cl}} = \operatorname{Re}(2\pi i\tau)\operatorname{Tr}(ra)^2 + 2\pi i\tau n - 2\pi i\overline{\tau n}, \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \tag{5.20}$$

where the radius r of the (squashed) sphere (5.8) makes the first term dimensionless.

The pedestrian way to compute one-loop determinants is to decompose fluctuations into modes (spherical harmonics), essentially diagonalizing the operators whose determinant we are computing. There are lots of cancellations between bosons and fermions. A more clever way is to avoid ever writing the factors that cancel: this is done by pairing bosonic and fermionic modes and only computing contributions from modes that are not paired. Finally, this latter calculation can be much simplified by using powerful index theorems. We refer to [117] for details in our S^4_b setting.

One eventually finds that the one-loop determinant (Gaussian integral) for quadratic fluctuations around the saddle-point with zero (anti-)instantons is

$$Z_{\text{one-loop}} = Z^{\text{vector}}_{\text{one-loop}} Z^{\text{hyper}}_{\text{one-loop}} \quad \text{for } n = \overline{n} = 0. \tag{5.21}$$

Here, the vector multiplet one-loop determinant is a product over roots α of all gauge group factors (non-zero weights of the adjoint representation),[40]

$$Z^{\text{vector}}_{\text{one-loop}} = \prod_{\alpha\in\Delta} \Upsilon_b(ira\cdot\alpha), \tag{5.22}$$

where Υ_b is a special function defined in Appendix A. The hypermultiplet one-loop determinant is a product over weights w (with multiplicity) of the representation in which the hypermultiplet transforms,

$$Z^{\text{hyper}}_{\text{one-loop}} = \prod_{w\in R} \frac{1}{\Upsilon_b(\frac{b+1/b}{2} + ira\cdot w)}. \tag{5.23}$$

As explained previously, hypermultiplet masses are simply background values for vector multiplet scalars corresponding to flavour symmetries, so adding a mass m in (5.23) simply changes

$$\Upsilon_b\Big(\frac{b+1/b}{2} + ira\cdot w\Big) \to \Upsilon_b\Big(\frac{b+1/b}{2} + ira\cdot w + irm\Big). \tag{5.24}$$

Importantly, hypermultiplets in the representations R and $\overline{R}$ are equivalent (with mass $m \to -m$) and one checks that the symmetry $\Upsilon_b(b + 1/b - x) = \Upsilon_b(x)$ ensures that the one-loop determinant (5.23) computed with both presentations is the same. For a half-hypermultiplet in a pseudoreal representation $R \simeq \overline{R}$ one should keep only one factor for each pair of conjugate weights; thanks to the same symmetry of Υ_b it does not matter which weight one selects in each pair. As expected all factors are invariant under $b \to 1/b$ thanks to $\Upsilon_b = \Upsilon_{1/b}$.

One-loop determinants can be further understood as products of contributions from both hemispheres, essentially by decomposing each Υ_b function as $\Upsilon_b(x) = 1/(\Gamma_b(x)\Gamma_b(b+$

[40] To be precise, we have included here the Vandermonde determinant $\prod_{\alpha\in\Delta}(ira\cdot\alpha)$ that arises when converting from an integral over the whole gauge algebra $\mathfrak{g}$ to its Cartan subalgebra.

$1/b - x)$). (Anti-)instantons at each pole only affect one-loop contributions from the corresponding hemisphere, which leads to a factorization property of the form

$$Z_{\text{one-loop}}(a, n, \xi, \overline{n}, \xi) = Z_{\text{one-loop}}(a) Z_{\text{one-loop,inst}}(a, n, \xi) Z_{\text{one-loop,inst}}(a, \overline{n}, \overline{\xi}) \tag{5.25}$$

where $Z_{\text{one-loop}}(a)$ is the ratio of Υ_b written above.

Altogether, collecting all (anti-)instanton contributions together, including the classical contributions expressed in terms of $q = \exp(2\pi i \tau)$, the partition function reads

$$Z_{S^4_b} = \int da\, Z_{\text{cl}}(a, q\overline{q}) Z_{\text{one-loop}}(a) Z_{\text{inst}}(a, q) Z_{\text{inst}}(a, \overline{q}) \tag{5.26}$$

$Z_{\text{one-loop}}$ is given above, $Z_{\text{cl}} = \exp(-S_{\text{cl}}) = |q|^{\operatorname{Tr}(ra)^2}$, and there remains to compute the instanton partition functions.

5.3 Instantons

The point-like configurations in (5.19) are only sensitive to the leading expansion of the supergravity background around the poles. This supergravity background has a flat metric and a non-trivial graviphoton, and coincides with the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ discovered by Nekrasov [10], with parameters $\epsilon_1 = b/r$, $\epsilon_2 = 1/(rb)$. Pestun thus naturally conjectured in [8] that the function $Z_{\text{inst}}(a, q) = 1 + O(q^1)$ is the partition function of the 4d $\mathcal{N} = 2$ theory on the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$, called Nekrasov's instanton partition function [10, 11]. We refer to reviews [107, 118] for a more detailed introduction to Z_{inst}.

General features of the instanton partition function. The instanton partition function is an integral over the moduli space of instantons with an integrand depending on matter. As mentioned above, this moduli space decomposes as a product over simple gauge group factors, and the moduli space for each gauge group factor has one connected component for each instanton number $k \geq 0$, with increasing dimensions. For classical gauge groups SU, USp, SO the n-instanton moduli space can be realized as a symplectic quotient through the ADHM construction [150], while no such constructions are available in general for the exceptional groups $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$.

The Omega background tends to $\mathbb{R}^4$ as $\epsilon_1, \epsilon_2 \to 0$, and can be understood as a regulator for IR divergences due to non-compactness of $\mathbb{R}^4$. In fact, as explained in [10, 11] and the appendix of [151], the partition function gives the low-energy prepotential of the gauge theory:

$$F(a, q) = F_{\text{pert}}(a, q) + \lim_{\epsilon_1,\epsilon_2 \to 0} \Big(\epsilon_1 \epsilon_2 \log Z_{\text{inst}}(a, q; \epsilon_1, \epsilon_2) \Big), \tag{5.27}$$

where F_{pert} results from a one-loop computation. In this way, the instanton partition function gives access to the low-energy dynamics of the 4d $\mathcal{N} = 2$ theory at a point a along the Coulomb branch.

The Omega background can also be obtained as the $\beta \to 0$ limit of a 5d background $S^1 \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$ defined as the quotient of $\mathbb{R} \times \mathbb{C} \times \mathbb{C}$ under the identification $(x, z_1, z_2) \sim (x + \beta, e^{i\beta\epsilon_1} z_1, e^{i\beta\epsilon_2} z_2)$. Any 4d $\mathcal{N} = 2$ Lagrangian theory can be lifted to a 5d $\mathcal{N} = 1$ theory and the instanton partition function Z_{inst} has a 5d analogue defined as the partition function on $S^1 \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$.

Computation methods. There are many methods to compute the instanton partition function.[41]

- For classical gauge groups, the ADHM construction expresses the instanton moduli space as a symplectic quotient, or physically as the vacuum moduli space of a supersymmetric quantum mechanics theory. When hypermultiplets are in suitable representation (such as the fundamental or bifundamental), Z_{inst} can be computed by supersymmetric localization with respect to U(1) symmetries. This reduces Z_{inst} from an integral over the whole moduli space down to a discrete sum of contributions from collections of point-like instantons respecting these symmetries. We give the resulting formula for $\mathrm{U}(N)$ gauge theories in (5.29), below. This Losev–Moore–Nekrasov–Shatashvili (LMNS) formula was obtained in [152–154], derived in [10], and extended to other classical groups in [155, 156] and to supergroup theories in [157].

- For exceptional gauge groups there is no ADHM construction. The instanton partition function for pure SYM theory with E_n gauge group can be determined from the Hall–Littlewood index of the E_n Minahan-Nemeschansky theory, calculated using the TQFT realization of this index discussed in subsection 9.2. Alternatively, one uses the 3d $\mathcal{N} = 4$ mirror of the E_n theory, whose Coulomb branch index leads to the Nekrasov partition function. An appropriate "folding" construction yields the result for F_4 and G_2 pure SYM.

- One can write recursion relations for the instanton partition function by comparing it with the partition function on the blow-up of $\mathbb{C}^2$ (see [158] and references therein). For hypermultiplets (but not half-hypermultiplets) in a large class of representations of both classical and exceptional gauge groups, one can solve these recursion relations and deduce Z_{inst} from the perturbative (one-loop) partition function.

Some matter representations foil all the available methods: most notably, general Lagrangians constructed from SU(2) vector multiplets and trifundamental half-hypermultiplets, which are the Lagrangian descriptions of SU(2) class S theories. Among these theories, Z_{inst} is known whenever the theory can be written with gauge groups SU(2) and $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$ and bifundamental matter (of two SU(2) groups or of an SU(2) and an SO(4) group); see [88, 90, 159].

[41] I thank Jaewon Song for correspondence on some methods.

Explicit formula for linear quiver gauge theories. We focus now on the linear quiver gauge theory

$$\begin{array}{ccccccc} \boxed{M_1} & & \boxed{M_2} & & & & \boxed{M_p} \\ | & & | & & & & | \\ \mathrm{SU}(N_1) & - & \mathrm{SU}(N_2) & - & \cdots & - & \mathrm{SU}(N_p) \end{array} \tag{5.28}$$

This theory has gauge group $\prod_{i=1}^{p} \mathrm{SU}(N_i)$, one hypermultiplet in each bifundamental representation $N_i \otimes \overline{N_{i+1}}$, and M_i hypermultiplets transforming in the fundamental representation N_i of each group.

More precisely, the instanton partition function of (5.28) is computed by extending the gauge groups from $\mathrm{SU}(N_i)$ to $\mathrm{U}(N_i)$, then removing spurious factors due to the additional $\mathrm{U}(1)$ gauge groups. Each gauge factor has is own coupling constant and correspondingly the dynamical scale Λ_i. We denote the Coulomb branch parameter $a^i = \{a^i_1, \ldots, a^i_{N_i}\}$ for $1 \leq i \leq p$. In the IIA construction, instantons are realized as D0 branes and the instanton moduli space is the vacuum moduli space of their world-volume theory. The partition function is the matrix model contour integral

$$Z_{\text{inst}} = \int d\phi \frac{\prod\limits_{i,F,I}(m^i_F - \phi^i_I) \prod\limits_{i=1}^{p-1}\left[\prod\limits_{I,J} S(\phi^{i+1}_J - \phi^i_I) \prod\limits_{A,J}(\phi^{i+1}_J - a^i_A + \epsilon_1 + \epsilon_2) \prod\limits_{I,B}(a^{i+1}_B - \phi^i_I)\right]}{\prod\limits_{i}\left[(\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2})^{k_i} \prod\limits_{I\neq J} S(\phi^i_I - \phi^i_J) \prod\limits_{A,I}(\phi^i_I - a^i_A + \epsilon_1 + \epsilon_2)(a^i_A - \phi^i_I)\right]} \tag{5.29}$$

where $S(\phi) = -(\phi+\epsilon_1)(\phi+\epsilon_2)/[\phi(-\phi-\epsilon_1-\epsilon_2)]$, indices run over the ranges that are natural given where they appear ($1 \leq i \leq p$, $1 \leq F \leq M_i$, $1 \leq I \leq k_i$, $1 \leq J \leq k_{i+1}$, $1 \leq A \leq N_i$, $1 \leq B \leq N_{i+1}$), and $d\phi$ denotes a product of all $d\phi^i_I$. The first factor in the numerator captures the effect of fundamental hypermultiplets, the rest of the numerator comes from bifundamental hypermultiplets, and the denominator from vector multiplets.

The contour in (5.29) is such that it enclose poles at

$$\{\phi^i_I \mid 1 \leq I \leq k_i\} = \{a^i_A + (r-1)\epsilon_1 + (s-1)\epsilon_2 \mid (r,s) \in \lambda^i_A\} \tag{5.30}$$

for each collection of Young diagrams λ^i_A, $1 \leq i \leq p$, $1 \leq A \leq N_i$ with a total number of boxes equal to the instanton number, $\sum_{A=1}^{N_i} |\lambda^i_A| = k_i$. The set of poles (5.30) can also be understood from a Jeffrey–Kirwan (JK) residue prescription, and for this purpose the odd placement of signs in $S(\phi)$ is important.

The explicit formula for Z_{inst} can also be understood from the IIA string theory construction of the $\prod_i \mathrm{U}(N_i)$ gauge theory, depicted in Figure 3. In the IIA construction, instantons are D0 branes stretching between pairs of neighboring NS5 branes. The vector multiplet contribution (denominator) arises from the interaction of D0 branes in a given interval ($1/S$ factors) and D0 and D4 branes (the remaining factors). The fundamental hypermultiplet contribution arises from the interaction with D6 branes. The bifundamental hypermultiplet contribution arises from the interaction of D0 and D4 branes on one side of an NS5 brane and D0 and D4 branes on the other side, *without* the D4-D4 interaction because that is already taken into account in $Z_{\text{one-loop}}$.

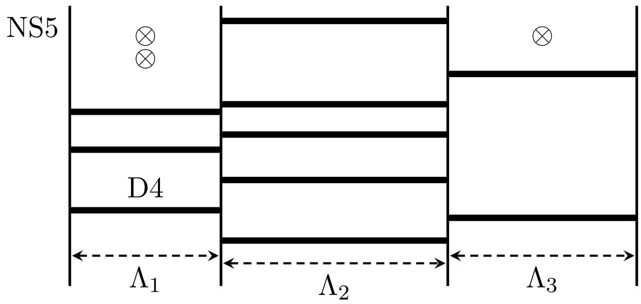


Figure 3: Brane construction of a $\mathrm{U}(3) \times \mathrm{U}(5) \times \mathrm{U}(2)$ gauge theory. The leftmost $\mathrm{U}(3)$ factor has two fundamental hypermultiplets (inserted by transverse D6 branes) and the rightmost $\mathrm{U}(2)$ factor has one fundamental hypermultiplet.

> **Exercise 5.3.** *Specialize these formulas to* $\mathrm{U}(2)$ *SQCD with* $N_f = 4$ *flavours* ($p = 1$, $N_1 = 2$, $M_1 = 4$)*. Note that the four mass parameters* m_B^1, $B = 1, \ldots, 4$ *can all be shifted by shifting the Coulomb branch parameters* a_A^1, $A = 1, 2$*. This reflects the fact that in the* $\mathrm{U}(2)$ *theory we have only* $\mathrm{SU}(N_f)$ *flavour symmetry, not* $\mathrm{SO}(2N_f)$ *like for* $\mathrm{SU}(2)$ *SQCD. Compute the one-instanton contribution in* Z_{inst}*. Start computing the two-instanton contribution to get an idea of the complexity: write the integrand and list the poles.*

$\mathrm{U}(1)$ **factor and renormalization schemes.** [35]

5.4 Cutting by localization

We have presented supersymmetric localization so far for Lagrangian theories, as it relies on a path-integral formulation.[35] Higher-rank class S theories are typically non-Lagrangian, obtained by gauging common flavour symmetries of certain isolated SCFTs called tinkertoys (most notably T_N). In particular, they involve usual vector multiplets, and we explain now how to apply localization to these vector multiplets.

In the absence of a path-integral formulation we cannot deform the action $S \to S + tQV$ as before, nor insert e^{-tQV} in the path integral, but we can still consider the expectation value $\langle e^{-tQV} \rangle$ or more generally $\langle \mathcal{O} e^{-tQV} \rangle$. In the same way as in (5.12) this expectation value is t-independent. The idea then is to add the deformation term (5.11) for the vector multiplet.[35]

6 AGT for $\mathrm{SU}(2)$ quivers

We now explain 2d Liouville CFT, namely the right-hand side of the AGT relation (1.1) for $\mathfrak{g} = \mathfrak{su}(2)$, which states

$$Z_{S^4}(\mathrm{T}(\mathfrak{su}(2), C, m)) = \langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n) \rangle_{\overline{C}}^{\text{Liouville}}. \tag{6.1}$$

6.1 Liouville CFT

We have seen how reducing $\mathcal{X}(\mathfrak{su}(2))$ on a Riemann surface $C = \overline{C} \setminus \{z_1, \dots, z_n\}$ yields a 4d $\mathcal{N} = 2$ $\mathfrak{su}(2)$ gauge theory. Likewise, we expect that reducing $\mathcal{X}(\mathfrak{su}(2))$ on S_b^4 should yield a 2d theory with a coupling constant b. The 4d theory depends on codimension 2 defects of $\mathcal{X}(\mathfrak{su}(2))$ inserted at punctures z_i of C, and in the 2d theory these should be local operators.

We can thus expect a relation of the form (6.1) for any number n of puncture, where $V_{\alpha_i}(z_i)$ are the reductions of codimension 2 operators down to points. As we will show, the 2d theory is the well-known Liouville CFT described below, and V_α are conformal primary operators. We refer to reviews such as [119, 120] for more on Liouville CFT.

Spectrum and three-point functions. We have seen that the 4d theory $\mathrm{T}(\mathfrak{su}(2), C, m)$ only depends on the complex structure of C, hence only on the conformal class of the metric on C. This means that the 2d theory we seek should be a CFT. Another indication is that the partition function $\mathrm{T}(\mathfrak{su}(2), C, m)$ admits a factorized form for each decomposition of C into three-punctured spheres, reminiscent of how 2d CFT correlators are decomposed into three-point functions and conformal blocks.

A 2d CFT is characterized by its spectrum (left and right conformal dimensions of primary operators) and OPE structure constants (equivalently, three-point functions of conformal primary operators). When constructing class S theories from $\mathcal{X}(\mathfrak{su}(2))$, the data associated to a puncture is a mass parameter $m \in \mathbb{R}/\mathbb{Z}_2$. We thus want local operators V with a continuous parameter. For consistency with later notations we denote this (dimensionless) parameter as $\alpha = (b + 1/b)/2 + irm$.

Determining the conformal dimension of V_α will have to wait; let us begin with three-point functions. We know that the theory associated to a three-punctured sphere is a trifundamental half-hypermultiplet. Its partition function is a hypermultiplet one-loop determinant (5.23), so that the three-point function is

$$\begin{aligned} C_{\alpha_1\alpha_2\alpha_3} &:= \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle = \prod_{\pm\pm} \frac{1}{\Upsilon_b(\alpha_1 \pm (\alpha_2 - (b+1/b)/2) \pm (\alpha_3 - (b+1/b)/2))} \\ &= \frac{1}{\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-b-1/b)} \end{aligned} \tag{6.2}$$

using the invariance $\Upsilon_b(x) = \Upsilon_b(b + 1/b - x)$.

Second, to determine the conformal dimension of V_α we consider a four-punctured sphere and cut it in a channel suitable for the $q \to 0$ limit, where q is the cross-ratio of the four punctures. The gauge theory corresponding to a four-punctured sphere is $\mathfrak{su}(2)$ $N_f = 4$ SQCD, and its partition function, computed using supersymmetric localization, takes the form (5.26)

$$Z_{S_b^4} = \int_{\mathbb{R}/\mathbb{Z}_2} da\, |q|^{\operatorname{Tr}(ra)^2} Z_{\text{one-loop}}(a) Z_{\text{inst}}(a,q) \overline{Z_{\text{inst}}}(a,\overline{q}). \tag{6.3}$$

In the $q \to 0$ limit, $Z_{\text{inst}} \to 1$. This expression should be compared to the decomposition of a four-point function in 2d CFT,

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(q) V_{\alpha_3}(1) V_{\alpha_4}(\infty)\rangle = \int d\alpha\, |q|^{\Delta(\alpha)} C_{\alpha_1\alpha_2\alpha} C_{\alpha\alpha_3\alpha_4} \langle V_\alpha V_\alpha\rangle^{-1} \mathcal{F}(\alpha_i, \alpha; q) \mathcal{F}(\alpha_i, \alpha; \overline{q}) \tag{6.4}$$

in which $\mathcal{F}$ are conformal blocks that depend (anti-)holomorphically on the cross-ratio q, and tend to 1 as $q \to 0$.

We have already identified the three-point functions C to hypermultiplet one-loop determinants, so it is natural to identify the inverse two-point function $\langle V_\alpha V_\alpha\rangle^{-1}$ to the vector multiplet one-loop determinant, the conformal blocks to instanton partition functions, and to identify $\Delta(\alpha)$ and $\operatorname{Tr}(ra)^2$, the powers of q.

We refer to Marilena's lectures and to [119, 120] for an explanation of conformal blocks in 2d CFT.

Summary of Liouville CFT. The spectrum and three-point functions we have found coincide with those of Liouville CFT, which is the 2d CFT constructed as follows. It describes a single scalar field subject to the action

$$S[\phi] = \frac{1}{4\pi} \int d^2 z\, \sqrt{g}\, (\partial_\nu \phi \partial^\nu \phi + QR\phi + 4\pi\mu\, e^{2b\phi}) \tag{6.5}$$

where R is the Ricci scalar. Provided $Q = b + 1/b$ this theory is conformal, with holomorphic stress-tensor $T = (\partial\phi)^2 + Q\partial^2\phi$. One can check that $:e^{2\alpha\varphi}:$ are conformal primary operators of dimension $\Delta(\alpha) = \alpha(b+1/b-\alpha)$, which is exactly what we obtained above from the classical Yang–Mills action. The three-point function is known to be given by the Dorn–Otto–Zamolodchikov–Zamolodchikov (DOZZ) formula

$$\langle :e^{2\alpha_1\varphi}::e^{2\alpha_2\varphi}::e^{2\alpha_3\varphi}:\rangle = \frac{(b^{2/b-2b}\lambda)^{Q-\alpha_1-\alpha_2-\alpha_3} \Upsilon_b'(0) \Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}, \tag{6.6}$$

where $\lambda = \bigl(\frac{\pi\Gamma(b^2)}{\Gamma(1-b^2)}\mu\bigr)^{1/b}$. The denominator here is identical to the three-point function (6.2) obtained from gauge theory. The numerator factorizes into factors that each depend on a single α_i, hence can be absorbed by a suitable normalization of the primary operators,

$$V_\alpha = (b^{2/b-2b}\lambda)^{\alpha-Q/3} \Upsilon_b'(0)^{-1/3} \Upsilon_b(2\alpha)^{-1} :e^{2\alpha\varphi}:. \tag{6.7}$$

It is also know that $:e^{2\alpha\varphi}: = R(\alpha) :e^{2(Q-\alpha)\varphi}:$ with the reflection coefficient

$$R(\alpha) = -\lambda^{Q-2\alpha} \frac{\Gamma(b(2\alpha-Q))\Gamma(\frac{1}{b}(2\alpha-Q))}{\Gamma(b(Q-2\alpha))\Gamma(\frac{1}{b}(Q-2\alpha))}. \tag{6.8}$$

Using the shift relations given in Appendix A, this translates to the symmetry

$$V_\alpha = \frac{1}{b^2(Q-2\alpha)^2} V_{Q-\alpha} \tag{6.9}$$

The two-point function $\langle :e^{2\alpha\varphi}: :e^{2(Q-\alpha)\varphi}:\rangle = 1$ is normalized and this translates to

$$\langle V_\alpha V_{Q-\alpha}\rangle = \frac{(b^{2/b-2b}\lambda)^{Q/3}\Upsilon_b'(0)^{-2/3}}{\Upsilon_b(Q-2\alpha)\Upsilon_b(Q+2\alpha)}. \tag{6.10}$$

One can match this to the vector multiplet one-loop determinant.

6.2 Liouville from 6d

We have argued that the 6d theory $\mathcal{X}(\mathfrak{su}(2))$ reduces to Liouville CFT upon dimensional reduction along S_b^4, and the many checks of the AGT correspondence validate this. It is natural to ask for a direct derivation of the dimensional reduction. This was done to some extent by Córdova and Jafferis in [16].[35]

They treat S_b^4 as a squashed three-sphere S_b^3 fibered over an interval: in the notations of (5.8) the interval is parametrized by $y_5 \in [-r, r]$ and the S_b^3 degenerates to a point at both ends. The 6d theory $\mathcal{X}(\mathfrak{su}(2))$ (and more generally $\mathcal{X}(\mathfrak{g})$) reduced on S_b^3 yields complex Chern–Simons theory with gauge group $G_\mathbb{C}$ on a warped product $C \times \mathbb{R}$. The fact that S_b^3 collapses to a point at each pole $y_5 = \pm r$ translates to boundary conditions at the extremities of $\mathbb{R}$ in $C \times \mathbb{R}$. The edge modes coming from each extremity are then understood to be described by chiral complex Toda theory. Combining these two chiral theories gives complex Toda CFT. Finally, there is some evidence that complex Toda CFT is dual to the ordinary Toda CFT, thus completing the derivation.

7 General class S theories

In this section we extend the story in two ways.[35] First, in subsection 7.1, we consider interesting limits where two punctures collide while the parameters describing the defects are appropriately scaled. The resulting class S theories include asymptotically-free gauge theories (such as SU(2) SQCD with $N_f < 4$), and AD theories [2]. Second, in subsection 7.2, we generalize from $\mathfrak{su}(2)$ to $\mathfrak{su}(N)$ gauge groups, in which case the Liouville CFT is replaced by the Toda CFT.

7.1 Irregular punctures and Argyres–Douglas theories

Wild punctures from collisions. We recall the massive tame puncture (4.2)

$$\varphi(z) \sim \left(\frac{\mathrm{diag}(m,-m)}{z-z_i} + O(1)\right)dz \implies \phi_2(z) = \left(\frac{m^2}{(z-z_i)^2} + O\left(\frac{1}{z-z_i}\right)\right)dz^2 \tag{7.1}$$

and its massless version (4.4). Colliding l such simple poles of φ, while scaling appropriately the mass parameters, leads to a pole of order l, hence generically to $\phi_2 \sim dz^2/(z-z_i)^{2l}$.

Just as their tame counterparts, the resulting *wild punctures* of level l can be partially closed by imposing some relations between eigenvalues in the series expansion of Φ_z, so that the pole of ϕ_2 has an order lower than $2l$. The collision limits can have two main effects on the 4d gauge theory: decoupling some hypermultiplets by making them massive while keeping the dynamical scale Λ fixed, or tuning the theory to an AD point on the Coulomb branch [2, 160–162] at which point the theory becomes a strongly-coupled isolated SCFT.

For the case $\mathfrak{g} = \mathfrak{su}(2)$ that we consider for now, wild punctures are labeled by the order of the pole of ϕ_2 (which is 2 for a tame puncture), and of course by coefficients of the expansion at these poles. By cutting the Riemann surface along circles as in the tame case, $\mathfrak{su}(2)$ class S theories can be constructed by gauging SU(2) flavour symmetries of the trifundamental half-hypermultiplet T_2 (corresponding to a sphere with three tame punctures) and of theories X_p corresponding to a sphere with a tame puncture and a wild puncture at which ϕ_2 has a pole of order $p > 2$. Spheres with a single wild puncture cannot be cut into these building blocks and lead to other interesting theories Y_p. This exhausts $\mathfrak{su}(2)$ class S.

Exercise 7.1. *Recall from the SW curve* (4.13) *of the* $\mathfrak{su}(2)$ *class S theory for a four-punctured sphere:* $x^2 = u_2(z)$ *with*

$$u_2(z) = \frac{\frac{q}{z}m_1^2 + \frac{q(q-1)}{z-q}m_2^2 + \frac{z-q}{z-1}m_3^2 + zm_4^2 - u}{z(z-q)(z-1)}. \tag{7.2}$$

This theory has a description as SU(2) *SQCD with gauge coupling* $\tau = (\log q)/(2\pi i)$ *and* $N_f = 4$ *flavours of masses* $m_1 \pm m_2$ *and* $m_3 \pm m_4$*.*

1. Decouple one hypermultiplet: take $m_1 + m_2 \to \infty$*, keeping* $m_1 - m_2$ *and* $m_3 \pm m_4$ *and* $\Lambda = q(m_1 + m_2)$ *fixed. You should get* $u_2 = P(z)/(z^4(z-1)^2)$ *for some quartic polynomial* P*.*

2. Decouple a second hypermultiplet in two ways. First, take $m_1 - m_2 \to \infty$*, keeping* $\Lambda'^2 = \Lambda(m_1 - m_2)$ *and other masses fixed. Second, instead, take* $m_3 + m_4 \to \infty$*, keeping* $\tilde{z} = z(m_3 + m_4)$ *and* $\tilde{x} = x/(m_3 + m_4)$ *and* $\Lambda'^2 = \Lambda(m_3 + m_4)$ *and other masses fixed. Map one SW curve to the other and check the difference of SW differentials* λ *is inessential (residues are masses, no Coulomb branch dependence).*

3. Decouple a third and a fourth hypermultiplet and rescale $z \to z\Lambda^2$ *to get the well-known curve of pure* SU(2) *SYM:* $z^2x^2 = u + \Lambda^2(z + 1/z)$ *with* $\lambda = xdz$*.*

Examples of theories with wild punctures. Just for this explanation we denote by $(p_1\, p_2 \dots p_k)$ the class S theory obtained for a sphere with k punctures at which ϕ_2 has poles of order $p_1, \dots, p_k$, respectively. Let us exemplify both effects above starting from SU(2) $N_f = 4$ SQCD, realized as (2 2 2 2) in these notations, namely by taking C to be a sphere with four tame punctures. We first decouple hypermultiplets.

- SU(2) $N_f = 3$ SQCD arises from (2 2 4), a sphere with two tame punctures and one wild puncture of order 4, obtained as a collision of two tame punctures.
- SU(2) $N_f = 2$ SQCD appears in two ways in class S. First, as (4 4) obtained from (2 2 4) by colliding the two tame punctures. Alternatively, as (2 2 3): one can decouple the hypermultiplet by tuning a mass parameter of the wild puncture in the (2 2 4) description of the $N_f = 3$ theory, and this reduces the pole of ϕ_2 at the wild puncture from order 4 to order 3. A consistency check is that the two constructions lead to equivalent SW geometry.
- SU(2) $N_f = 1$ SQCD then appears as (4 3).
- Pure SU(2) SYM appears as (3 3) with two minimally wild punctures.

There are further collision limits, which turn out to realize AD theories. By colliding the two wild punctures in the (4 3) realization of SU(2) $N_f = 1$ SQCD we get a single wild puncture of rather high order (7). The AD point (most singular point) of the Coulomb branch of SU(2) $N_f = 2$ is obtained by colliding punctures $(4\,4) \to (8)$ or $(2\,2\,3) \to (2\,5)$, both punctured curves C turning out to give the same 4d SCFT. For SU(2) $N_f = 3$ we find the collision $(2\,2\,4) \to (2\,6)$. Of course, these limits all translate to tuning parameters on the gauge theory side and were thus found a long time ago [2, 160], but the class S realization embeds these in a broader setting.

CFT side. On the 2d CFT side, the limits that produce wild punctures correspond to colliding primary vertex operators V_α. This yields so-called irregular operators, whose Ward identities with the stress tensor involve poles of the same order p as the pole of ϕ_2 on the gauge theory side, consistent with the fact that ϕ_2 can be understood as the semiclassical limit of T:

(7.3)

By the state-operator correspondence, these operators give coherent states of the Virasoro algebra [87] (alternatively called Whittaker vectors or Gaiotto states) and generalizations thereof called irregular states (or BMT states) [163, 164].

7.2 Higher rank and Toda CFT

Higher-rank gauge group. The AGT correspondence [5] extends beyond $\mathfrak{su}(2)$ to $\mathfrak{su}(N)$ [86] and all simply-laced $\mathfrak{g}$ [88, 90, 91]. Liouville CFT is generalized to the Toda CFT associated to $\mathfrak{g}$. This theory has a $W_{\mathfrak{g}}$ symmetry algebra generalizing Virasoro: it has one spin k current $W^{(k)}(z)$ for each k-th differential ϕ_k that shows up in the construction (1.4) of the SW curve. In particular $W^{(2)}(z) = T(z)$ is the holomorphic stress tensor. Instead of $P \in \mathbb{R}/\mathbb{Z}_2$, the momenta of vertex operators V_α now belong to the Cartan subalgebra of $\mathfrak{g}$ modulo the Weyl group. These momenta are (diagonalizations of) r times the mass parameter $m \in \mathfrak{g}_{\mathbb{C}}$ at a given tame puncture. Ward identities of $W^{(k)}$ with primary vertex operators V_α reduce in the classical limit $r \to \infty$ to singularities of ϕ_k near tame punctures as in (1.10).

The main building block $T_{\mathfrak{g}}$ of class S theories, which is the tinkertoy for three full tame punctures, is non-Lagrangian for $\mathfrak{g} \neq \mathfrak{su}(2)$. Partition functions of class S theories can in general be reduced as in (1.7), but one-loop factors and instanton contributions cannot be computed in presence of $T_{\mathfrak{g}}$ factors. On the CFT side, once correlators are decomposed into sums of products of three-point functions to get (1.9), we get stuck because only some of these three-point functions can be written in terms of three-point functions of primary vertex operators, which themselves are not known. These issues make defining and studying the AGT correspondence much more challenging.

Besides full tame punctures, there are other tame punctures (and of course a host of wild punctures). For some choices, such as the one depicted in Figure 1 in the introduction, the resulting class S theory has a Lagrangian description. The corresponding Toda CFT correlators are typically such that all three-point functions can be deduced from those of primary operators. In these cases, the AGT correspondence can be checked explicitly by comparing instanton expansions to conformal block expansions.

Inserting twist operators into correlators corresponds to suitably orbifolding the gauge theory. In this way one realizes non-simply-laced gauge groups.

8 Operators of various dimensions

Wilson [168] and 't Hooft [169] loop operators, and dyonic loops combining them [170], play an important role in studying phases of 4d gauge theories. Surface operators are less studied but capture a lot of interesting structure of some gauge theories. Finally, domain walls describe interfaces between two 4d theories, or boundary phenomena. A large source of such operators in the AGT correspondence are the half-BPS codimension 2 and codimension 4 defects of the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$. Another source is to orbifold the 6d setup, with an orbifold group that must respect orientation since $\mathcal{X}(\mathfrak{g})$ is a chiral theory. These defects can be inserted into the AGT correspondence with various orientations relative to the product spacetime $M_4 \times C$. We refer to Table 1 in the introduction for a list of possibilities. Here, we organize our discussion by increasing dimension on the 4d side, starting with a discussion of point-like operators in subsection 8.1, then line and loop operators in subsection 8.2 (see the review [171]), 2d operators in subsection 8.3 (see the review [172]), and 3d walls and interfaces in subsection 8.4. The case of 4d "operators" in the 4d spacetime simply corresponds to punctures of C that we discuss throughout the review.

8.1 Local operators in 4d

We have already covered at length the case where a codimension 2 defect inserted at a point in C wraps the 4d spacetime: indeed these are simply the punctures and twist operators described throughout this review, especially in section 7.

Coulomb branch operators The order k holomorphic differentials $\phi_k(z) = u_k(z)dz^k$ that define the SW curve can be calculated from the classical limit of Toda CFT correlators. In this limit, where the radius of S_b^4 is large, or $\epsilon_1, \epsilon_2 \to 0$, or equivalently 2d CFT conformal dimensions are large, u_k is given by the insertion of the spin k current W_k:[42]

$$u_k(z) \propto \frac{\langle W_k(z) V_{\mu_1} \dots V_{\mu_n} \rangle}{\langle V_{\mu_1} \dots V_{\mu_n} \rangle} \qquad \text{as } \epsilon_1, \epsilon_2 \to 0. \tag{8.1}$$

For $\mathfrak{su}(2)$ see the original AGT paper [5] or the more explicit [178] for instance.[43]

Consider now a pants decomposition of C, and the corresponding description of $\mathrm{T}(\mathfrak{g}, C, D)$ as a collection of tinkertoys and vector multiplets gauging some symmetries. Let ϕ be the scalar in one of the vector multiplets (corresponding to a tube), and consider gauge-invariants such as $\operatorname{Tr} \phi^l$ in the A-type case, and more generally all Casimirs of all gauge groups. Classically, they appear as coefficients of the differentials ϕ_k and can thus be retrieved as certain weighted integrals of ϕ_k. Going back to general ϵ_1, ϵ_2, the *operator* $\operatorname{Tr} \phi^l$ on the 4d side corresponds to a suitable weighted integral of currents $\tilde{W}_l$ [17].[44] For instance, inserting $\operatorname{Tr} \phi^2$ takes a derivative of $Z_{S_b^4}$ with respect to gauge couplings [179–181], namely to the shape of C, which indeed translates to an integrated insertion of the holomorphic stress-tensor $T = \tilde{W}_2$.

Correlation functions on S_b^4 with (products of) $\operatorname{Tr} \phi^j$ inserted at one pole and $\operatorname{Tr} \bar{\phi}^k$ at the other can be computed by supersymmetric localization, although the operators complicate instanton counting. By a conformal transformation the round case $b = 1$ leads to results on flat space correlators with exactly one antichiral operator [182–187], which provide detailed checks of various field theory ideas such as resurgence [182, 188], large-charge expansions [189–196], and more [197]. These specific correlators have not been pursued on the 2d CFT side of the correspondence.[45]

Orbifold $\mathbb{C}^2/\mathbb{Z}_M$ Next we consider another operation whose effect is to make one point singular inside $\mathbb{R}^4$, or both poles of S_b^4: orbifolding by a group $\mathbb{Z}_M$ acting as $(\exp(2\pi i/M), \exp(-2\pi i/M))$ on $\mathbb{C}^2$. Supersymmetric localization still works: one must simply restrict all modes to $\mathbb{Z}_M$-invariant ones, and instanton counting to $\mathbb{Z}_M$-invariant instanton counting. This corresponds to the coset CFT

$$\frac{\widehat{\mathfrak{su}}(N)_k \times \widehat{\mathfrak{su}}(N)_M}{\widehat{\mathfrak{su}}(N)_{k+M}} \times \frac{\widehat{\mathfrak{su}}(M)_N}{\widehat{\mathfrak{u}}(1)^{M-1}}, \quad k = -N - \frac{Mb^2}{1+b^2}, \tag{8.2}$$

which for $M = 1$ reduces nontrivially to the usual Toda CFT. The case $N = M = 2$, essentially super-Liouville CFT, is studied in [18, 21–23, 25, 27, 28, 30, 198, 199], see also [200] with a surface operator. Instanton counting on $\mathbb{C}^2/\mathbb{Z}_2$ and on its blow up, where one has instantons at two fixed point of an U(1) isometry, are related [201–204]. This

[42]The numerical coefficient depends on conventions.

[43]More references welcome.

[44]The spin l current $\tilde{W}_l$ is polynomial in the W_k to account for the difference between Casimirs $\operatorname{Tr} \varphi^l$ and coefficients in a characteristic polynomial $\det(x - \varphi)$.

[45]More references welcome.

leads to a decomposition of super-Liouville CFT into a Liouville and time-like Liouville pieces [205–208] The general N, M extension is partially worked out in [20, 24, 26, 29, 32] and the coset (8.2) studied further in [209–211]. Another perspective is to realize $\mathbb{R}^4/\mathbb{Z}_M$ by dimensional reduction of $\mathbb{R}^4 \times S^1$, which corresponds to taking q to a root of unity [19, 31, 33–35] in the q-deformed AGT correspondence we explain later.

8.2 Line operators

We now[35] place a codimension 4 operator of the 6d theory along $L \times \gamma$, where L is one of the circles $\{y_3 = y_4 = 0, y_5 = \text{const}\}$ or $\{y_1 = y_2 = 0, y_5 = \text{const}\}$ in S_b^4 allowed by supersymmetry, while γ is a closed loop in C with no self-intersection. Upon dimensional reduction, this inserts loop operators in the AGT correspondence (1.1): a loop operator labeled by γ and placed on $L \subset S_b^4$ in the partition function, and a loop operator on γ in the Toda CFT correlator.

Wilson loop operators. Since γ has no self-intersection we can cut C along it and get a (possibly disconnected) surface C' with two additional punctures (with some data, say D_1, D_2). As discussed near (1.2), the corresponding class S theory $\mathrm{T}(\mathfrak{g}, C, D)$ is obtained from the theory $\mathrm{T}(\mathfrak{g}, C', \{D, D_1, D_2\})$ corresponding to C' by gauging a diagonal subgroup of the flavour symmetries F_1, F_2 associated to D_1, D_2 as in (1.2). In this way each loop γ is associated to a gauge group $G_\gamma = (F_1 \times F_2)_{\text{diag}}$ in some description of $\mathrm{T}(\mathfrak{g}, C, D)$. The loop operator in 4d then turns out to be a half-BPS Wilson loop measuring the holonomy of the corresponding gauge field A_γ along L (plus some contribution from scalar superpartners to ensure supersymmetry):

$$W_{\gamma,R} = \mathrm{Tr}_R\Big(\mathrm{Pexp} \int_L (A_\gamma + \text{scalars}_\gamma)\Big). \tag{8.3}$$

This depends additionally on a choice of representation R of $\mathfrak{g}$. In fact several approaches suggest that the codimension 4 operator we started with in 6d carries such a label.

The Wilson loop is invariant under deformations of γ and only depends on its homotopy class. On the 2d CFT side the corresponding object is a certain 1d topological defect along γ called a degenerate Verlinde loop operator. Verlinde loops are constructed as monodromies of a vertex operator V_α. The specific choice corresponding to $W_{\gamma,R}$ is to take a momentum $\alpha = -b^{\pm 1}\Omega_R$, where $\pm$ depends on the choice of circle L, while Ω_R is the highest weight[46] of the representation R. For this α, the vertex operator V_α is degenerate in the sense that it is annihilated by various combinations of W-algebra generators. Incidentally, the most general degenerate momentum $\alpha = -b\Omega - b^{-1}\Omega'$ corresponds to inserting Wilson loops along both allowed circles.

Concrete checks of the correspondence are straightforward. The Wilson loop is compatible with supersymmetric localization [8] and inserts a simple a-dependent factor in (1.7). The Verlinde loop $\mathcal{L}_\gamma$ acts diagonally on a complete set of states inserted along γ

[46] Weights are in $\mathfrak{g}^*$, which we identify with $\mathfrak{g}$ using the Killing form.

hence appears in (1.9) simply as a function of the internal momentum α (related to a). They match:

$$\begin{aligned}\langle W_{\gamma,R}\rangle^{\mathrm{T}(\mathfrak{g},C,D)}_{S^4_b} &= \int da\ \mathrm{Tr}_R(e^{a_\gamma})\, Z_{\mathrm{cl}}(a,q,\bar q)\, Z_{\text{one-loop}}(a) Z_{\mathrm{inst}}(a,q) Z_{\mathrm{inst}}(a,\bar q)\\ &= \int d\alpha\, f(\alpha)\, C(\alpha)\mathcal{F}(\alpha,q)\mathcal{F}(\alpha,\bar q)\\ &= \langle V_{\mu_1}\dots V_{\mu_n}\mathcal{L}_\gamma\rangle^{\mathrm{Toda}(\mathfrak{g})}_{C}.\end{aligned}\tag{8.4}$$

Other loop operators. Now consider a pants decomposition of C that does not include γ among its cuts. The 2d CFT side is still given by a Verlinde loop along γ, but its expression in the given basis of conformal blocks is no longer diagonal: it is

$$\langle V_{\mu_1}\dots V_{\mu_n}\mathcal{L}_\gamma\rangle^{\mathrm{Toda}(\mathfrak{g})}_{C} = \int d\alpha\, C(\alpha)\mathcal{F}(\alpha,q)\sum_h \mathcal{L}_\gamma(\alpha,\alpha+h)\mathcal{F}(\alpha+h,\bar q)\tag{8.5}$$

where h ranges over a finite collection of momenta related to the weights of R. The corresponding loop operator in 4d is described in this S-duality frame as a 't Hooft or dyonic loop instead of a Wilson loop. Rather than being defined by an insertion in the path integral like the Wilson loop (8.3), a 't Hooft loop on L is defined by imposing a singular boundary condition on the gauge field that prescribes a non-zero monopole charge $\frac{1}{2\pi}\int F$ on a two-sphere S^2 surrounding L. Dyonic loops involve additionally a Wilson loop insertion along the same circle L. The path integral ranges over such singular field configurations instead of the usual smooth ones, and supersymmetric localization still applies [173]. It reproduces (8.5). Interestingly, Verlinde loops must be generalized to Verlinde networks (involving fusion of degenerate vertex operators) to reproduce half-BPS dyonic loops with the most general electric and magnetic charges.

In light of (8.5), dyonic loops or Verlinde networks can be understood as difference operators acting on functions of internal momenta α, or equivalently acting on functions on the Coulomb branch $\mathcal{B}$. Dyonic loops can be inserted along a one-parameter family of circles on S^4_b (at different latitudes y^5), and inserting several dyonic loops yields a product of difference operators. The OPE of loop operators provides skein relations that express these products as linear combinations of dyonic loops. The resulting algebra of loop operators has an interesting limit (Nekrasov–Shatashvili (NS) limit $b\to\infty$) where the difference operators become differential operators, and a further classical limit ($r\to\infty$) where the differential operators become coordinates on a torus fibration over $\mathcal{B}$.[47] This torus fibration is the Coulomb branch of the 4d theory on $\mathbb{R}^3\times S^1$; it can be seen from several points of view as the Hitchin moduli space on C, or the moduli space of flat $G_{\mathbb{C}}$ connections (where the Lie algebra of $G_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$), etc. These considerations fit nicely with the 3d/3d correspondence discussed later in the review.

Verlinde loops

[47] Cf. how momentum $p\sim i\partial_x$ in quantum mechanics is a coordinate in classical mechanics.

8.3 Surface operators

Surface operators compatible with supersymmetric localization on S_b^4 can be inserted along two squashed spheres intersecting at the poles or some two-tori, expressed in Cartesian coordinates of an $\mathbb{R}^5$ as follows:

$$\begin{aligned} S_b^4 &\coloneqq \{y_5^2 + b^2(y_1^2+y_2^2) + b^{-2}(y_3^2+y_4^2) = r^2\}, \\ S_b^2 &\coloneqq \{y_1 = y_2 = 0, y_5^2 + b^{-2}(y_3^2+y_4^2) = r^2\}, \\ S_{1/b}^2 &\coloneqq \{y_5^2 + b^2(y_1^2+y_2^2) = r^2, y_3 = y_4 = 0\}, \\ T_{\theta,\varphi}^2 &\coloneqq \{y_5 = r\cos\theta, y_1^2+y_2^2 = (rb^{-1}\sin\theta\cos\varphi)^2, y_3^2+y_4^2 = (rb\sin\theta\sin\varphi)^2\}. \end{aligned} \tag{8.6}$$

The latter case has not been studied so we concentrate in subsection 8.3 on the spheres, on which the surface operators preserve a 2d $\mathcal{N} = (2,2)$ subalgebra of 4d $\mathcal{N} = 2$.

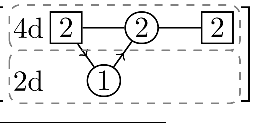
Vortex string operators. In this paragraph we discuss surface operators arising from a codimension 4 operator of the 6d theory placed at a point $z \in C$.

As we have learned from studing loops, codimension 4 operators carry a choice of representation R of $\mathfrak{g}$. On the 2d CFT side we thus want a point operator labeled by R: the natural guess is a degenerate vertex operator V_α with $\alpha = -b^{\pm 1}\Omega_R$, the sign $\pm$ being determined by which squashed two-sphere we use on the 4d side. This suggests an equality

$$\langle \text{surface operator} \rangle_{S_b^4}^{\mathrm{T}(\mathfrak{g},C,D)} = \langle V_{\mu_1} \dots V_{\mu_n} V_{-b^{\pm 1}\Omega_R} \rangle_{C}^{\mathrm{Toda}(\mathfrak{g})}. \tag{8.7}$$

The right-hand side can be written as an analytic continuation of an $(n+1)$-point correlator $\langle V_{\mu_1} \dots V_{\mu_{n+1}} \rangle$ of non-degenerate vertex operators. The analytic continuation in the corresponding class S theory T' was first understood in [212] in Lagrangian cases[48]: it amounts to considering a supersymmetric "vortex string" configuration in T' in which certain hypermultiplet scalars acquire space-dependent VEVs concentrated in codimension 2. In the low-energy limit, the non-zero scalars Higgs some gauge symmetries of T' down to those of T, and the configuration is effectively described by a surface operator in the theory T.

Besides this vortex string construction of surface operators obtained from codimension 4 operators of $\mathcal{X}(\mathfrak{g})$, these surface operators can be described by coupling to the 4d theory a 2d $\mathcal{N} = (2,2)$ gauge theory living on the defect. In this context the left-hand side of (8.7) is the partition function of the 4d-2d coupled system on squashed spheres. The simplest example is that of SU(2) SQCD with $N_f = 4$ and a defect labeled by the fundamental representation. The 2d theory then consists of chiral multiplets in doublet representations of 4d flavour and gauge groups, and with charges ± 1 under a 2d U(1) gauge group:

$$Z_{S_b^2 \subset S_b^4} \Bigg[$$


$$\Bigg] = \Big\langle V_{\mu_1} V_{\mu_2} V_{\mu_3} V_{\mu_4} V_{-b/2} \Big\rangle_{S^2}^{\text{Liouville}}. \tag{8.8}$$

[48]It would be good to clarify the situation for the most general class S theories.

The position z of $V_{-b/2}$ matches the Fayet–Iliopoulos (FI) parameter of the 2d U(1) gauge group. Such a 2d description of the most general R in $\mathfrak{su}(N)$ Lagrangian theories is conjectured in [61] and checked by comparing limits $z \to z_i$ in gauge theory to the known OPE of $V_{-b^{\pm 1}\Omega_R}$ and V_{μ_i}. More general degenerate insertions $V_{-b\Omega-\Omega'/b}$ translate to intersecting defects with extra 0d fields living at the poles [62]. An important difficulty in checking equalities like (8.8) is to compute contributions $Z_{\text{inst,vort}}$ from the poles of S_b^4, which involve both instantons of the 4d theory and vortices of the 2d theory [213]. Incidentally, in an $\epsilon_1, \epsilon_2 \to 0$ limit this 4d-2d analogue of Nekrasov's partition function gives both the 4d theory's effective prepotential F and the 2d theory's effective twisted superpotential $\mathcal{W}$, obtained earlier in [50, 214]:

$$\log Z_{\text{inst,vort}} = \frac{F}{\epsilon_1\epsilon_2} + \frac{\mathcal{W}}{\epsilon_1} + O(1). \tag{8.9}$$

Gukov–Witten operators: monodromy defects and orbifolds. We have already encountered codimension 2 defects of $\mathcal{X}(\mathfrak{g})$, since they are the origin of tame punctures that impose certain boundary conditions on the differentials ϕ_k. Wrapping these codimension 2 operators on C thus gives surface operators that impose certain boundary conditions on the 4d fields. Specifically, this yields $\mathcal{N}=2$ versions of Gukov–Witten (GW) surface operators [215], which impose that 4d gauge fields A behave as $A \sim \alpha d\theta$ as $r \to 0$ with a prescribed $\alpha \in \mathfrak{t}$ in the Cartan algebra of $\mathfrak{g}$, where (r, θ) are polar coordinates transverse to the defect. GW defects can also be described by coupling suitable 2d $\mathcal{N}=(2,2)$ gauge theories to the 4d theory, as we explain in the main text.

If A is an SU(N) gauge field, say, denote eigenvalues of $\alpha = \operatorname{diag}(\alpha_1, \dots)$ as α_i with multiplicities N_i, $i = 1, \dots, M$ so that $\sum_i N_i\alpha_i = 0$ and $\sum_i N_i = N$. Then the 4d gauge group breaks to $(\prod_i \mathrm{U}(N_i))/\mathrm{U}(1)$ at the defect. The instanton moduli space with such a monodromy defect is equivalent as a complex manifold to the moduli space of instantons on an orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ [216]. Here, $\mathbb{Z}_M$ embeds into both rotations of $\mathbb{C}$ with charge $+1$ and the gauge group SU(N) with charges i with multiplicity N_i, thus reproduce the expected symmetry breaking. The Nekrasov partition function Z_{inst} is obtained from the usual one by restricting to $\mathbb{Z}_M$-invariant instantons. It matches conformal blocks of the affine SL(N) algebra (for the full defect that has all $N_i = 1$) [65, 66], and of its Drinfeld–Sokolov (DS) reductions [67]. The (non-chiral) CFT with affine SL(N) symmetry or its DS reductions is not known.

Interestingly, conformal blocks of the affine SL(2) algebra are related to conformal blocks of the Virasoro algebra with additional degenerate vertex operators, as pointed out early on in an AGT setting in [217]. This, and its $N > 2$ analogues, leads to some identifications between the two types of surface operators up to a suitable integral kernel [218, 219].

The "codimension 2" orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ considered here should not be confused with the "codimension 4" orbifold $\mathbb{C}^2/\mathbb{Z}_M$ where $\mathbb{Z}_M$ rotates both factors: the latter orbifold also changes the 2d CFT, but to a certain coset CFT that reduces in the simplest case to super-Liouville CFT.

8.4 Domain walls

The AGT correspondence also allows for half-BPS 3d operators that separate the 4d spacetime into two parts or give it a boundary.[35]

Janus wall. Our first construction does not involve codimension 2 or 4 operators. Instead, we place $\mathcal{X}(\mathfrak{g})$ on $S_b^4 \times C$ with the complex structure of C varying with the latitude of S_b^4 [41]. This preserves half of the supersymmetry and in the limit where the variation happens sharply at the equator (or a parallel) we get a so-called Janus domain wall [174] in the 4d theory. This is a half-BPS interface between class S theories with different gauge couplings. The partition function with this interface has the usual factorized form (1.7) with holomorphic and antiholomorphic contributions from the poles, but the gauge couplings used in each factor are not complex conjugates. Correspondingly, the CFT correlator (1.9) changes to using different complex structures for the holomorphic and antiholomorphic factors.

S-duality wall. Tuning gauge couplings we can get theories that are S-dual. By switching to the same S-duality frame on both sides we get a 3d operator called the S-duality wall that has the same theory (and same gauge couplings) on both sides. Inserting an S-duality wall in a 4d theory then amounts on the 2d side to performing a modular transformation (fusion, braiding, S-move) on holomorphic (or antiholomorphic) conformal blocks. Using this relation to 2d CFT modular kernels, S-duality walls of a handful of 4d $\mathcal{N}=2$ theories have known descriptions as 3d $\mathcal{N}=2$ gauge theories coupled to the 4d theories on both sides of the wall [79, 81, 83, 175, 176]. This is related to special cases [80, 177] of the 3d/3d correspondence we discuss later. The S-duality wall has an interplay with loop operators: it translates in a suitable sense from Wilson loops on one side of the wall to 't Hooft loops on the other side.

Symmetry-breaking wall. Another construction [41] is to place a (tame) codimension 2 defect of $\mathcal{X}(\mathfrak{g})$ on the equator of S_b^4, times a closed loop $\gamma \subset C$. Roughly speaking this codimension 2 defect amounts to inserting an additional puncture and moving it around γ as in the Janus wall construction. On the CFT side one gets a non-degenerate Verlinde loop on γ. On the gauge theory side the wall can be described as breaking gauge symmetries to a subgroup. Toda CFT considerations show that tame codimension 2 defects that are not full can be dressed with additional codimension 4 defects living on their world-volume. In the present construction this leads to loop operators stuck on the domain wall.

Boundary CFT. We mentioned orbifolds earlier. Instead of orbifolding the 4d space one can orbifold by a $\mathbb{Z}_2$ symmetry that acts as a reflection with respect to the equator of S_b^4 and a reflection on the Riemann surface (so as to preserve chirality of $\mathcal{X}(\mathfrak{g})$). This leads to an AGT correspondence for Riemann surfaces with boundaries and for non-orientable surfaces [84, 85].

9 Other dimensions and geometries

Class S theories $\mathrm{T}(\mathfrak{g}, C, D)$ are obtained by compactifying the 6d $(2,0)$ theory of type $\mathfrak{g}$ on $M_4 \times C$ with C a Riemann surface with punctures which extra data D. So far we have extensively discussed the case $M_4 = S_b^4$ and its building block $M_4 = \mathbb{R}^4_{\epsilon_1,\epsilon_2}$, for which the partition function is equal to a 2d CFT correlator or conformal block, respectively.

We first discuss the 5d lifts of these 4d observables to (deformations of) $\mathbb{R}^4 \times S^1$, $S^4 \times S^1$, and S^5, which are connected to q-deformations[49] (subsection 9.1). We then change the geometry, first relating the supersymmetric index, which is the partition function on $M_4 = S^3 \times S^1$, to a q-Yang–Mills TQFT correlator (subsection 9.2), then compactifying instead on products $M_3 \times C_3$ and $M_2 \times C_4$ in which the "internal" manifold C is a hyperbolic three-manifold (subsection 9.3) or a four-manifold (subsection 9.4). Then we mention generalizations with less supersymmetry (subsection 9.5).

9.1 Lift to 5d and q-Toda

Here we briefly survey how lifting the 4d $\mathcal{N} = 2$ theories to 5d $\mathcal{N} = 1$ amounts to a q-deformation of the 2d theories. For a review, see [141].

Instanton partition functions. The 4d $\mathcal{N} = 2$ Omega background used to define Nekrasov's instanton partition function [10, 11] is conveniently expressed in terms of a 5d $\mathcal{N} = 1$ lift: placing the theory on $\mathbb{R}^4 \times S^1$ with twisted boundary conditions around S^1 such that $\mathbb{R}^4$ rotates by q and t in two two-planes, and with an additional twist by an R-symmetry to preserve some supersymmetry. More precisely, this definition for $|q| = |t| = 1$ can be extended to complex q, t by turning on additional supergravity fields. The 5d lift deforms all factors in Z_{inst} from rational functions to trigonometric functions of masses and Coulomb branch parameters. It is natural to ask how the Toda CFT side of the AGT correspondence can be deformed to accomodate for this.

One finds that the 5d (also called K-theoretic) Z_{inst} is a chiral block for a q-deformed W-algebra [33, 101, 220–222]: the relevant deformations of the Virasoro algebra and of W-algebras [223–226] were constructed long ago.[50] When mass and Coulomb branch parameters are suitably quantized the equality can be proven using Dotsenko–Fateev integral representations of qW_N conformal blocks [228–231] (also used in [232]). See also [233, 234].

The 5d $\mathcal{N} = 1$ quiver gauge theories admit realizations in terms of webs of (p, q) five-branes in IIB string theory. Applying S-duality exchanges the role of D5 and NS5 branes, thus equating Z_{inst} for a $\mathrm{SU}(N)^{M-1}$ linear quiver gauge theory to an $\mathrm{SU}(M)^{N-1}$ one (see [235] for a proof for $M = N = 2$). This 5d spectral duality (also called fiber-base

[49] The parameter q appearing in the 5d lift is unrelated to the gauge couplings parameters describing the complex structure of C. We will actually not need a notation for this gauge coupling any longer.

[50] As in 4d, working with $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$ gauge groups makes Z_{inst} more tractable; correspondingly one works with the Ding–Iohara algebra, a slight extension of (the universal envelopping algebra of) the qW-algebra of type $\mathfrak{g}$ [227].

duality) relates in general chiral blocks of different qW_N theories [236, 237], it implies certain instances of 3d mirror symmetry [238], and relations between spin chains [239].

The 4d case is retrieved as the limit $q \to 1$ with $t = q^{-\beta}$ and fixed $-\beta = b^2 = \epsilon_1/\epsilon_2$ giving the 4d deformation parameters. Other interesting limits than $q, t \to 1$ exist, especially taking q and t a k-th roots of unity one obtains Nekrasov partition functions on $\mathbb{C}^2/\mathbb{Z}_k$ ALE space studied in [27, 31, 33–35, 78, 240]. Another simplifying limit is the Hall-Littlewood limit $q \to 0$ [241, 242].

An unrelated application of Z_{inst} and qW_N conformal blocks is to construct solutions of q-Painlevé equations [243–246], as in the 4d case.

Compact partition functions. Let us now glue instanton partition functions together. While the partition function on S^4 involves a pair of instanton contributions from the two fixed point of the supercharge squared, the partition function of 5d $\mathcal{N} = 1$ theories on S^5 combines three K-theoretic instanton partition functions because the supercharge has three fixed circles. Schematically,

$$Z_{S^5} = \int da \, Z_{\text{pert}} Z_{\text{inst},1} Z_{\text{inst},2} Z_{\text{inst},3}. \tag{9.1}$$

The squashed S^5 has three axis lengths $\omega_1, \omega_2, \omega_3$ and here the different $Z_{\text{inst},i}$ are computed in the Ω background with parameters (q, t) given by $(\frac{\omega_2}{\omega_1}, \frac{\omega_3}{\omega_1})$, $(\frac{\omega_1}{\omega_2}, \frac{\omega_3}{\omega_2})$, $(\frac{\omega_1}{\omega_3}, \frac{\omega_2}{\omega_3})$, respectively.

The picture that emerges [58, 60, 247] is that there exists a q-deformed version of Toda CFT, called q-Toda theory,[51] that has qW_N symmetry and whose correlators should match with S^5 partition functions. The fact that three chiral factors need to be combined leads to a remarkable “modular triple” of q-Virasoro algebras [102] (for $N = 2$), similar to the modular double combining $U_q(\mathfrak{sl}(2))$ with $q = e^{2\pi i b^2}$ and $q = e^{2\pi i/b^2}$ in 2d CFT. A non-local Lagrangian for q-Liouville is proposed in [102].

Half-BPS operators with 3d $\mathcal{N} = 2$ supersymmetry played an important early role right from the start. On the squashed S^5 they can be inserted along three distinct S^3 that intersect pairwise along S^1. The first explorations of the correspondence for Z_{S^5} concerned the case of a single 3d operator in a simple 5d bulk theory, which can be obtained by Higgsing a larger 5d theory [58, 60]. Just as some surface operators in the standard AGT correspondence, these 3d operators correspond to degenerate q-Toda CFT vertex operators. They are useful to bootstrap structure constants of q-Toda, and show up in a Higgs branch localization expression of the instanton and S^5 partition functions [248–250], again completely analogous to the 4d story [251, 252], albeit more technically involved. A mathematical take on this is in [253].

Codimension 4 operators of the 5d theory, specifically Wilson loops, are studied in [254]; they translate in q-Toda to stress tensor and higher-spin operator insertions.

[51] One should be careful that many papers talk about q-Liouville or q-Toda theory even when they only consider chiral blocks, which only involve the q-Virasoro and qW_N symmetry algebras.

Elliptic lift. Lifting one dimension up, 6d partition functions on $\mathbb{R}^4 \times T^2$ and $S^5 \times S^1$ (superconformal index) are related to the elliptic deformation (q,t)-Toda: see [234, 255–269].

9.2 Superconformal index and 2d q-YM

We now move on to partition functions of 4d $\mathcal{N}=2$ class S theories on $M_4 = S^3 \times S^1$.

Supersymmetric index. See the review [270] for superconformal class S theories and [271] for general 4d $\mathcal{N}=1$ theories. We shall not write too much here, but we rather point to another course in this school [272]. The AGT relation to q-YM is also surveyed briefly in [109].

The $S^3 \times S^1$ partition function is defined and computable for 4d $\mathcal{N}=1$ theories with an anomaly-free U(1) R-symmetry. Up to a factor involving the Casimir energy of the theory, expressible in terms of a and c anomalies, the partition function coincides with the supersymmetric index, defined to be the Witten index of the theory quantized on $S^3 \times \mathbb{R}$. Once refined by fugacities u_i for mutually commuting rotations, flavour, and R-symmetries (with charges K_i), the index is written as

$$\mathcal{I}(u) = \operatorname{Tr}\biggl[(-1)^F e^{-\beta\tilde{H}} \prod_i u_i^{K_i}\biggr] \tag{9.2}$$

where $(-1)^F$ counts bosonic and fermionic states with opposite signs, $\tilde{H} = \{Q, Q^\dagger\}$ for some supercharge Q, and $\mathcal{I}$ is β-independent thanks to cancellations between bosonic and fermionic states when $\tilde{H} \neq 0$. This simplification means that $\mathcal{I}(u)$ counts (with signs) short representations of the supersymmetry algebra. Fugacities u_i are encoded in the $S^3 \times S^1$ partition function as holonomies around the S^1 for background gauge fields coupled to the given symmetry: in particular, fugacities (p,q) for two combinations of rotations and R-symmetries can be understood as a non-trivial fibration of S^3 over S^1.

The index formally does not depend on any continuous parameter beyond these:[52] it is an renormalization group (RG) flow invariant and is independent of gauge couplings for instance, thus can be easily computed in any weakly-coupled Lagrangian description. In this way $\mathcal{I}(u)$ reduces to a simple signed count of local gauge-invariant operators built from the elementary fields in any given Lagrangian description (in other dimensions nonperturbative objects must be included). Being an eminently computable RG flow invariant makes the index a powerful window into nonperturbative physics of 4d $\mathcal{N}=1$ gauge theories, especially their IR dualities.

Computing the index is much harder if we have no Lagrangian description, but part of the structure remains: if a theory T is defined by gauging a common flavour symmetry G of two theories T_1, T_2, then the indices are related schematically as

$$\mathcal{I}[T](a_1,a_2) = \int [dz]_G\, \mathcal{I}_{\text{vec}}(z)\, \mathcal{I}[T_1](a_1,z)\, \mathcal{I}[T_2](a_2,z), \tag{9.3}$$

[52] In some contexts, this formal invariance of the index fails, which gives wall-crossing phenomena.

where we hid the p, q dependence but kept explicit the fugacities a_1 and a_2 for flavour symmetries of T_1 and T_2 commuting with G, which become flavour symmetries of T. The integral over the fugacity z for the symmetry G is done with a suitable measure $\mathcal{I}_{\text{vec}}$, which from the localization point of view is the vector multiplet one-loop determinant. In fact, (9.3) gives a way to compute the index of a non-Lagrangian theory: embed it into a larger theory that is dual to a Lagrangian gauge theory, whose index is computable [273].

Class S. We now return to class S theories, and specifically to superconformal ones. Since the index cannot depend on gauge couplings, it only depends on the topology of the Riemann surface C and the type of punctures. Thus, compared to the standard AGT correspondence, the 2d CFT side should be replaced by a TQFT, as worked out in [274]. Consider the theory $\mathrm{T}(\mathfrak{g}, C, D)$. A flavour symmetry is associated to each puncture z_i, $i = 1, \ldots, n$, and we turn on corresponding fugacities a_i. For any pants decomposition of C we can express $\mathrm{T}(\mathfrak{g}, C, D)$ as the result of gauging flavour symmetries of a collection of tinkertoys (isolated SCFTs) associated to three-punctured spheres. Through (9.3), the index then reduces to an integral of superconformal indices of tinkertoys. This precisely mimics the structure of correlators in a TQFT:

$$\mathcal{I}[\mathrm{T}(\mathfrak{g}, C, D)](a_i) = \langle \mathcal{O}_{D_1}(a_1) \ldots \mathcal{O}_{D_n}(a_n) \rangle_{\text{some TQFT}} \tag{9.4}$$

for suitable operators $\mathcal{O}_D$ that depend on the type of puncture.

In analogy to Liouville CFT bootstrap, which relied on using degenerate vertex operators that correspond in gauge theory to surface operators, one can bootstrap the index of all tinkertoy building blocks using surface operators [212]. Adding a surface operator to the index corresponds to acting with a difference operator Θ on fugacities associated to any one of the punctures, and by topological invariance it does not matter which puncture. Expressing the result in an eigenbasis of Θ labeled by representations λ of $\mathfrak{g}$ eventually gives

$$\mathcal{I}[\mathrm{T}(\mathfrak{g}, C, D)](a_i) = \sum_{\lambda} (C_\lambda)^{2g-2} \phi_\lambda^{D_1}(a_1) \ldots \phi_\lambda^{D_n}(a_n) \tag{9.5}$$

for some structure constants $C_\lambda(p, q, t)$ and wave functions $\phi_\lambda^D(p, q, t; a)$.[53] Wave functions are related to those for full punctures by taking suitable residues in flavour fugacities [275]. The sum may diverge if punctures are too "small", signalling either that the given class S theory does not exist or that the index is not sufficiently refined because there are additional flavour symmetries not associated to any of the punctures.

The wave functions can be computed order by order in p, q, t, but are not known in closed form. In the Schur limit $q = t$ correlators are functions of q only (p-dependent terms are Q-exactness), wavefunctions are proportional to Schur polynomials, and the corresponding TQFT is q-deformed 2d Yang-Mills theory [103], as derived from the 6d description in [276, 277]. In the more general Macdonald limit $p = 0$, wavefunctions are essentially Macdonald polynomials in q, t and the TQFT must be deformed by changing

[53] Here we introduced a fugacity t for the additional R-symmetry of 4d $\mathcal{N} = 2$ theories.

the measure in the path integral of q-YM theory [278]. The Coulomb limit ($t, p \to 0$ with fixed p/t and q) is also interesting.

The correspondence is tested and extended in natural ways: inserting Wilson–'t Hooft loops at the poles of S^3 and correspondingly loops in q-YM [279, 280], inserting general surface operators [59, 281–283], replacing S^3 by the Lens space $L(p,1) = S^3/\mathbb{Z}_p$ [284, 285], taking C to have non-zero area [286], generalizing to D-type gauge groups and non-simply-laced ones (using outer automorphism twists) [287, 288]. The relation with Hilbert series of instanton moduli spaces is explored in [289, 290]. The superconformal index of many AD type theories is also known by now in the Macdonald limit [291–295]. The key open question in this direction seems to be getting a handle on the full parameter space (p, q, t) rather than its $p = 0$ Macdonald slice.

9.3 3d/3d correspondence

For reviews, see [140, 296]. So far we have reduced the 6d $(2,0)$ theory of type $\mathfrak{g}$ with a partial topological twist along a Riemann surface. Reducing it instead on $M_3 \times C_3$, with a twist along a three-manifold C_3, gives a 3d $\mathcal{N} = 2$ gauge theory on M_3. One can add codimension 2 operators of the 6d theory to get analogues of punctures: boundary conditions along knots $K_1 \subset C_3$ (we leave additional data implicit). This defines a large class of 3d $\mathcal{N} = 2$ gauge theories[54] $\mathrm{T}(\mathfrak{g}, C_3, K_1)$. Natural building blocks for $C_3 \setminus K_1$ are tetrahedra, and there is an explicit gauge theory description of the 3d $\mathcal{N} = 2$ theory for each triangulation of $C_3 \setminus K_1$. In contrast to the 4d/2d correspondence, the theories $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ are not given exactly by 3d $\mathcal{N} = 2$ gauge theories, but rather by their IR limits (and deformations by masses or other parrameters). Pachner's 3-2 move for triangulated 3-manifolds, which exchanges two neighboring tetrahedra with three tetrahedra covering the same part of the manifold, yields 3d $\mathcal{N} = 2$ IR dualities.

3d/3d correspondence. The 3d/3d AGT correspondence was formulated in [104, 299], after several papers treating less general geometries: either reducing $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ further on S^1 [300] (getting a 2d $\mathcal{N} = (2,2)$ theory), or taking C_3 to be a mapping cylinder or torus (Riemann surface fibered over an interval or circle) [80, 177, 301, 302]. The partition function of $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ on certain M_3 is equal to the partition function on $C_3 \setminus K_1$ of Chern–Simons theory with a gauge group $G_{\mathbb{C}}$ whose Lie algebra is the complexification of $\mathfrak{g}$. This, in turn, provides invariants of knots and of 3-manifolds. Complex Chern–Simons theory depends on levels (k, σ), one quantized $k \in \mathbb{Z}$ and one continuous $\sigma \in \mathbb{R}$, related to the choice of M_3. Its action is straightforward,

$$S = \mathrm{Re}\biggl(\frac{k + i\sigma}{4\pi} \int_{C_3} \mathrm{Tr}\Bigl(\mathcal{A} \wedge d\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Bigr)\biggr), \tag{9.6}$$

where $\mathcal{A} = A + i\Phi$ is a complex gauge field, but defining Chern–Simons theory completely is subtle when the gauge group is noncompact [303], and in fact the 3d/3d correspondence

[54] This is not a standard notation; sometimes $\mathrm{T}(\mathfrak{su}(N), C_3, K_1)$ is denoted $T_N[C_3 \setminus K_1]$. On a separate note, it is worth mentioning that some papers associating gauge theories to three-manifolds do not actually obtain the full theory $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ but rather only a subsector [297, 298].

helps define it for complex gauge group $G_{\mathbb{C}}$ [140, 304, 305].

The squashed sphere partition function ($M_3 = S_b^3$) corresponds to Chern–Simons at level $k = 1$ [104, 306]. The supersymmetric index ($M_3 = S^2 \times_q S^1$) corresponds to Chern–Simons at level $k = 0$ [307–309].[55] The partition function on a squashed lens space $M_3 = L(k,1)_b$ corresponds to a general Chern–Simons level k [304, 305].

$$\begin{aligned} Z_{S^2\times_q S^1}[\mathrm{T}(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (0,\sigma)\big],\ q = e^{2\pi/\sigma} \\ Z_{S_b^3}[\mathrm{T}(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (1, \tfrac{1-b^2}{1+b^2})\big] \\ Z_{L(k,1)_b}[\mathrm{T}(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (k, k\tfrac{1-b^2}{1+b^2})\big] \end{aligned} \tag{9.7}$$

All three can be decomposed into holomorphic blocks [310, 311], which are partition functions on the Omega background $\mathbb{R}^2 \times_q S^1$ or equivalently the cigar $D^2 \times_q S^1$, and are wave functions on the Chern–Simons side. A semi-classical version of this is that the set of supersymmetric vacua on $\mathbb{R}^2 \times S^1$ matches the space of flat $G_{\mathbb{C}}$ connections on $C_3 \setminus K_1$ with suitable boundary conditions along K_1.

Boundaries and generalizations. When C_3 has 2d boundaries (on which the knot K_1 can end), the 3d $\mathcal{N} = 2$ theory lives at the boundary of (and is coupled to) the 4d $\mathcal{N} = 2$ class S theory associated to ∂C_3 [41, 82, 104, 280]. In particular, when C_3 is a cobordism, namely ∂C_3 consists of two disconnected components, it is more natural to think of the 3d $\mathcal{N} = 2$ theory as a domain wall between the two corresponding 4d $\mathcal{N} = 2$ class S theories which are only coupled through their common 3d boundary. The construction is thus functorial with respect to gluing. One particularly simple example of the setup was described in [41]: consider $C_3 = \mathbb{R} \times C$ where the complex structure of C varies along the $\mathbb{R}$ direction; then on the gauge theory side we have the 4d $\mathcal{N} = 2$ theory $\mathrm{T}(\mathfrak{g}, C)$ with a Janus domain wall defined by varying the 4d gauge couplings along one direction.

The 3d/3d correspondence can be refined using higher-form symmetries [312]. A different twist realizes homological invariants of knots and three-manifolds (monopole/ Heegaard Floer and Khovanov–Rozansky homology) in terms of 3d $\mathcal{N} = 2$ theory $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ partially topologically twisted on a Riemann surface [297, 313]. Holographic calculations [314] probe the correspondence at large N. Dimensional reduction from the 4d $\mathcal{N} = 2$ superconformal index to the 3d $\mathcal{N} = 2$ sphere partition function translates to dimensional oxydation from 2d q-YM to a hyperbolic manifold [284, 315]. Half-BPS 2d $\mathcal{N} = (0,2)$ boundary conditions and domain walls of 3d $\mathcal{N} = 2$ theories were studied in [316] and subsequent papers, and one offshoot is the 2d/4d correspondence [105] discussed next.

9.4 2d/4d correspondence

Reducing the 6d $(2,0)$ theory on $T^2 \times C_4$ with a partial topological twist along the four-manifold C_4 gives 2d $\mathcal{N} = (0,2)$ supersymmetric gauge theories. This setting has

[55] I don't know if the differences between [308, 309] have been resolved.

been somewhat less studied, owing to how the topology of these manifolds is more complicated than 4d/2d and 3d/3d correspondences.

A dictionary à la AGT is proposed in [105]: the Vafa–Witten partition function on C_4 [317] is the superconformal index of the 2d $(0,2)$ theory, while 4d Kirby calculus translates to dualities of 2d $\mathcal{N}=(0,2)$ theories such as [318]. Four-manifolds with a boundary $\partial(C_4)$ correspond to domain walls between the 3d $\mathcal{N}=2$ theories associated to $\partial(C_4)$ by the 3d/3d correspondence [316] (see also [319]). This can be enriched further by inserting defects of the 6d $(2,0)$ theory. There are several variants: compactifying on $S^2 \times C_4$ [320], generalizing to 6d $(1,0)$ theories [321], and using a different twist to connect 4-manifold invariants to 2d $\mathcal{N}=(0,2)$ chiral correlators [322, 323]. The abelian case is studied in detail in [324].

9.5 Less supersymmetry

One can learn properties of strongly-coupled 4d $\mathcal{N}=1$ theories, for instance analogues of Seiberg dualities, from supersymmetry-breaking deformations of 4d $\mathcal{N}=2$ theories and S-duality [325–332]. Another approach to getting 4d $\mathcal{N}=1$ theories is to consider more general compactifications of 6d $\mathcal{N}=(2,0)$ SCFTs that amount to placing M5 branes on a complex curve inside a Calabi–Yau three-fold [328, 333–342]. Generalizations of SW geometry appear to exist [343].

It is quite natural as well to consider orbifolds of the M-theory setup. In particular, the 6d $(1,0)$ theory of M5 branes at a $\mathbb{Z}_k$ singularity has very interesting reductions to 4d $\mathcal{N}=1$ theories called class S_k [344–355]. In principle this leads to an analogue of the AGT correspondence, but the Z_{S^4} partition function of $\mathcal{N}=1$ theories suffers some ambiguities, and the instanton partition function is now known (see however a very interesting proposal [356]).

While 6d $(2,0)$ theories have an ADE classification, there exists a zoo of 6d $(1,0)$ theories constructed from F-theory, reviewed in [110] and more generally [357]. Compactifying them on a torus gives 4d $\mathcal{N}=2$ supersymmetry, reproducing many class S theories [347]: this is similar to how reducing the 6d $(2,0)$ SCFT on a torus yields $\mathcal{N}=4$ SYM. The set-up has also been studied with fluxes turned on [358] and with surface operators [359].

These developments have led to finding 4d $\mathcal{N}=1$ Lagrangian descriptions for 4d $\mathcal{N}=2$ AD theories that admit no 4d $\mathcal{N}=2$ Lagrangian description, as done for instance in [342, 360–370] (see also [371] with more supersymmetry). They also lead to new 4d $\mathcal{N}=2$ that may be "minimal" in the sense of having the smallest central charges (a,c) [366].

10 Conclusions

There is plenty more to be said about the AGT correspondence. Most obviously we have not placed this correspondence in the wider context of the BPS/CFT correspondence between 4d $\mathcal{N}=2$ gauge theories and integrable models underlying SW geometry. We have also omitted the connections to refined topological strings and matrix models.

Integrable systems. The SW solutions of many 4d $\mathcal{N} = 2$ theories can be realized as the spectral curve of known integrable systems [372–375], such as the periodic Toda spin chain, Calogero-Moser, Ruijsenaars, sine-Gordon etc. As a quite general example, for 4d $\mathcal{N} = 2$ theories of class S it is the Hitchin integrable system [147]. Placing the 4d theory on the Omega background with $\epsilon_2 = 0$ (the Nekrasov–Shatashvili limit) corresponds to quantizing the integrable system, and turning on ϵ_2 yields a further refinement [376]. It is also understood how to include surface operators in these discussions.

Nekrasov has advocated for seeing these considerations as a BPS/CFT correspondence, reviewed in [64, 377–380] (see also [63, 233, 248, 249, 381–385]), which relates supersymmetric gauge theories with 8 supercharges (e.g., 4d $\mathcal{N} = 2$) to integrable models and 2d CFT. Since this applies beyond class S theories, one can consider the AGT correspondence as merely an instance of it in which one can make further progress.

Another instance is the Bethe/gauge correspondence, which roughly speaking arises in the Nekrasov–Shatashvili limit. In this limit, the Omega-deformed 4d $\mathcal{N} = 2$ theories reduce to a 2d $\mathcal{N} = (2, 2)$ theory whose properties match with those of quantum integrable systems. For instance the twisted chiral ring of the 2d theory gives quantum Hamiltonian, supersymmetric vacua correspond to Bethe states, and the 2d twisted superpotential is the Yang–Yang function of the integrable system. The limit was studied in [376, 386–404] among others.

For further unsorted references regarding integrable models and class S theories, see [52, 157, 239, 376, 405–472].

Topological strings. Topological strings and their relation with AGT are reviewed in [473].

For 4d $\mathcal{N} = 2$ theories realized from IIB geometric engineering [474–476] or as dimensional reductions of 5d $\mathcal{N} = 1$ theories living on (p, q) fivebrane webs, instanton partition functions can be expressed as partition functions Z_{top} of topological strings, as reviewed in [477]. As advocated in [478] to explain the AGT correspondence, Z_{top} can be further expressed in terms of Penner-like matrix models with logarithmic potentials [479–485], which match with Dotsenko–Fateev integral representations of conformal blocks. We return to these matrix models shortly.

The topological string partition function Z_{top} is computed through the topological vertex formalism, developped in [486–489] in the unrefined case $\epsilon_1 = -\epsilon_2$, and for general ϵ_1, ϵ_2 in two formulations in [490, 491] and [492, 493], whose equivalence is explained in [494] by realizing the refined topological vertex as an intertwiner of the Ding–Iohara–Miki (DIM) algebra. See also [495–499] for further tests and subtleties, [500, 501] for a world-sheet perspective, [502, 503] for a discussion of dualities that ensure that Z_{top} is independent of the so-called preferred direction, and [504] for a generalization beyond A-type quivers by introducing new topological vertices. Sometimes, Z_{top} can instead be bootstrapped using holomorphic anomaly equations [392, 477, 505–510], blowup equations [511], or the quantum curve [512, 513].

The calculation in [247, 514] of the partition function of T_N theories, hence the three-point function of Toda CFT, is particularly interesting. Surface operators and their relation to geometric transition and qq-characters are discussed in [51, 53, 54, 269, 457, 495, 515–518] etc. Other developments based on the refined topological vertex and somewhat related to AGT include [159, 512, 519–529].

Symmetries and special polynomials. The renewed interest in refined topological strings in the context of AGT led to developping many families of special polynomials generalizing Jack and Macdonald polynomials, including Macdonald-Kerov functions, generalized Schur functions, elliptic generalized Macdonald polynomials and more: see [530–545] for recent developments.

These developments are based on various underlying symmetry algebras that generalize W-algebras [122, 546] and would deserve a review of their own by someone more qualified ([250] looks like a good starting point): qW-algebra symmetries and quiver W-algebras [77, 126, 263, 268, 323, 382, 384, 385, 547–562], the spherical Hecke central algebra [563–565], the Ding–Iohara–Miki algebra [34, 241, 242, 250, 256, 260, 449, 467, 494, 503, 504, 535, 536, 540, 566–575], quantum toroidal $\mathfrak{gl}_k$ and $W_{k+\infty}$ for uses when there is an orbifold [576–578], the cohomological Hall algebra (COHA) [557, 579], double affine Hecke algebra [382, 580], elliptic quantum algebras [35, 552, 581] etc.

Based on these generalizations there exists an elliptic version of the refined topological vertex (an intertwiner of the elliptic DIM algebra) [447, 582, 583] for use in the 6d lift of the AGT correspondence, as well as a Macdonald refined topological vertex [265, 269] and an analogue when the 4d spacetime is orbifolded [577].

Matrix models. The AGT correspondence (including its q-deformed version) can be explored by studying matrix models [478] since both instanton partition functions and conformal blocks are Penner-like matrix model integrals with logarithmic potentials. See the reviews [131, 584].

On the 2d CFT side the matrix model representation arises as Dotsenko–Fateev free-field representations of conformal blocks, which are available provided the sum of Liouville/Toda momenta in each three-punctured sphere is suitably quantized. It was clarified in [585–587] how internal momenta translate to choices of contours in the matrix model integral. Moving away from these quantized slices in parameter space requires analytic continuation, which is only completely under control in the $\mathfrak{g} = \mathfrak{su}(2)$ case since coefficients in various expansions are known to be rational functions of all parameters in this case.

The link between 4d gauge theory and matrix models shows up as an equality of Z_{inst} with the matrix model partition functions [588, 589] (directly without going through Z_{top}), a matching of the SW curve with the matrix model's spectral curve and of the SW differential with the 1-point resolvent [55, 407, 590–592]. Both this link and the one with 2d CFT generalize to $b^2 = \epsilon_1/\epsilon_2 \neq -1$ (β-deformed matrix model) [588, 592], to 5d [220, 588, 593], to asymptotically free theories [591, 594–598], to $\mathfrak{su}(N)$ theories [593, 599], to quiver gauge theories [600] and generalized quivers [601] (higher genus C), to an orbifold of $\mathbb{R}^4$ [19].

The relations are tested in various limits in [596, 602–606] and proven in some cases [229, 230]. Since the Nekrasov–Shatashvili limit $\epsilon_2 \to 0$ of Z_{inst} quantizes the integrable models underlying a 4d $\mathcal{N} = 2$ theory's SW solution, matrix models give useful information about integrable models, see for instance [301, 434, 607–610]. Matrix models have also been studied for applications to irregular punctures and AD theories, especially in the classical limit $c \to \infty$ (Nekrasov–Shatashvili limit on the gauge theory side) in [611–627].

In another direction, modular properties of (properly normalized) instanton partition functions under S-duality are studied in [522, 523, 628–644] through matrix model and other techniques. Other works and reviews abound [235, 246, 645–659], as well as PhD theses [660–662].

Other connected topics. We list disparate subjects that are connected in various ways with the AGT correspondence.[56]

First, interplays with other properties of 4d $\mathcal{N} = 2$ theories.

- Topological anti-topological fusion (tt^* equations) [525, 663–665].
- The chiral algebra that appears as a protected subsector of 4d $\mathcal{N} = 2$ theories [558, 666–674].
- Gauge/Yang–Baxter equation (YBE) [675–682].
- Conformal bootstrap: besides the constructions discussed in this review, another interesting method to find QFTs, specifically unitary CFTs, is the conformal bootstrap program started in [683] and applied to 4d $\mathcal{N} = 2$ theories in [684].

The way class S theories are built by combining building blocks through gauging suggests to introduce a notion of theory space [685, 686].

The correspondence has also increased the interest in several 2d CFT questions:

- Computing conformal blocks, correlators, and fusion matrices, either through recursion relations [221, 611, 641, 687–697] or using holography in the large c limit [199, 404, 698–716].
- Studying variants of Toda CFT, parafermionic Liouville CFT etc. [717–719].
- Some (disputed) links to the fractional quantum Hall effect [720–724].
- Isomonodromy problems, as it is now known that conformal blocks (and hence Nekrasov partition functions), Fourier transformed with respect to internal momenta, give solutions to Painlevé equations arising in isomonodromy problems for Fuchsian connections [203, 424, 444, 469, 513, 529, 625, 644, 655, 656, 700, 713, 725–764]; likewise the chiral blocks of the q-deformed Virasoro algebra and qW-algebras give solutions of q-Painlevé equations [243–245].

[56] Reference lists are both less complete and less properly filtered here than elsewhere in the review.

Incidentally, the Liouville CFT has finally been defined mathematically from its path integral: see [765, 766] and references therein. Other mathematical references include the study of 6j symbols of (the modular double of) $U_q(\mathfrak{sl}_2)$ [81], relations to the geometric Langlands correspondence or deformations thereof [554, 555, 767–770], and more [75, 771–774].

For holographic duals of the 6d $\mathcal{N} = (2,0)$ theory, of 4d $\mathcal{N} = 2$ class S theories, and in the presence of extended operators, see [314, 576, 721, 775–795].

Supersymmetric localization applies to many background geometries, and for a sample of interesting cases see [353, 356, 796–808] and reviews [799, 803]. Resurgence and Borel summability of various expansions of Z_{inst} (and of other exact results from supersymmetric localization) are studied in [188, 733, 760, 809–813]. These give some insight on how applicable resurgence techniques are in QFT.

As known since Witten's [814], many knot invariants can be expressed as partition functions or other observables of gauge theories. Through the 3d/3d correspondence and other string dualities this relates to the AGT correspondence [425, 541, 815–834].

Final thoughts. The construction of new theories by dimensionally reduction in various geometrical setups has proven very fruitful. It has led to many new quantum field theories that can be used as building blocks for yet more discoveries. The large number of dualities uncovered in this way can be further enriched by considering extended operators (loops, surfaces, walls) in their various incarnations.

I hope that readers will participate in this exciting journey charting the space of quantum field theories!

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Table of acronyms

ABJM Aharony–Bergman–Jafferis–Maldacena (M2 brane worldvolume theory)

AD Argyres–Douglas (strongly coupled 4d $\mathcal{N} = 2$ theories)

AGT Alday–Gaiotto–Tachikawa (4d/2d correspondence)

ALE asymptotically locally Euclidean space (resolution of $\mathbb{C}^2/\Gamma$)

BPS Bogomol'nyi–Prasad–Sommerfield (supersymmetric)

CFT conformal field theory

DIM Ding–Iohara–Miki (algebra)

DOZZ Dorn–Otto–Zamolodchikov–Zamolodchikov (three-point function in Liouville CFT)

DS Drinfeld–Sokolov (reduction of W-algebras)

FI Fayet–Iliopoulos (parameter in supersymmetric action)

GW Gukov–Witten (surface defect)

IR infra-red (low energy/long distance)

JK Jeffrey–Kirwan (residue prescription)

KK Kaluza–Klein (reduction on circle)

LMNS Losev–Moore–Nekrasov–Shatashvili (formula for the instanton partition function)

NS Nekrasov–Shatashvili ($b \to 0$ limit, i.e., $\epsilon_1 \to 0$)

OPE operator product expansion

QFT quantum field theory

RG renormalization group

SCFT superconformal field theory

SQCD super-QCD (SYM plus matter)

SW Seiberg–Witten (curve Σ and differential λ giving IR description and prepotential F of 4d $\mathcal{N}=2$ theories)

SYM super-Yang–Mills (for us, 4d $\mathcal{N}=2$ vector multiplet)

TQFT topological quantum field theory

UV ultra-violet (high energy/short distance)

VEV vacuum expectation value

YBE Yang–Baxter equation

YM Yang–Mills (non-supersymmetric)

YRISW the Young Researchers Integrability School and Workshop

A Special functions

In the main text we use the following special functions.

- The Gamma function $\Gamma(x) = \prod_{n\geq 0}^{\text{reg}} \frac{1}{x+n}$ has poles at $-\mathbb{Z}_{\geq 0}$ and obeys the shift formula $\Gamma(x+1) = x\Gamma(x)$.
- The Barnes Gamma function $\Gamma_b(x) = \prod_{m,n\geq 0}^{\text{reg}} \frac{1}{x+mb+n/b}$ has poles at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$ and obeys the shift formula $\Gamma_b(x+b)/\Gamma_b(x) = \sqrt{2\pi}b^{xb-1/2}/\Gamma(xb)$.
- The double-sine function $S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(b+1/b-x)}$ has poles at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$, zeros at $b\mathbb{Z}_{\geq 1} + b^{-1}Z_{\geq 1}$, and obeys the shift formula $S_b(x+b)/S_b(x) = 2\sin(\pi bx)$.
- The Upsilon function $\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(b+1/b-x)}$ has zeros at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$ and $b\mathbb{Z}_{\geq 1}+b^{-1}Z_{\geq 1}$, and obeys the shift relation $\Upsilon_b(x+b)/\Upsilon_b(x) = b^{1-2bx}\Gamma(bx)/\Gamma(1-bx)$.

Note that $\Gamma_b, S_b, \Upsilon_b$ are invariant under $b \to 1/b$.

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