**Motivation**

The response of an oscillating circuit like a cavity to a Gaussian or even arbitrary beam current is not trivial, in contrast to its response to sine or cosine waves. The dynamical model should therefore use the Fourier coefficients of the beam current as states. The deduced model satisfies this in a natural way. Apart from that the approach follows the moment method [2, 4].

The Fourier decomposition is especially suited for modeling beam loading effects. The common approach of K. W. Robinson [6] for narrow-band is extended to broad-band (e.g. lattice) cavities by including higher harmonics in addition to the fundamental.

**Approach**

**Background - Longitudinal Single Particle Dynamics**

Equations of a single particle in $|1, 2, \ldots, M|$ with relative phase $\phi_m$ and scaled energy deviation $E_m$:
\[
\dot{\phi}_m = \omega_m, \quad \dot{E}_m = 0 \quad (0 = m) \quad \text{with} \quad \frac{\hbar \omega_m}{2Q},
\]
and
\[
\omega_m^2 = \sum_{k=1}^M \cos(\omega_{mk} + b k \Phi) E_k, \quad \omega_m^2 = \sum_{k=1}^M \sin(\omega_{mk} + b k \Phi) E_k
\]

Equilibrium w/o beam loading: $u_m = 1$, $n = 1$.

**Model - Beam Dynamics**

State Definition
\[
c_{0} = \sum_{k=1}^{N} \cos(\omega_{mk} + b k \Phi) E_k, \quad s_{0} = \sum_{k=1}^{N} \sin(\omega_{mk} + b k \Phi) E_k
\]

Remarks
- No significant simplifications / approximations are made.
- The present class of bi-linear (control-affine) systems is well investigated regarding characteristics and control theory.
- Infinite-dimensional model → truncation necessary
- The input is no mere control input but may come from the feedback through beam loading (cf. block diagram).

**Example**

- stationary bucket ($\Phi = 0$)
- sinusoidal gap voltage, i.e. high Q value or negligible beam loading

Closed-Loop System

**Add-on - Cavity Dynamics**

Transfer functions of the cavity are deduced by inserting
\[
\hat{u}(t) = (1 + s t)^{\Theta(1)} \sin(\omega_d t + \Phi_d) \cos(\omega_s t)
\]
and
\[
\hat{u}(t) = \hat{u}(t) + s t \sin(\omega_d t + \Phi_d) \cos(\omega_s t)
\]
into the equation of the equivalent parallel RLC resonator circuit.

For consistency with the beam model a representation with sines and cosines is shown. Representation by phases and amplitudes requires an additional vector diagram (see e.g. [1, Fig. 11]).

The transfer functions for the fundamental harmonic $k = 1$ given in [1, 5] follow from approximations $s < \omega$ and $\omega < \omega_d$ (resulting from $Q > 1$), assuming a harmonic excitation with a frequency close to resonance. Therefore it is not possible to simply replace $\omega$ by $k \omega_d$ to extend the model.

**Outlook**

- More general steady state conditions may follow from superposition examining every harmonic component separately.
- Lumped damping can be included by time-dependent truncation to analyze the response to larger disturbances.
- The model may be used for controller design
  - with and w/o relevant beam loading
  - with single and higher-harmonic gap voltage / control inputs.

**Application - Stability Considerations**

- long bunch, narrow-band cavity
  - similar to Pedersen model
  - truncation could not be adapted

- short bunch, narrow-band cavity
  - Robinson diagram for comparison
  - limiting case of the adjoint figures

- short bunch, broad-band cavity ($Q = 50$)
  - single particle (association)
  - harmonic components considered

References


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