BFKL at next-to next-to leading order

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Talk based on:
I present recent results for an approximation of the $O(\alpha_s^3)$ contribution $\chi_2$ to the kernel of the BFKL equation which includes all the singular contributions and it’s derived using duality between GLAP and BFKL.

1. The BFKL equation

2. The construction of the collinear approximation
   - Duality relations
   - Leading order computation

3. Beyond leading order
   - Running coupling duality
   - Integrated / unintegrated parton distribution
   - Scheme dependence
   - Choice of kinematic variables and symmetrisation

4. Discussion of the results

5. Conclusions
LHC will explore regions of large $Q^2$ and small-$x$.

It is important to understand the high energy (small-$x$) limit of QCD.
The BFKL equation describes the high energy evolution of the gluon distribution $G$:

$$\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha_s) G(\xi, M),$$

where $\xi = \ln \frac{s}{Q^2} = \ln \frac{1}{x}$ and $M$ is the Mellin moment w.r.t. $Q^2$.

The introduction of the running coupling is nontrivial:

$$\alpha_s(t) = \frac{\alpha_s}{1 + \alpha_s \beta_0 t} \implies \hat{\alpha}_s = \frac{\alpha_s}{1 - \alpha_s \beta_0 \frac{\partial}{\partial M}},$$

where $t = \ln \frac{Q^2}{\mu^2}$.

Different arguments for the running coupling correspond to different orderings of the operators.
The BFKL kernel is known exactly at NLO accuracy:

\[ \chi(M, \hat{\alpha}_s) = \hat{\alpha}_s \chi_0(M) + \hat{\alpha}_s^2 \chi_1(M) + \ldots \]  

[Fadin and Lipatov, 1998]

NLO computed studying the Regge limit of parton - parton scattering.

The NLO corrections are large and they change the qualitative shape of the kernel.
Consider the GLAP and the BFKL equations:

\[
\frac{d}{dt} G(N, t) = \gamma(N, \alpha_s) G(N, t)
\]

\( t \) - evolution for \( N \) moments of \( G \)

\[
\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha_s) G(\xi, M)
\]

\( \xi \) - evolution for \( M \) moments of \( G \)

Duality is the statement that these two equations admit the same leading twist solution when:

\[
\chi(\gamma(N, \alpha_s), \alpha_s) = N
\]

\[
\gamma(\chi(M, \alpha_s), \alpha_s) = M
\]

and the boundary conditions are suitably matched.
The LO computation

- Duality can be used to compute the BFKL collinear singularities from the small-$x$ singularities of the GLAP anomalous dimension:

$$\gamma = \frac{\alpha_s}{N} + \ldots$$

- Duality states that $\gamma$ is the inverse of $\chi$:

$$\gamma(\chi(M)) = M \implies \frac{\alpha_s}{\chi} = M$$

$$\implies \chi = \frac{\alpha_s}{M} + \ldots$$

- The LO anomalous dimension determines the collinear singularity of the LO BFKL kernel.
- Diagrams for BFKL processes are symmetric upon the exchange of the virtualities at the top and the bottom of the ladder. In Mellin space the kernel is symmetric upon:

$$M \leftrightarrow 1 - M,$$

so the result can be extended to the anticollinear region.
The collinear approximation is very close to the full kernels both at LO (max difference of 0.8%) and NLO (~ 1.5%).
The GLAP anomalous dimension has been computed at $O(\alpha_s^3)$ [Vogt, Moch and Vermaseren, 2004].

This determines the collinear singularities of the BFKL kernel at next-to next-to leading order:

$$\chi_2(M) = \frac{c_{2,-3}}{M^3} + \frac{c_{2,-2}}{M^2} + \frac{c_{2,-1}}{M} \quad \text{1}.$$  

Things are more complicated than this: beyond leading order different contributions arise.

- Running coupling effects.
- Integrated / unintegrated issue.
- Scheme dependence.
- Choice of kinematic variables (responsible for higher order poles).

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$^1$Kernel in asymmetric variables
The frozen coupling hypothesis is no longer valid beyond leading order. Duality relations still hold but they receive \textit{running coupling contributions} [Altarelli, Ball, Forte, 2002].

Running coupling duality has been proven to all orders using an operator method [Ball and Forte, 2006].

Because of non-vanishing commutation relations the inversions of the kernels is not trivial.

This formalism enables us to compute the running coupling corrections in an algebraic way calculating commutators of the relevant operators

\[
[\hat{\alpha}_s^{-1}, M] = -\beta_0 + \alpha_s \beta_0 \beta_1 + \ldots
\]

and express the result in terms of the fixed coupling duals.
The BFKL kernel describes the evolution of a parton density $G$ unintegrated over the transverse momenta, while GLAP of the integrated one $G$.

The relation between $G$ and $G$ is given by:

$$ G(N, t) = \frac{d}{dt} G(N, t) \Rightarrow G(N, M) = MG(N, M). $$

The relation between the integrated and the unintegrated kernels can be computed with the operator formalism.
The GLAP anomalous dimension and the BFKL kernel are naturally computed in different factorisation schemes: $\overline{\text{MS}}$ and $Q_0$.

The NLO normalisation factor (which gives the NNLO scheme dependence of the kernels) can be written as:

$$R(N, t) = \mathcal{N}(N, t)\mathcal{R}(N, t) , \quad [\text{Ciafaloni and Colferai, 2005}]$$

$\mathcal{N}$ contains the running coupling and the integrated / unintegrated contributions.

$\mathcal{R}$ is due to the fact that in the $\overline{\text{MS}}$ scheme the anomalous dimension is defined as the residue of the simple $\varepsilon$ pole in the partonic cross section:

$$
\gamma(\alpha_s, N, \varepsilon) \over \beta(\alpha_s, \varepsilon) = \frac{1}{\alpha_s\varepsilon} \left( 1 - \frac{\beta(\alpha_s)}{\alpha_s \varepsilon} + \ldots \right) (\gamma(\alpha_s, N) + \varepsilon \gamma(\alpha_s, N) + \ldots )
$$

$$
= \frac{1}{\alpha_s \varepsilon} \left( \gamma(\alpha_s, N) - \frac{\beta(\alpha_s)}{\alpha_s} \gamma(\alpha_s, N) + \ldots \right).
$$
The auxiliary scheme $\overline{MS}^*$

- The scheme change $\mathcal{R}$ is calculated from the $d$-dimensional GLAP kernel.
- The NLO $O(\varepsilon)$ contribution is not available in the literature, though it can be in principle extracted from the $d$-dimensional splitting amplitudes.
- Using fixed coupling duality on $\gamma^{\overline{MS}^*}$ one gets the collinear part of $\chi^{Q_0}$. 

\[ \chi^{Q_0} \quad rcd + \text{int/unint} \quad \chi^{\overline{MS}^*} \]
\[ \chi^{\overline{MS}^*} \quad rcd + \text{int/unint} \quad \chi^{\overline{MS}} \]
\[ \chi^{\overline{MS}} \quad rcd + \text{int/unint} \quad \chi^{Q_0} \]

\[ \gamma^{Q_0} \quad nd \quad \tilde{\chi}^{Q_0} \]
\[ \gamma^{\overline{MS}^*} \quad nd \quad \tilde{\chi}^{\overline{MS}^*} \]
\[ \gamma^{\overline{MS}} \quad nd \quad \tilde{\chi}^{\overline{MS}} \]

\[ \mathcal{N} \quad rcd + \text{int/unint} \quad \mathcal{R} \]
Exchange symmetry

- The exchange symmetry \( M \equiv 1 - M \) is broken by choice of kinematic variables (e.g. DIS \( x = \frac{Q^2}{s} \)), argument of the running coupling \( \alpha_s(Q^2) \).

- The collinear approximation of the BFKL kernel can be written in symmetric variables thanks to the relation:

\[
\chi^{sym}(\hat{\alpha}_s, M) = \chi^{DIS}(\hat{\alpha}_s, M + \frac{1}{2} \chi^{sym}(\hat{\alpha}_s, M)).
\]

- The result can be extended to the anticollinear region \( M \sim 1 \):

\[
\chi^{sym}_2(\hat{\alpha}_s, M) = \sum_{j=1,5} c_{2,-j} \left[ \hat{\alpha}_s^3 \frac{1}{M^j} + \frac{1}{(1 - M)^j} \hat{\alpha}_s^3 \right] + O(M^0).
\]

- The different order of the operators in the collinear and anticollinear regions corresponds to a symmetric choice for the running coupling.

- After the symmetrisation one can express the results canonically ordered with all the powers of \( \hat{\alpha}_s \) on the left.
The result for $\chi_2$

The NNLO kernel for evolution of the unintegrated distribution, with the argument of the strong coupling chosen as $\alpha_s(Q^2)$, and the symmetric choice of kinematic variables is:

$$\chi_2(M) = \frac{C_A^3}{2\pi^3} \left( \frac{1}{M^5} + \frac{1}{(1-M)^5} \right) - \frac{C_A^2}{2\pi^2} \left( - \frac{11C_A}{4\pi} - \frac{n_f}{2\pi} + \frac{C_F n_f}{\pi C_A} + \beta_0 \right) \left( \frac{1}{M^4} + \frac{1}{(1-M)^4} \right)$$

$$+ 4\beta_0 \frac{C_A^2}{\pi^2} \left( \frac{1}{(1-M)^4} + \frac{C_A}{\pi} \left[ \left( - \frac{11C_A}{12\pi} - \frac{n_f}{6\pi} + \frac{C_F n_f}{3\pi C_A} \right)^2 - \frac{C_A}{9C_A^3} \frac{C_F n_f^2}{18C_A^2} - \frac{11C_F n_f}{36C_A^2} - \frac{C_A}{6} + \frac{67C_A}{36\pi} \right] - \frac{13C_F n_f}{18\pi^2} - \frac{23C_A n_f}{36\pi^2} \right)$$

$$\times \left( \frac{1}{M^3} + \frac{1}{(1-M)^3} \right) + 2\frac{C_A}{\pi} \beta_0 \left( \beta_0 + \frac{11C_A}{6\pi} + \frac{n_f}{3\pi} - \frac{2C_F n_f}{3\pi C_A} \right) \frac{1}{(1-M)^3}$$
The result for $\chi_2$

\[
\begin{align*}
+ & \left[ - \frac{1}{2} \left( \frac{C_A^3 \zeta(3)}{2\pi^3} + \frac{11C_A^3}{72\pi} - \frac{395C_A^3}{108\pi^3} + \frac{C_A n_f}{36\pi} - \frac{71C_A^2 n_f}{108\pi^3} - \frac{C_A C_F n_f}{18\pi} + \frac{71C_A C_F n_f}{54\pi^3} \right) \\
+ & \left( - \frac{11C_A}{12\pi} - \frac{n_f}{6\pi} + \frac{C_F n_f}{3\pi C_A} \right) \left( \frac{13C_F n_f}{18\pi^2} - \frac{23C_A n_f}{36\pi^2} \right) \\
+ & \frac{C_A}{\pi} \left( - \frac{2\zeta(3)C_A^2}{\pi^2} + \frac{1643C_A^2}{216\pi^2} - \frac{11C_A^2}{36} + \frac{43C_A n_f}{54\pi^2} + \frac{C_F n_f}{18} - \frac{547C_F n_f}{216\pi^2} + \frac{13C_F n_f^2}{108\pi^2 C_A} \right) \\
+ & \frac{C_A^2 n_f}{4\pi^2 C_A} - \frac{13C_F n_f^2}{54\pi^2 C_A^2} + \beta_0 \left( \frac{67}{12\pi} + \frac{7n_f}{81\pi} \right) + \frac{3\zeta(3)C_A^3}{2\pi^3} \right] \left( \frac{1}{M^2} + \frac{1}{(1-M)^2} \right) \\
- & \beta_0 \left( \frac{C_A}{\pi} \beta_1 + 2 \left( \frac{13C_F n_f}{18\pi^2} - \frac{23C_A n_f}{36\pi^2} \right) \right) \frac{1}{(1-M)^2} \\
+ & \left[ - \frac{143\zeta(3)C_A^3}{24\pi^3} - \frac{29\pi C_A^3}{720} - \frac{389C_A^3}{432\pi} + \frac{73091C_A^3}{2592\pi^3} - \frac{11C_A^2 \zeta(3)n_f}{12\pi^3} - \frac{C_A^2 n_f}{9\pi} \\
+ & \frac{301C_A^2 n_f}{81\pi^3} + \frac{8\zeta(3)C_A C_F n_f}{3\pi^3} + \frac{35C_A C_F n_f}{108\pi} + \frac{59C_A n_f^2}{648\pi^3} - \frac{28853C_A C_F n_f}{2592\pi^3} \right. \\
- & \frac{2\zeta(3)C_F n_f^2}{3\pi^3} - \frac{65C_F n_f^2}{324\pi^3} + \frac{11C_F n_f^2}{12\pi^3} - \beta_0 \delta_1 - \beta_0 \frac{\pi^2}{12} - 2\beta_0 \frac{\zeta(3)C_A^2}{\pi^2} \right] \left( \frac{1}{M} + \frac{1}{1-M} \right).
\end{align*}
\]
The BFKL expansion is not well behaved (due to poles of increasing order).

$\chi_2$ has a minimum for every value of the coupling (here $\alpha_s = 0.2$).

The uncertainty due to the unknown part of the scheme change is comparable to what is expected in the collinear approximation ($\sim 5\%$).
The NNLO curve is closer to the LO for higher values of $\alpha_s$ than the NLO one. The convergence is still poor in the region interesting for phenomenology.
The collinear approximation of the BFKL kernel has been discussed. Results on duality relations and factorisation schemes, with a complete inclusion of the running coupling enable us to construct an approximation of the BFKL kernel at NNLO. Because the collinear approximation of $\chi_0$ and $\chi_1$ are in excellent agreement with the full results, $\chi_2$ is likely to be close to the complete unknown NNLO kernel. Because of the slow convergence of the series, the NNLO contribution is important for an accurate assessment of the NLO uncertainty.