About epsilon-expansion of Hypergeometric Functions

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Motivation

Oleg Tarasov (1996), and Davydychev-Delbourgo (1997), have suggested two elegant approaches for construction of hypergeometric representation of one-loop Feynman Diagrams. One of main achievement of these approaches are the essential reduction of independent variables. In accordance with Fleischer-Jegerlehner-Tarasov, 2003.

Type of 1-loop diagram	# 1	# 2
N=2 (propagator)	3	1
N=3 (vertex)	6	2
N=4 (box)	10	3
N = 5 (pentagon)	15	4
N = k	k(k+1)/2	k-1

where
#1 the number of kinematic invariants (non-zero masses/momenta)
and
#2 the number of variables in hypergeometric representation.
In this approach, the one-loop N-point function is expressible in terms of hypergeometric functions of $N-1$ variables.
New results for 1-loop Feynman integrals: Kniehl-Tarasov, 2009, 2010.

1-loop diagrams: Finite part

The program of constructing the analytical coefficients of the ε-expansion is a more complicated matter.
The finite parts of one-loop diagrams in $d = 4$ dimension are expressible in terms of the Spence dilogarithm function
t'Hooft, Veltman, 1979;
Denner, Nierste, Scharf, 1991;
Ellis, Zanderighi, 2007;
Denner, Dittmaier, 2010

Solved 1-loop: beyond finite part

Only partial results for higher-order terms in the ε-expansion are known at one loop. The all-order ε-expansion of the one-loop propagator with an arbitrary values of masses and external momentum has been constructed in terms of Nielsen polylogarithms:
Davydychev, 1999; Davydychev-Kalmykov., 2000, 2001;
The term linear in ε for the one-loop vertex diagram with non-exceptional kinematics has also been constructed in terms of Nielsen polylogarithms
Nierste-Müller-Böhm, 1993
The all-order ε expansion for the one-loop vertex with non-exceptional kinematics is expressible in terms of multiple polylogarithms of two variables:
Tarasov, 2000; Davydychev, 2005; Tarasov, 2008
Beyond these examples, the situation is less complete. The term linear in ε for the box diagram is still under construction. Some cases for particular masses have been analyzed
Fleischer-Riemann-Tarasov, 2003;
Körner-Merebashvili-Rogal, 2005,2006;
Del Duca, Dixon, Drummond, Duhr, Henn, Smirnov, 2009-2011;

Our goal

Local:
Elaboration of algorithm for construction of analytical coefficients of ε-expansion of multiple hypergeometric (Horn-type) functions.

Final:
Package for Numerical Evaluation of finite, $O(\varepsilon)$ and $O(\varepsilon^2)$ parts of one-loop Feynman Diagrams with an arbitrary set of kinematic invariants

Comments 1:
Weinzierl, 2001
"Does one need the $O(\varepsilon)$- and $O(\varepsilon^2)$-terms of one-loop amplitudes in an NNLO calculation ?"

Comments 2:
Kurihara, 2006; Kaneko, 2010, 20111
Numerical approach to evaluation of finite part of 1-loop Feynman Diagrams on the basis of hypergeometric representation

Horn-type Hypergeometric Functions

In accordance with Horn definition, a formal (Laurent) power series in r variables,

$$\Phi(\vec{z}) = \sum C(\vec{m})\vec{z}^m \equiv \sum_{m_1,m_2,\cdots,m_r} C(m_1,m_2,\cdots,m_r)x_1^{m_1}\cdots x_r^{m_r},$$

is called hypergeometric if for each $i = 1,\cdots,r$ the ratio

$$\frac{C(\vec{m}+e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \Rightarrow C(\vec{m}) = \Pi_{i=1}^r \lambda_i^{m_i} R(\vec{m}) \left(\frac{\Pi_{j=1}^N \Gamma(\mu_j(\vec{m})+\gamma_j)}{\Pi_{k=1}^M \Gamma(\nu_k(\vec{m})+\delta_k)}\right) .$$

P, Q, R are the rational functions in the index of summation: $\vec{m} = (m_1,\cdots,m_r)$, and $\vec{e}_j$ is unit vector with unity in the jth place.

The Horn hypergeometric function satisfies the following system of equation

$$Q_j\left(\sum_{k=1}^r x_k\frac{\partial}{\partial x_k}\right)\frac{1}{x_j}\Phi(\vec{z}) = P_j\left(\sum_{k=1}^r x_k\frac{\partial}{\partial x_k}\right)\Phi(\vec{z}) .$$

Current status

The systematic algorithms for construction of analytical coefficients of ε-expansion for a large class of Horn-type hypergeometric functions around integer values of parameters was suggested by Moch-Uwer-Weinzierl, 2001.

Hypergeometric function	Algorithm
${}_pF_{p-1}$	A
F_1	A, B
F_2, F_3	A,C,D
F_4	does not work

Two of these algorithms $A \Rightarrow {}_pF_{p-1}$ and $B \Rightarrow F_1$
was extended for zero-balance case: Weinzierl, 2004.
It was a few attempts to extend these approach
to hypergeometric functions like F_4
Del Duca,Duhr,Glover,Smirnov, 2009
or to another (different from zero-balance case) set of parameters
Rottmann-Reina, 2011
Ablinger-Blümlein-Schneider , 2011

Alternative technique

Direct integration of one-, or two-fold integrals:

Anastasiou-Glover-Oleari, 1999

Körner-Merebashvili-Rogal, 2005, 2006;

Aglietti-Del Duca-Duhr-Somogyi-Trocsanyi, 2008

Del Duca-Duhr-Glover-Smirnov, 2009

Multi-variables generalization of Generating Function approach:
Jegerlehner-Kalmykov-Veretin, 2003
Davydychev-Kalmykov, 2003;
Kalmykov-Ward-Yost, 2007;
Kalmykov-Kniehl, 2008.

Our technology: starting point

We suggested a new technology for construction of analytical coefficients of ε-expansion of any hypergeometric function:

Kalmykov-Ward-Yost, 2006-2007

Let us consider the function $F(\vec{a};\vec{z})$ where

$\{\vec{a}\}$ are a set of discrete variables and $\{\vec{z}\}$ are a set of continuous variable.

Let us suggested that $F(\vec{a};\vec{z})$ satisfy to the system of differential equations:

$$\sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z})\frac{\partial}{\partial z_k}F(\vec{a};\vec{z}) = 0 \ ,$$

and $P_{\vec{J};k}(\vec{a};\vec{z})$ are polynomial. The coefficients of ε-expansion for function $F(\vec{a};\vec{z})$ are derivative of function with respect to parameters A:

$$F(\vec{A}_0+\vec{a}\varepsilon;\vec{z}) \ = \ F_0(\vec{A}_0;\vec{z}) + \sum_{j=1}^{\infty}\varepsilon^j f_j(\vec{z}) \ ,$$

$$f_j(\vec{z}) \ = \ \frac{\partial}{\partial\vec{A}}F(\vec{A};\vec{z})\Big|_{A\equiv A_0} \ .$$

Our technology: direct solution of PDE for coefficients functions

The system of differential equations for coefficient's functions $f_j(z)$ follows from original system of PDE for function $F(\vec{a};\vec{z})$

$$\sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z})\frac{\partial}{\partial z_k}F(\vec{a};\vec{z}) = 0 \Rightarrow \frac{\partial}{\partial \vec{a}}\left[\sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z})\frac{\partial}{\partial z_k}F(\vec{a};\vec{z})\right] = 0$$

$$\Rightarrow \sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z})\Bigg|_{\vec{a}=\vec{a_0}} \frac{\partial}{\partial z_k}f_j(z) = -\sum_{J,k}\left\{\frac{\partial}{\partial \vec{a}}P_{\vec{J};k}(\vec{a};\vec{z})\right\}\frac{\partial}{\partial z_k}F(\vec{a};\vec{z})\Bigg|_{\vec{a}=\vec{a_0}}$$

$P_{\vec{J};k}(\vec{a};\vec{z})$ are polynomial and its derivative are polynomial.

The coefficients functions $f_j(z)$ satisfy to the system of non-homogeneous PDE. and non-homogeneous part is defined by function $F(\vec{a};\vec{z})$ at rational values of parameters $\vec{a} = \vec{a}_0$.

Equivalent to Multiple Residues

When non-homogeneous PDE are solvable?

When the system of PDE with non-zero non-homogeneous part are solvable in terms of multiple polylogarithms?

(i) the differential operator(s) are factorisable into product of differential operators of the first order;

(ii) there is linear parametrization;

(iii) non-homogeneous part belongs to the function of the special type.

One-variable example:

$$\sum_{k=0}^{r} P_k(z;\varepsilon)\left(\frac{d}{dz}\right)^k = F(z;\varepsilon)$$

where $P_k(z;\varepsilon), F(z;\varepsilon)$ are rational functions or iterated integrals over rational 1-form with rational function:

$$P_k(z;\varepsilon) = \frac{\Pi_j(z-\alpha_j-\beta_j\varepsilon)}{\Pi_r(z-A_r-B_r\varepsilon)}\,, \quad F(z;\varepsilon) = \int^z \frac{dt}{t-\sigma}\frac{\Pi_j(t-\mu_j-\nu_j\varepsilon)}{\Pi_r(t-M_r-N_r\varepsilon)}\,,$$

Iterated integral

We are looking for solution in terms of iterated integrals.

The generalized iterated integral is defined as

$$I(z; R_k, R_{k-1}, \cdots, R_1) = \int_0^z \frac{dt}{R_1(t)} I(t; R_{k-1}, \cdots, R_1)$$

$$= \int_0^z \frac{dt_k}{R_k(t)} \int_0^{t_k} \frac{dt_{k-1}}{R_{k-1}(t)} \cdots \int_0^{t_2} \frac{dt_1}{R_1(t_1)} .$$

where $R_k(t)$ are some functions (may be not polynomial, not elementary).

Multiple polylogarithm corresponds to $R_k(t) = t - a_k$.

$$I(z; a_k, a_{k-1}, \cdots, a_1) = \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1}, \cdots, a_1)$$

$$= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1}$$

Illustration

(I) Factorization :

$$\sum_{k=0}^{r} P_k(z;0)\left(\frac{d}{dz}\right)^k = \Pi_{k=1}^{r}\left[R_k(z;0)\frac{d}{dz} + Q_k(z;0)\right]$$

(II) Linear parametrization :

The solution of differential equation

$$\left[R_1(z)\frac{d}{dz} + Q_1(z)\right]\left[R_2(z)\frac{d}{dz} + Q_2(z)\right] f(z) = F(z)$$

is iterated integral

$$f(z) = \int^{z} \frac{dt_2}{R_2(t_2)}\left[\exp^{-\int_0^{t_2}\frac{Q_2(t_3)}{R_2(t_3)}dt_3}\right]\int^{t_2}\frac{dt_1}{R_1(t_1)}\left[\exp^{-\int_0^{t_1}\frac{Q_1(t_2)}{R_1(t_2)}dt_2}\right]F(t_1)$$

Linear parametrization

The building block are Abel's type integrals:

$$\int^z \frac{Q_i(t)}{R_i(t)} dt = \ln \frac{M_i(\xi)}{N_i(\xi)} ,$$

$$\frac{dt}{R_2(t)}\bigg|_{t=t(\xi)} \frac{N_i(\xi)}{M_i(\xi)} = dx \frac{K_i(x)}{L_i(x)} ,$$

where M_i, N_i, K_i, L_i are polynomial functions.

When such parametrization exist?

(III) Special Form of non-homogeneous part's of PDE

Should satisfy to homogeneous PDE with Factorization and Linear parametrization

In one-variable case, the system should posses by Liouville solution.

Hypergeometric vs. Feynman Diagram

Two main building block: Factorization and Linear parametrization are sufficient to write solution in terms of multiple polylogarithms.

It is a cornerstone of all modern multiloop analytical evaluations of master-integrals performed on the basis of differential equation technique: Kotikov, Remiddi, Anastasiou, Argeri, Aglietti, Bonciani, Caffo, Czyz, Ferroglia, Gehrmann, Laporta, Mastrolia, Manteuffel, Maitre, Melnikov, Studerus, $\cdots$

We applied this technique for construction of ε-expansion of Horn-type hypergeometric functions.

Horn-type Hypergeometric Functions: reduction to the basis

Let us consider the series

$$\Phi(\vec{\gamma};\vec{\sigma};\vec{z}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\Pi_{j=1}^{K}\Gamma\left(\sum_{a=1}^{r}\mu_{ja}m_a+\gamma_j\right)}{\Pi_{k=1}^{L}\Gamma\left(\sum_{b=1}^{r}\nu_{kb}m_b+\sigma_k\right)} \right) x_1^{m_1}\cdots x_r^{m_r} \ ,$$

The sequences $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively. Two functions with sets of parameters shifted by a unit, $\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{z})$ and $\Phi(\vec{\gamma};\vec{\sigma};\vec{z})$, are related by a linear differential operator:

$$\Phi(\vec{\gamma}+\vec{e_c};\vec{\sigma};\vec{z}) = \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{z})$$

$$\Phi(\vec{\gamma};\vec{\sigma}-\vec{e_c};\vec{z}) = \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c \right) \Phi(\vec{\gamma};\vec{\sigma};\vec{z}) \ .$$

Horn-type Hypergeometric Functions: Inverse Operators

Staring from homogeneous system of PDE and direct differential operators, the inverse differential operators can be constructed:

$$\Phi(\vec{\gamma}-\vec{e_c};\vec{\sigma};\vec{z}) = \sum_a S_a(\vec{z},\vec{\partial_x})\Phi(\vec{\gamma};\vec{\sigma};\vec{z})$$

$$\Phi(\vec{\gamma};\vec{\sigma}+\vec{e_c};\vec{z}) = \sum_b L_b(\vec{z},\vec{\partial_x})\Phi(\vec{\gamma};\vec{\sigma};\vec{z}) \ .$$

In this way, any Horn-type function can be written as follows:

$$P_0(\vec{z})\Phi(\vec{\gamma}+\vec{k};\vec{\sigma}+\vec{l};\vec{z}) = \sum_{m_1,\cdots,m_p=0} P_{m_1,\cdots,m_r}(\vec{z})\left(\frac{\partial}{\partial\vec{z}}\right)^{\vec{m}}\Phi(\vec{\gamma};\vec{\sigma};\vec{z}) \ ,$$

where $P_0(\vec{z})$ and $P_{m_1,\cdots,m_p}(\vec{z})$ are polynomials with respect to $\vec{\gamma},\vec{\sigma}$ and $\vec{z}$ and $\vec{k},\vec{l}$ are lists of integers.

Simplify the procedure of Factorization.

Example: ${}_pF_{p-1}$ with unbalance set of parameters

$$ {}_pF_{p-1}(\vec{A};\vec{B};z)=\sum_{j=0}^{\infty}\frac{\Pi_{i=1}^{p}(A_i)_j}{\Pi_{k=1}^{p-1}(B_k)_j}\frac{z^j}{j!} $$

where $(A)_j=\Gamma(A+j)/\Gamma(A)$.

Problem: Construct the ε expansion for hypergeometric function

$$ {}_pF_{p-1}\left(\vec{I}+\vec{a}\varepsilon, A+c\varepsilon; \vec{K}+\vec{b}\varepsilon, B+f\varepsilon; z\right) $$

$\vec{I}$ and $\vec{K}$ - integers, A, B, $\vec{a}$, $\vec{b}$, c, f - arbitrary rational numbers.

Step I: reduction

$$ R_{p+1}{}_pF_{p-1}\left(\vec{I}+\vec{a}\varepsilon, A+c\varepsilon; \vec{K}+\vec{b}\varepsilon, B+f\varepsilon; z\right) $$
$$ =\sum_{k=1}^{p}R_k\left(z\frac{d}{dz}\right)^{k-1}{}_pF_{p-1}\left(\vec{a}\varepsilon, A+c\varepsilon; \vec{1}+\vec{b}\varepsilon, B+f\varepsilon; z\right) $$

Example: ${}_pF_{p-1}$ with one unbalance set of parameters: II

Step II: PDE & Factorization:

$$\left[z\,(\theta+A+c\varepsilon)\,\Pi_{j=1}^{p-1}(\theta+a_j\varepsilon)-\theta\,(\theta+B-1+f\varepsilon)\,\Pi_{k=1}^{p-2}(\theta+b_k\varepsilon)\right]\omega(z)=0\ ,$$

and writing its ε expansion as $\omega(z)=1+\sum_{j=1}^{\infty}w_k(z)\varepsilon^k$, we obtain

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right]\theta^{p-1}w_m=\left[P_1^{(p)}(\vec{a},c)-\frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\theta^{p-1}w_{m-1}$$
$$+\sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c)-\frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right]\theta^{p-j}w_{m-j}(z)+AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}$$
$$+\sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a})-\frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right]\theta^{p-1-k}w_{m-k}+P_p^{(p)}(\vec{a},c)w_{m-p}\ ,$$

where $\theta=zd/dz$ and $P_j^{(k)}(\vec{a})$ as some rational polynomial.

Example: ${}_pF_{p-1}$ with one unbalance set of parameters: III

Step III: Still Factorization :

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right] \Rightarrow h(z) = (-1)^A z^{1-B}(z-1)^{B-A-1} \ ,$$

In terms of $p-1$ new functions (redefinition)

$$f^{(k)} = (\omega, \theta\omega, \cdots, \theta^{p-2}\omega, h(z)\phi^{(p-1)})$$

$$h(z)(1-z)\frac{d}{dz}\phi_m^{(p-1)}(z) = h(z)\left[P_1^{(p)}(\vec{a},c)-\frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\phi_{m-1}^{(p-1)}$$

$$+\sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c)-\frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right]f_{m-j}^{(p-j)} + AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}$$

$$+\sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a})-\frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right]f_{m-k}^{(p-1-k)}+P_p^{(p)}(\vec{a},c)w_{m-p} \ ,$$

$$\theta f_m^{(p-2)}(z) = h\phi_m^{(p-1)} \ , \quad s\theta f_m^{(j-1)}(z) = f_m^{(j)} \ , \qquad j = 1, \cdots, p-2 \ .$$

Example: ${}_pF_{p-1}$ with one unbalance set of parameters: IV

Step IV: Linear parametrization:

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right] \Rightarrow h(z) = (-1)^A z^{1-B}(z-1)^{B-A-1} \ ,$$

Three integrals should posses linear parametrization:

$$\frac{dz}{z} = P(\xi)d\xi \ , \quad h(z) = Q(\xi) \ , \quad \frac{z}{(z-1)} = R(\xi) \ ,$$

where $Q(\xi), R(\xi), P(\xi)$ are rational functions of ξ.

For $A = r/q$ and $B = 1 - p/q$, where p, r, q are integers, the following parametrization exist in accordance with Weinzierl:

A	B	h(z)	ξ	$h(\xi)$	$z(\xi)$
0	1 - p/q	$[(z/(z-1)]^{p/q}$	$\xi = [z/(z-1)]^{1/q}$	ξ^p	$-\frac{z^q}{1-z^q}$
r/q	1	$(1-z)^{-r/q}$	$\xi = (1-z)^{1/q}$	ξ^{-r}	$1-\xi^q$
A	B-A = k	$(1-z)^{k-1}z^{p/q}$	$\xi = z^{1/q}$	$\xi^p(1-\xi^q)$	ξ^q

Horn hypergeometric function of few variables:

Generalization of previos technique for the multivariable linear PDE:

(I) Convert system of linear PDE with polynomial coefficients into Pfaffian form (matrix)

$$\sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z})\frac{\partial}{\partial z_k}F(\vec{a};\vec{z}) = 0 \Rightarrow d_k\omega_i(\vec{z}) = \Omega^k_{ij}(\vec{z})\omega_j(\vec{z})dz_k \ .$$

$$d_r\left[d_k\omega_i(\vec{z})\right] = 0 \ .$$

(II) Convert Pfaffian matrix into triangular form (new functions)

$$\omega_j = \Delta_{jk}v_k \ .$$

(III) Linear parametrization (solution in terms of multiple polylogarithms)

$$P_k(\vec{z})\frac{d}{dz_k} + Q_k(\vec{z}) \longrightarrow \text{linear form for each variable}$$

Horn hypergeometric function of two variables:

Step I: system of PDE from series represenation

Let us consider the system of linear differential equations of the second order for the functions $\omega(\vec{z})$:

$$\theta_{11}\omega(\vec{z}) = \left\{ P_0(\vec{z})\theta_{12} + P_1(\vec{z})\theta_1 + P_2(\vec{z})\theta_2 + P_3(\vec{z}) \right\}\omega(\vec{z}) \ ,$$

$$\theta_{22}\omega(\vec{z}) = \left\{ R_0(\vec{z})\theta_{12} + R_1(\vec{z})\theta_1 + R_2(\vec{z})\theta_2 + R_3(\vec{z}) \right\}\omega(\vec{z}) \ ,$$

$\vec{z} = (z_1, z_2)$, z_1, z_2 are variables, $\{P_j, R_j\}$ are rational functions,

$$\theta_j = z_j \partial_{z_j} \ , j = 1, 2 \ , \quad \theta_{i_1 \cdots i_k} = \theta_{i_i} \cdots \theta_{i_k}.$$

P_j,R_j are defined as ratio of coefficients $C(m_1, m_2)$ of series represenation

Horn hypergeometric function of two variables:

Step II: prolongation of system of PDE (first step)

From these two equation two another equations can be deduced:

$$\{\theta_1 - P_0\theta_2\}\,\theta_{12}\omega(\vec{z}) = \left\{ F_{12}\theta_{12}(\vec{z}) + F_1\theta_1(\vec{z}) + F_2\theta_2(\vec{z}) + F_0(\vec{z}) \right\}\omega(\vec{z}) \ ,$$

$$\{-R_0\theta_1 + \theta_2\}\,\theta_{12}\omega(\vec{z}) = \left\{ E_{12}\theta_{12}(\vec{z}) + E_1\theta_1(\vec{z}) + E_2\theta_2(\vec{z}) + E_0(\vec{z}) \right\}\omega(\vec{z}) \ ,$$

where

$$\begin{array}{l||l}
F_{12} = \theta_2 P_0 + P_1 + P_2 R_0 & E_{12} = \theta_1 R_0 + R_2 + R_1 P_0 \\
F_1 = P_2 R_1 + \theta_2 P_1 & E_1 = P_1 R_1 + \theta_1 R_1 + R_3 \\
F_2 = P_2 R_2 + \theta_2 P_2 + P_3 & E_2 = P_2 R_1 + \theta_1 R_2 \\
F_0 = P_2 R_3 + \theta_2 P_3 & E_0 = R_1 P_3 + \theta_1 R_3
\end{array}$$

Horn hypergeometric function of two variables:

Step II: Case I

Under condition

$$1 - P_0 R_0 \neq 0 \ ,$$

there are 4 independent solutions of differential system (F_2, F_3, F_4)

The original system can be reduced to the Pfaff system of four differential equations

$$d\vec{f} = R\vec{f} \ ,$$
$$\vec{f} = \omega(\vec{z}), \theta_1\omega(\vec{z}), \theta_2\omega(\vec{z}), \theta_{12}^2\omega(\vec{z}).$$

In this case there are two additional equation

$$(1-P_0R_0)\theta_{112}\omega(\vec{z}) = A_{12}(\vec{z})\theta_{12}^2 + A_1(\vec{z})\theta_1 + A_2(\vec{z})\theta_2 + A_0(\vec{z}) \ ,$$
$$(1-P_0R_0)\theta_{122}\omega(\vec{z}) = B_{12}(\vec{z})\theta_{12}^2 + B_1(\vec{z})\theta_1 + B_2(\vec{z})\theta_2 + B_0(\vec{z}) \ ,$$

Horn hypergeometric function of two variables:

Step II: Case II

Under conditions

$$1 - P_0 R_0 = 0 \ ,$$

In this case, the system has 3 solutions (F_1). and the original system of differential equations can be reduced to the Pfaff system of three differential equations

$$\vec{f} = (\omega(\vec{z}), \theta_1 \omega(\vec{z}), \theta_2 \omega(\vec{z})) \ ,$$

In this case there are one equation

$$\theta_{12}\omega(\vec{z}) = \left\{ Q_1(\vec{z})\theta_1 + Q_2(\vec{z})\theta_2 + Q_0(\vec{z}) \right\} \omega(\vec{z}) \ .$$

Example: Appell Hypergeometric Function F_3

Let us consider Appell hypergeometric function F_3

Series representation:

$$\omega_0 \equiv F_3(a_1, a_2, b_1, b_2, c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} ,$$

$$= \sum_{m} \frac{(a_1)_m (b_1)_m}{(c)_m} \frac{x^m}{m!} {}_2F_1\left(\left. \begin{array}{c} a_2, b_2 \\ c+m \end{array} \right| y \right) .$$

Symmetries It is symmetric with respect to simultaneous exchange

$$a_1 \Leftrightarrow a_2 , \quad b_1 \Leftrightarrow b_2 , \quad x \Leftrightarrow y .$$

F_3: Integral representation

Integral representation:

$$F_3(a_1,a_2;b_1,b_2;c;x_1,x_2)=\frac{\Gamma(c)}{\Gamma(c-b_1-b_2)\Gamma(b_1)\Gamma(b_2)}\int\int du_1du_2$$
$$\times\left[u_1^{b_1-1}u_2^{b_2-1}(1-x_1u_1)^{-a_1}(1-x_2u_2)^{-a_2})\right](1-u_1-u_2)^{(c-b_1-b_2-1)}\ ,$$

with

$$u_1,u_2\leq 0;1-u_1-u_2\leq 0$$

$$F_3(a_1,a_2;b_1,b_2;c;x_1,x_2)=\frac{\Gamma(c)\Gamma(c-b_1)}{\Gamma(b_1)}$$
$$\times\int_0^1 dtt^{b_1-1}(1-t)^{c-b_1-1}(1-x_1t)^{-a_1}\ {}_2F_1(a_2,b_2;c-b_1;(1-t)x_2)\ ,$$

F_3: System of differential equation

Step I: system of differential equations

$$\frac{C(m+1)}{C(m)} = \frac{P_1}{Q_1} = \frac{(a_1+m)(b_1+m)}{(c+m+n)(m+1)}$$

$$\begin{aligned}
&P(m \to \theta_x, n \to \theta_y)\frac{1}{x}w_0 = Q(m \to \theta_x, n \to \theta_y)w_0 \\
&\Rightarrow (\theta_x + a_1)(\theta_x + b_1)w = \partial_x(\theta_x + \theta_y + c - 1)w_0 \\
&\Rightarrow (1-x)\theta_{xx}\omega_0 = -\theta_{xy}\omega_0 + [(a_1+b_1)x-(c-1)]\,\theta_x\omega_0 + xa_1b_1\omega_0 \ ,
\end{aligned}$$

where

$$\theta_x = x\frac{\partial}{\partial x} \ , \quad \theta_y = y\frac{\partial}{\partial y} \ .$$

Step II: prolongation equations

$$\begin{aligned}
(xy-x-y)\theta_{xxy}\omega_0 \ = \ & [(1-y)(a_1+b_1)x-y(a_2+b_2+1-c)]\,\theta_{xy}\omega_0 \\
& +(1-y)xa_1b_1\theta_y\omega_0 - ya_2b_2\theta_x\omega_0 \ ,
\end{aligned}$$

Pfaff system is constructed.

F_3: Reduction procedure

Step III: differential reduction procedure: Implemented in HYPERDIRE

Direct Differential operators (from P,Q polynomial)

$$(c-1)F_3(a_1,a_2,,b_1,b_2,c-\mathbf{1};x,y) = (c-1+\theta_x+\theta_y)F_3(a,b_1,b_2,c_1,c_2;x,y) \ .$$

Inverse differential operators

$$\Delta F_3(a_1,a_2,b_1,b_2,c+\mathbf{1};x,y) =$$
$$c\left\{A - D_1\left(1-\frac{1}{x}\right)\theta_x - D_2\left(1-\frac{1}{y}\right)\theta_y + B\left(1-\frac{1}{x}-\frac{1}{y}\right)\theta_{xy}\right\}$$
$$\times F_3(a,b_1,b_2,c_1,c_2;x,y) \ ,$$
$$\delta_1 = c-a_1-b_1 \ , \quad \delta_2 = c-a_2-b_2 \ , \quad F = c-a_1-a_2-b_1-b_2 \ ,$$
$$A = \delta_1\delta_2 F + a_1b_1\delta_1 + a_2b_2\delta_2 \ , \quad B = \delta_1+\delta_2 \ ,$$
$$D_1 = \delta_2 F + a_1b_1 - a_2b_2 \ , \quad D_2 = \delta_1 F + a_2b_2 - a_1b_1 \ ,$$
$$\Delta = (c-b_1-b_2)(c-a_1-a_2)(c-a_2-b_1)(c-a_1-b_2) \ .$$

F_3: Triangular form of differential equations

Step IV: Conditions for triangular form For the most general values of parameters

$$a_1 = \frac{p_1}{q} + a_1\varepsilon \ , \quad a_2 = \frac{p_2}{q} + a_2\varepsilon \ ,$$
$$b_1 = \frac{r_1}{q} + b_1\varepsilon \ , \quad b_2 = \frac{r_2}{q} + b_2\varepsilon \ ,$$
$$c = 1 - \frac{p}{q} + c\varepsilon \ .$$

$$\sum_{m,n=0}^{\infty} \frac{\left(\frac{p_1}{q} + a_1\varepsilon\right)_m \left(\frac{p_2}{q} + a_2\varepsilon\right)_n \left(\frac{r_1}{q} + b_1\varepsilon\right)_m \left(\frac{r_2}{q} + b_2\varepsilon\right)_n}{\left(1 - \frac{p}{q} + c\varepsilon\right)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \ .$$

where p_i, r_i, p, q are integers,

The triangular form corresponds to

$$p_1 r_1 = 0 \ , \quad p_2 r_2 = 0 \ ,$$

F_3: Linear parametrization

$$\left[(1-x)\frac{d}{dx}-\frac{s_1}{q}-\frac{1}{x}\frac{p}{q}\right]\omega_1^{(r)}$$
$$=-\frac{1}{x}\omega_3^{(r)}+\left[(a_1+b_1)-\frac{c}{x}\right]\omega_1^{(r-1)}+\left[\frac{a_1r_1+b_1p_1}{q}\right]\omega_0^{(r-1)}+a_1b_1\omega_0^{(r-2)}\ ,$$
$$\left[(xy-x-y)\frac{d}{dx}-(1-y)\frac{s_1}{q}+\frac{y}{x}\frac{(s_2+p)}{q}\right]\omega_3^{(r)}$$
$$=\left[(1-y)(a_1+b_1)-\frac{y}{x}(a_2+b_2-c)\right]\omega_3^{(r-1)}$$
$$+(1-y)\left[\frac{a_1r_1+b_1p_1}{q}\omega_2^{(r-1)}+a_1b_1\omega_2^{(r-2)}\right]$$

where

$$s_1=p_1+r_1\ ,\quad s_2=p_2+r_2\ .$$

$$\{\omega_0,\omega_1,\omega_2,\omega_3\}\equiv\{\omega_0,\theta_x\omega_0,\theta_y\omega_0,\theta_{xy}\omega_0\}\quad \omega_j=\sum_{k=0}^{\infty}\omega_j^{(k)}\varepsilon^k\ ,\quad j=0,1,2,3,$$

F_3: Linear parametrization:

Redefinition

$$h_1(x) = (-1)^{\frac{s_1}{q}} \left[\frac{x^p}{(x-1)^{s_1+p}}\right]^{\frac{1}{q}} , \quad \omega_1^{(r)} = \theta_x \omega_0 \to h_1(x)\phi_1^{(r)} ,$$

$$h_2(y) = (-1)^{\frac{s_2}{q}} \left[\frac{y^p}{(y-1)^{s_2+p}}\right]^{\frac{1}{q}} , \quad \omega_2^{(r)} = \theta_y \omega_0 \to h_2(y)\phi_2^{(r)} ,$$

$$H(x,y) = (-1)^{\frac{s_1+s_2}{q}} \left[\frac{x^{s_2+p} y^{s_1+p}}{(xy-x-y)^{s_1+s_2+p}}\right]^{\frac{1}{q}}$$

$$\omega_3^{(r)} = \theta_x \theta_y \omega_0^{(r)} = H(x,y)\phi_3^{(r)} ,$$

Self-consistency condition

$$h_1(x)\theta_y \phi_1^{(r)} = H(x,y)\phi_3^{(r)} , \quad h_2(y)\theta_x \phi_2^{(r)} = H(x,y)\phi_3^{(r)} .$$

F_3: Final system

System of equation in terms of a new variables:

Suitable for construction of iterative solution

$$(1-x)\frac{d}{dx}\phi_1^{(r)} = -\frac{1}{x}\frac{H(x,y)}{h_1(x)}\phi_3^{(r)}$$
$$+\left[(a_1+b_1)-\frac{c}{x}\right]\phi_1^{(r-1)} + \left[\frac{a_1r_1+b_1p_1}{q}\right]\frac{1}{h_1(x)}\omega_0^{(r-1)} + \frac{1}{h_1(x)}a_1b_1\omega_0^{(r-2)} ,$$
$$(xy-x-y)\frac{d}{dx}\phi_3^{(r)} = \left[(1-y)(a_1+b_1)-\frac{y}{x}(a_2+b_2-c)\right]\phi_3^{(r-1)}$$
$$+\frac{a_1r_1+b_1p_1}{q}(1-y)\frac{h_2(y)}{H(x,y)}\phi_2^{(r-1)} + a_1b_1(1-y)\frac{h_2(y)}{H(x,y)}\phi_2^{(r-2)}$$
$$-\frac{a_2r_2+b_2p_2}{q}\frac{y}{x}\frac{h_1(x)}{H(x,y)}\phi_1^{(r-1)} - a_2b_2\frac{y}{x}\frac{h_1(x)}{H(x,y)}\phi_1^{(r-2)} ,$$

F_3: Linear parametrization: IV

First iteration corresponds to

$$ {}_2F_1\left(\frac{r_j}{q}, \frac{p_j}{q}, 1-\frac{p}{q}; z\right) \quad , \quad j = 1, 2, $$

It is expressible in terms of multiple polylogarithms when

$$ \begin{aligned} & s_j + p = 0 \quad , s_j, p \neq 0 \\ & s_j = 0, \quad p \neq 0 \ , \\ & p = 0 \ , \quad s_1 \neq 0 \ , \end{aligned} $$

The main problem is the function

$$ H(x,y) \sim \left[\frac{x^{s_2+p} y^{s_1+p}}{(xy-x-y)^{s_1+s_2+p}}\right]^{\frac{1}{q}} $$

Next iteration:

$$ s_1 + s_2 + p = 0. $$

F_3: Zero-balance case

Zero-balance case: $s_1 = -p\ , \quad s_2 = 0.$

$$h_1(x) \;=\; x^{\frac{p}{q}}\ , \quad h_2(y) = \left(\frac{y}{y-1}\right)^{\frac{p}{q}}\ , \quad H(x,y) = x^{\frac{p}{q}} \equiv h(x)\ ,$$

$$\xi_1 \;=\; x^{\frac{1}{q}}\ , \quad \xi_2 = \left(\frac{y}{y-1}\right)^{\frac{1}{q}}\ ,$$

$$H(x,y) \;\equiv\; h_1(x) = \xi_1^p\ , \quad h_2(y) = \xi_2^p\ , \quad x + y - xy = \frac{\xi_1^q - \xi_2^q}{1 - \xi_2^q}$$

F_3: Multiple binomial sums

Multiple binomial sums: $s_2 = p = 0\ , \quad s_1 \neq 0.$

$$h_1(x) = \left(\frac{1}{1-x}\right)^{\frac{s_1}{q}}\ , \quad h_2(y) = 1\ , \quad H(x,y) = \left(\frac{y}{xy-x-y}\right)^{\frac{s_1}{q}}\ ,$$

There is polynomial P, R such that

$$x = 1 - P^q\ , \quad y = \frac{(1-P^q)}{(1-R^q)}\frac{R^q}{P^q}\ , \quad h(x) = P^{-s_1}\ , \quad H(x,y) = \left(\frac{R}{P}\right)^{s_1}\ ,$$

$$P^q = \xi_1^q\ , \quad R^q = \xi_2^q\ ,$$

$$h(x) = \xi_q^{-s_1}\ , \quad H(x,y) = \left(\frac{\xi_1}{\xi_2}\right)^{s_1}\ ,$$

F_3: Inverse binomial sums

Inverse binomial sums: $s_1 = 0\,, \quad s_2 = 0\,, \quad p \neq 0 \;\;, s_1 + s_2 + p \neq 0$

$$F_3\left(a_1\varepsilon, a_2\varepsilon, b_1\varepsilon, b_2\varepsilon, 1 - \frac{p}{q} + c\varepsilon; x, y\right)$$

$$h_1(x) = \left(\frac{x}{x-1}\right)^{\frac{p}{q}}\,, \quad h_2(y) = \left(\frac{y}{y-1}\right)^{\frac{p}{q}}\,,$$

$$H(x,y) = \left[\frac{xy}{(x-1)(y-1)-1}\right]^{\frac{p}{q}}$$

Equivalent to solution of polynomial equation:

$$1 = Q_1^q + Q_2^q + Q_3^q\,.$$

F_3: Final result

For Appell function

$$F_3\left(\frac{p_1}{q}+a_1\varepsilon,\frac{p_2}{q}+a_2\varepsilon,\frac{r_1}{q}+b_1\varepsilon,\frac{r_2}{q}+b_2\varepsilon,1-\frac{p}{q}+c\varepsilon;x,y\right)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{p_1}{q}+a_1\varepsilon\right)_m\left(\frac{p_2}{q}+a_2\varepsilon\right)_n\left(\frac{r_1}{q}+b_1\varepsilon\right)_m\left(\frac{r_2}{q}+b_2\varepsilon\right)_n}{\left(1-\frac{p}{q}+c\varepsilon\right)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\,.$$

the coefficients of ε-expansion are expressible in terms of Goncharov polylogarithms only in two cases:

$$F_3\left(I_1+a_1\varepsilon,I_2+a_2\varepsilon,I_3+b_1\varepsilon,I_4+b_2\varepsilon,I_5+c\varepsilon;x,y\right)\ ,$$
$$]F_3\left(I_1+\frac{p_1}{q}+a_1\varepsilon,I_2+a_2\varepsilon,I_3+b_1\varepsilon,I_4+b_2\varepsilon,I_5+\frac{p_1}{q}+c\varepsilon;x,y\right)\ ,$$
$$F_3\left(I_1+\frac{p_1}{q}+a_1\varepsilon,I_2+a_2\varepsilon,I_3+b_1\varepsilon,I_4+b_2\varepsilon,I_5+c\varepsilon;x,y\right)\ .$$

Conclusion

- It was shown, that Differential equation approach proposed in 2006-2007 by Kalmykov-Yost-Ward to construction of coefficients of ε-expansion of hypergeometric functions is applicable to the case of multivariable Horn type hypergeometric functions.
- There are: (i) iterative solutions (ii) iterative solution expressible in terms of multiple polylogarithms
- Natural expectation, that any (multivariable) iterated solution can be (later) rewritten in terms of multiple polylogarithms (symbols?)
- The lest analyzed problem is existence of linear parametrization.
- The most cumbersome procedure: the differential reduction procedure for function of a few variables.
- For Appell functions F_1, F_2, F_3, F_4 the differential reduction procedure is implemented into package HYPERDIRE