Structure of Open and Closed Superstring Amplitudes

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Harmony from/in String Theory

Various **relations within or between** gravity and gauge theory scattering amplitudes suggest a **unification** within or between these theories of the sort **inherent to string theory** !

Tree–level: • Color decomposition of gauge theory amplitudes into sum over partial subamplitudes

• BCJ: relations within gauge theory between color and kinematics

• KLT: relation between gravity and gauge theory

⋮

• Motifs: natural appearance in string theory

$\hookrightarrow$ (String) world–sheet derivation of amplitude relations

(By applying world–sheet string techniques $\Longrightarrow$ new algebraic relations)

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Harmony continues to hold at

<u>One–loop:</u> *The N–point one–loop integrand on the cylinder*,

Mafra and Schlotterer: nice results to appear.

Outline: Superstring amplitudes

1. Open superstring $N-$point disk amplitude: striking compact form

2. Closed superstring $N-$point amplitude: results and observations

3. Structure of $\alpha'-$expansions: motivic multi zeta values

1. *Open superstring N−point disk amplitude*

Here I will report on my work with

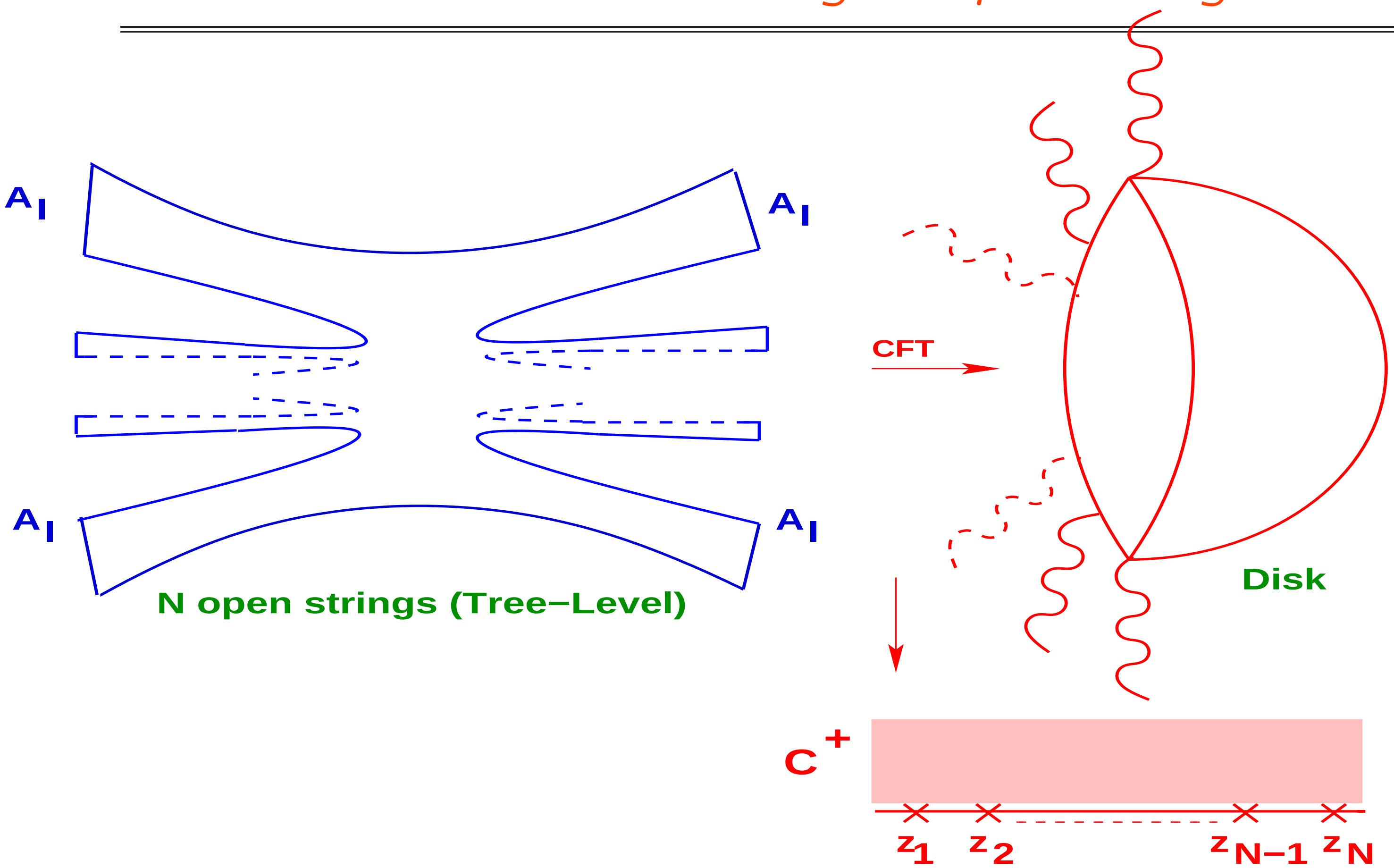
Mafra and Schlotterer:

- Complete N-Point Superstring Disk Amplitude I.
  Pure Spinor Computation, `arXiv:1106.2645`

- Complete N-Point Superstring Disk Amplitude II.
  Amplitude and Hypergeometric Function Structure, `arXiv:1106.2646`

- Explicit BCJ Numerators from Pure Spinors, `arXiv:1104.5224`

Disk scattering of open strings



$$\mathcal{A}(1,2,\ldots,N) = g_{YM}^{N-2} \sum_{\sigma\in S_N} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(N)}})\ A(\sigma(1),\sigma(2),\ldots,\sigma(N))$$

$A(1,2,\ldots,N)$ tree–level color–ordered N–leg partial amplitude (helicity subamplitude)

Disk scattering of open strings

N open string amplitude given by CFT computation with gluon vertex operators $V_g(z)$:

$$A(1,\ldots,N;\alpha') \;=\; V_{\mathrm{CKG}}^{-1} \int_{z_1<\ldots<z_N} \left(\prod_{j=1}^{N} dz_j \right) \langle V_g(z_1)\ldots\; V_g(z_N)\rangle$$

$$A(1,\ldots,N;\alpha') = V_{\mathrm{CKG}}^{-1} \int_{z_1<\ldots<z_N} \left(\prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i<j}^{N} |z_i - z_j|^{s_{ij}} \; (z_i - z_j)^{n_{ij}^I}$$

$$s_{ij} = \alpha'(k_i + k_j)^2$$

$\hookrightarrow$ It would be desirable to obtain **simple** and **compact** result !

Full N–point open superstring amplitude

Compact and short expression in terms of a *minimal basis* of $(N-3)!$ *building blocks*

$$A(1,2,\ldots,N;\alpha') = \sum_{\sigma\in S_{N-3}} A_{YM}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\ F^{\sigma}_{(1,\ldots,N)}(\alpha')$$

Mafra, Schlotterer, St.St., arXiv:1106.2645 and arXiv:1106.2646

A_{YM} Yang–Mills subamplitudes

$F^{\sigma}(\alpha')$ generalized Euler integral
multiple Gaussian hypergeometric functions

Pure spinor formalism allows for remarkable **simplifications** to package kinematics and α'–dependence

Structure of N–point superstring amplitude

Remarks:

- $N = 4$: $\begin{cases} A_{YM}(1,2,3,4) \;\rightarrow\; 2g_{YM}^2 \dfrac{1}{su}\, t_8(\xi_1,k_1,\xi_2,k_2,\xi_3,k_3,\xi_4,k_4) \\ F_{1234} \;\rightarrow\; \dfrac{\Gamma(1-s)\ \Gamma(1-u)}{\Gamma(1-s-u)} \end{cases}$ Green, Schwarz, 1982

- $D = 4$, MHV

$$A_{YM}(1^-,2^-,3^+,\ldots,N^+) = g_{YM}^{N-2} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle N1\rangle}\,,$$ Parke, Taylor, 1986; Berends, Giele, 1988

- $(N-3)!$ dimensional minimal basis of hypergeometric functions F^σ

- For any external state of SYM VM (FT = STTH Ward identities)

Set of $(N-3)!$ basis functions F^σ

$$F^{\sigma}_{(1,\ldots,N)}(\alpha') = \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \ \left(\prod_{i<l} |z_{il}|^{s_{il}} \right) \ \left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\} \quad , \quad \sigma \in S_{N-3}$$

$$F^{\sigma}_{(1,\ldots,N)}(\alpha') \ = \ \int\limits_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \ \left(\prod_{i<l} |z_{il}|^{s_{il}} \right)$$
$$\times \left(\prod_{k=2}^{[N/2]} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=[N/2]+1}^{N-2} \ \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right) \ , \ \sigma \in S_{N-3}$$

<u>$E.g.: N=5$</u>

$$\begin{cases} F^{(23)} &= \ s_{12}\ s_{34} \int_0^1 dx \int_0^1 dy \ x^{s_{45}} \ y^{s_{12}-1} \ (1-x)^{s_{34}-1} \ (1-y)^{s_{23}} \ (1-xy)^{s_{24}} \\ &= \ 1+\zeta(2) \ (s_1s_3-s_3s_4-s_1s_5) \\ &- \ \zeta(3) \ (s_1^2s_3+2s_1s_2s_3+s_1s_3^2-s_3^2s_4-s_3s_4^2-s_1^2s_5-s_1s_5^2)+\mathcal{O}(\alpha'^4) \ , \\ F^{(32)} &= \ s_{13}\ s_{24} \int_0^1 dx \int_0^1 dy x^{s_{45}} \ y^{s_{12}} \ (1-x)^{s_{34}} \ (1-y)^{s_{23}} \ (1-xy)^{s_{24}-1} \\ &= \ \zeta(2) \ s_{13}\ s_{24} - \zeta(3) \ s_{13}\ s_{24} \ (s_1+s_2+s_3+s_4+s_5)+\mathcal{O}(\alpha'^4) \ . \end{cases}$$

with: $s_i \equiv s_{i,i+1} = \alpha' \ (k_i+k_{i+1})^2 \ , \ i+5 \equiv i$

$$B_N[n] \;=\; \int_0^1 dx_1 \ldots \int_0^1 dx_{N-3} \prod_{a=1}^{N-3} x_a^{1+a-N+n_a} \prod_{b=a}^{N-3} x_a^{\;2\alpha' \, k_{b+3}\left(k_1 + \sum_{j=a+3}^{b+2} k_j\right)}$$

$$\times \;\left(1 - \prod_{j=a}^{b} x_j\right)^{2\alpha' \, k_{2+a} k_{3+b} + n_{ab}} \quad , \qquad b \geq a = 1, 2, \ldots, N-3 \;, \quad n_a, n_{ab} = 0, \pm 1$$

$\frac{1}{2}N(N-3)$ Laurent polynomials = number of kinematic invariants

N	dimension	function F^σ	reference
4	1	${}_2F_1$	Green, Schwarz, et al., 1982
5	2	${}_3F_2$	Medina, et al., `hep-th/0208121`
6	6	triple hypergeometric function $F^{(3)}$	Oprisa, St.St., `hep-th/0509042`
7	24	multiple hypergeometric function	St.St., Taylor, `arXiv:0708.0574`
N	$(N-3)!$	multiple hypergeometric function	Mafra, Schlotterer, St.St.

Structure of integrals is analyzed:

- ▷ *multiple pole* structure (dual channels)
- ▷ *transcendentality* properties
- ▷ *Gröbner basis* analysis to account for various integral relations

α'–expansion $\Longleftrightarrow$ multiple Euler–Zagier sums

E.g. N=7:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \ \frac{x^{s_2}\ (1-x)^{s_3}\ y^{t_2}\ (1-y)^{s_4}\ z^{t_6}\ (1-z)^{s_5}\ w^{s_7}\ (1-w)^{s_6}}{(1-xy)\ (1-wz)\ (1-yz)}\ (1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times\ (1-xy)^{-s_3-s_4+t_3}\ (1-wz)^{-s_5-s_6+t_5}\ (1-yz)^{-s_4-s_5+t_4}\ (1-wyz)^{s_5+t_1-t_4-t_5}\ (1-xyz)^{s_4-t_3-t_4+t_7}$$

$$= \mathcal{I}_0\ +\ \mathcal{I}_{1a}\ (s_1+s_2+s_3+s_4+s_5+s_6+s_7)\ +\ \mathcal{I}_{1b}\ (t_1+t_2+t_3+t_4+t_5+t_6+t_7) + \mathcal{O}(\alpha'^2)$$

with **Multiple Euler–Zagier sums** $\mathcal{I}_0$, $\mathcal{I}_{1a}$, $\mathcal{I}_{1b}$:

$$\mathcal{I}_0 = \int \frac{1}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_2=0\\ n_3=1}}^{\infty} \frac{1}{n_3\ (1+n_1)\ (n_1+n_2+1)\ (n_2+n_3)} = \frac{27}{4}\zeta(4)$$

$$\mathcal{I}_{1a} = \int \frac{\ln w}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_3=1\\ n_2=0}}^{\infty} \frac{1}{n_1\ n_3^2\ (n_1+n_2)\ (n_2+n_3)} = \frac{7}{2}\zeta(5) - 4\zeta(2)\zeta(3)$$

$$\mathcal{I}_{1b} = \int \frac{\ln y}{(1-xy)(1-wz)(1-yz)} = \sum_{\substack{n1,n_2=1\\ n_3=0}}^{\infty} \frac{1}{n_1\ n_2\ (n_2+n_3)\ (n_1+n_3)^2} = -\frac{9}{2}\zeta(5) + \zeta(2)\zeta(3)$$

Color vs. kinematics in string theory

String amplitude at fixed color ordering $(1,\ldots,N)$
combines two sectors or moduls

$$A(1,\ldots,N) \quad \in \quad \underbrace{\boxed{A_{YM,\pi}}}_{(N-3)!\ \text{basis}} \quad \otimes \quad \underbrace{\boxed{F^{\pi}_{(1,\ldots,N)}(\alpha')}}_{\text{dual basis ?}}$$

Can the integrals $F^{\pi}_{(1,\ldots,N)}$ be reduced to an $(N-3)!$ basis ?

Or more generally:

> Do we have an analog of
> Kleiss–Kuijf and BCJ relations
> in kinematic space ?

Recall: $(N-3)!$ (independent) building blocks A_{YM}, i.e. for any Π:

$$A_{YM}(1_\Pi,\ldots,N_\Pi)=\sum_{\sigma\in S_{N-3}} K^\sigma_\Pi\, A_{YM,\sigma}$$

- K^σ_Π can be derived from string theory monodromy relations $\alpha'\to 0$:

$$A(1,2,\ldots,N)\ +e^{i\pi s_{12}}\ A(2,1,3,\ldots,N-1,N)+e^{i\pi(s_{12}+s_{13})}\ A(2,3,1,\ldots,N-1,N)$$
$$+\ldots+e^{i\pi(s_{12}+s_{13}+\ldots+s_{1N-1})}\ A(2,3,\ldots,N-1,1,N)=0$$

(imaginary part) field–theory relations (BCJ relations):

$$s_{12}\ A_{YM}(2,1,3,\ldots,N-1,N)+\ldots+(s_{12}+s_{13}+\ldots+s_{1N-1})\ A(2,3,\ldots,N-1,1,N)=0$$

(real part) field–theory relations (Kleiss–Kuijf relations):

$$A_{YM}(1,2,\ldots,N)+A_{YM}(2,1,3,\ldots,N-1,N)+\ldots+A_{YM}(2,3,\ldots,N-1,1,N)=0$$

St.St., arXiv:0907.2211 & Bjerrum-Bohr, Damgaard, Vanhove, arXiv:0907.1425

- K^σ_Π can also be derived *directly* from string theory:

$$K^\sigma_\Pi=F^\sigma_\Pi(\alpha')\Big|_{\alpha'=0}$$

Color vs. kinematics in string theory

Further insights from looking at different representations (basis) for same amplitude:

$$A(1,\ldots,N)=\sum_{\pi\in S_{N-3}} A_{YM,\pi}\ F^{\pi}_{(1,\ldots,N)}(\alpha')$$

with some basis of $(N-3)!$ independent basis amplitudes $A_{YM,\pi}$ yields:

$$\boxed{F^{\sigma}_{(1,\ldots,N)}=\sum_{\pi\in S_{N-3}} (K^{-1})^{\sigma}_{\pi}\ F^{\pi}_{(1,\ldots,N)}}$$

$\Longrightarrow$ for a given **fixed color ordering** $(1,\ldots,N)$ any function F^{σ} (referring to the **kinematics** $\boldsymbol{\sigma}$) may be expressed in terms of a basis of $(N-3)!$ functions F^{π} referring to the **kinematics** $\boldsymbol{\pi}$.

Color vs. kinematics in string theory

$\Longrightarrow$ We have found a **dual system of equations** for the functions F

(**complementary** to BCJ and KK relations)

reducing the set of functions to a **minimal basis** of dimension $(N-3)!$

reduction to	$A_{YM,\pi}$	$F^\pi(\alpha')$
$(N-2)!$	Kleiss–Kuijf	partial fraction $\frac{1}{z_{12}z_{23}} - \frac{1}{z_{13}z_{23}} = \frac{1}{z_{12}z_{13}}$ $\Longrightarrow$ Gröbner basis analysis
$(N-3)!$	BCJ	tot. derivatives $0 = \int \partial_k \prod_{i<j} \lvert z_{ij}\rvert^{s_{ij}} \ldots$

2. *Closed superstring N–point amplitude*

(Color ordered) gluon amplitudes give rise to graviton amplitudes in type I or Type II superstring theory (field–theory for $\alpha' \to 0$)

At tree–level:

$$\textbf{gravity} = \textbf{gauge theory} \otimes \textbf{gauge theory}$$

- Spectrum:

$$\left|\begin{matrix}\mathcal{N}{=}8\\ SUGRA\end{matrix}\right\rangle = \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle \otimes \left|\begin{matrix}\mathcal{N}{=}4\\ SYM\end{matrix}\right\rangle$$
$$256 = 16 \times 16$$

E.g.: in $D=4$, N=8: Fock space decomposition of the 256 states of the $\mathcal{N}=8$ supergravity multiplet

- Vertex operators:

$$V_G(\epsilon, \overline{z}, z) \simeq V_g(\overline{\epsilon}, \eta) \otimes V_g(\epsilon, \xi)$$
$$\epsilon_{\mu\nu} = \overline{\epsilon}_\mu \otimes \epsilon_\nu$$

with $R_{\mu\nu\rho\sigma} = \kappa\ k_{[\mu} k_{[\rho}\ \overline{\epsilon}_{\nu]} \otimes \epsilon_{\sigma]}$ linearized Riemann tensor

String theory: Gauge vs. gravitational amplitudes

- Amplitudes (on–shell S–matrix): KLT relations: closed = open$\otimes$open

$$\begin{aligned}
\mathcal{M}_4(1,2,3,4)_{S^2} &= (2\alpha'\pi)^{-1}\ \sin(\pi s_{12})\ \overline{A}_4(1,2,3,4)_{D_2}\ A_4(1,2,4,3)_{D_2} \\
\mathcal{M}_5(1,2,3,4,5)_{S^2} &= (2\alpha'\pi)^{-2}\Big\{\ \sin(\pi s_{12})\ \sin(\pi s_{34})\ \overline{A}_5(1,2,3,4,5)_{D_2}\ A_5(2,1,4,3,5)_{D_2} \\
&\quad + \sin(\pi s_{13})\ \sin(\pi s_{24})\ \overline{A}_5(1,3,2,4,5)_{D_2}\ A_5(3,1,4,2,5)_{D_2}\Big\} \\
&\ \ \vdots
\end{aligned}$$

E.g.:

$$\mathcal{M}(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}\ \frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \longrightarrow \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8\ [12]}{N(4)\ \langle 34\rangle}$$

with: and: $\langle ij\rangle[ij] = s_{ij} = \alpha' k_i k_j, \quad N(n) = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle$

Tree–level higher order gravitational couplings

Type I or Type II superstring:

$$\mathcal{L}_{\text{tree}} = \frac{1}{2\kappa^2}\, R + \frac{\alpha'^3}{2^9\, 4!\, \kappa^2}\, \zeta(3)\; t_8 t_8\; R^4$$

Gross, Witten, 1986
Gross, Sloan, 1987

$$\mathcal{L}'_{\text{tree}} = \kappa^{-2} \sum_{n\geq 4}^{\infty} \sum_{m=0}^{\infty} \alpha'^{n-1+m} \sideset{}{'}\sum_{\substack{i_r\in\mathbf{N},\, i_1>1\\ i_1+\ldots+i_d=n-1+m}} \zeta(i_1,\ldots,i_d)\; c_{m,n,\vec{i}}\; (t^{\vec{i}}_{m,n}\, D^{2m} R^n)$$

Multi zeta values (MZVs):

$$\boxed{\zeta(i_1,\ldots,i_d) = \sum_{n_1>\ldots>n_d>0} \prod_{r=1}^{d} n_r^{-i_r}, \quad i_r\in\mathbf{N},\; i_1>1}$$

transcendentality degree $\sum_{r=1}^{d} i_r = n-1+m$ and depth d

Multi zeta values (MZVs)

The set of integral linear combinations of MZVs is a ring ($\to$ algebra)

e.g.: $\zeta(m)\,\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$

Many relations over $\mathbf{Q}$, e.g.:

$$\zeta(2,1) = 2\,\zeta(3) \quad , \quad \zeta(4,1) = 2\,\zeta(5) - \zeta(2)\,\zeta(3)$$

$$\zeta(5,3) = -\frac{5}{2}\,\zeta(6,2) - \frac{21}{25}\,\zeta(2)^4 + 5\,\zeta(3)\,\zeta(5)$$

$$\vdots$$

Zagier: For a given weight $w \in \mathbf{N}$ the dimension d_w of the space spanned by MZVs: $\boldsymbol{d_w = d_{w-2} + d_{w-3}}$, $d_0, d_1 = 0$,

w	d_w	basis
2	1	$\zeta(2)$
3	1	$\zeta(3)$
4	1	$\zeta(2)^2$
5	2	$\zeta(5)$, $\zeta(2)\zeta(3)$
6	2	$\zeta(2)^3$, $\zeta(3)^2$
7	3	$\zeta(7)$, $\zeta(2)\zeta(5)$, $\zeta(3)\zeta(2)^2$
8	4	$\zeta(2)^4$, $\zeta(2)\zeta(3)^2$, $\zeta(3)\zeta(5)$, $\zeta(5,3)$
9	5	$\zeta(9)$, $\zeta(2)\zeta(7)$, $\zeta(2)^2\zeta(5)$, $\zeta(2)^3\zeta(3)$, $\zeta(3)^3$
10	7	$\zeta(5)^2$, $\zeta(2)^5$, $\zeta(2)^2\zeta(3)^2$, $\zeta(2)\zeta(3)\zeta(5)$, $\zeta(3)\zeta(7)$, $\zeta(2)\zeta(5,3)$, $\zeta(7,3)$

Structure of α'–expansions

Why should we be interested in these higher order gravitational terms ?

- constraints and transcendentality properties
  of curvature couplings

  St.St., `arXiv:0910.0180`

- information on candidate counter terms
  satisfying SUSY Ward identities

  Beisert, Elvang, Freedman, Kiermaier, Morales, St.St. `arXiv:1009.1643`

- learn more about the structure of
  α'–expansions of open superstring amplitude

  work to appear

Tree–level higher order gravitational couplings

<u>Task:</u> Compute graviton amplitudes $\mathcal{M}(1,\ldots,N)$ and extract their power series expansion in α'

	$N=4$	$\boldsymbol{N=5}$	$\boldsymbol{N=6}$	$N=7$	$N=8$
$\alpha'^3\ \zeta(3)$	R^4				
α'^4 ~~$\zeta(4)$~~	~~D^2R^4~~	~~$\boldsymbol{R^5}$~~			
$\alpha'^5\ \zeta(5)$	D^4R^4	$\boldsymbol{D^2R^5}$	$\boldsymbol{R^6}$		
α'^5 ~~$\zeta(2)\zeta(3)$~~	~~D^4R^4~~	~~$\boldsymbol{D^2R^5}$~~	~~$\boldsymbol{R^6}$~~		
$\alpha'^6\ \zeta(3)^2$	D^6R^4	$\boldsymbol{D^4R^5}$	$\boldsymbol{D^2R^6}$	R^7 ?	
$\alpha'^6\ \zeta(6)$	~~D^6R^4~~	~~$\boldsymbol{D^4R^5}$~~	~~$\boldsymbol{D^2R^6}$~~	R^7 ?	
$\alpha'^7\ \zeta(7)$	D^8R^4	$\boldsymbol{D^6R^5}$	$\boldsymbol{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(3)\zeta(4)$	~~D^8R^4~~	~~$\boldsymbol{D^6R^5}$~~	~~$\boldsymbol{D^4R^6}$~~	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(2)\zeta(5)$	~~D^8R^4~~	~~$\boldsymbol{D^6R^5}$~~	~~$\boldsymbol{D^4R^6}$~~	D^2R^7 ?	R^8 ?
$\alpha'^8\ \zeta(3)\zeta(5)$	$D^{10}R^4$	$\boldsymbol{D^8R^5}$	$\boldsymbol{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(8)$	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(2)\zeta(3)^2$	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(5,3)$	~~$D^{10}R^4$~~	~~$\boldsymbol{D^8R^5}$~~	~~$\boldsymbol{D^6R^6}$~~	D^4R^7 ?	D^2R^8 ?

St.St. arXiv:0910.0180

Tree–level higher order gravitational couplings

Further evidence: verified also α'^9 and α'^{10}

	$N=4$	$\boldsymbol{N=5}$	$\boldsymbol{N=6}$
$\alpha'^9\ \zeta(9)$	$D^{12}R^4$	$\boldsymbol{D^{10}R^5}$	$\boldsymbol{D^8R^6}$
$\alpha'^9\ \zeta(3)^3$	$D^{12}R^4$	$\boldsymbol{D^{10}R^5}$	$\boldsymbol{D^8R^6}$
$\alpha'^9\ \zeta(2)\ \zeta(7)$	~~$D^{12}R^4$~~	~~$\boldsymbol{D^{10}R^5}$~~	~~$\boldsymbol{D^8R^6}$~~
$\alpha'^9\ \zeta(2)^2\ \zeta(5)$	~~$D^{12}R^4$~~	~~$\boldsymbol{D^{10}R^5}$~~	~~$\boldsymbol{D^8R^6}$~~
$\alpha'^9\ \zeta(2)^3\ \zeta(3)$	~~$D^{12}R^4$~~	~~$\boldsymbol{D^{10}R^5}$~~	~~$\boldsymbol{D^8R^6}$~~

	$N=4$	$\boldsymbol{N=5}$	$\boldsymbol{N=6}$
$\alpha'^{10}\ \zeta(3)\ \zeta(7)$	$D^{14}R^4$	$\boldsymbol{D^{12}R^5}$	$\boldsymbol{D^{10}R^6}$
$\alpha'^{10}\ \zeta(5)^2$	$D^{14}R^4$	$\boldsymbol{D^{12}R^5}$	$\boldsymbol{D^{10}R^6}$
$\alpha'^{10}\ \zeta(2)^5$	~~$D^{14}R^4$~~	~~$\boldsymbol{D^{12}R^5}$~~	~~$\boldsymbol{D^{10}R^6}$~~
$\alpha'^{10}\ \zeta(2)^2\ \zeta(3)^2$	~~$D^{14}R^4$~~	~~$\boldsymbol{D^{12}R^5}$~~	~~$\boldsymbol{D^{10}R^6}$~~
$\alpha'^{10}\ \zeta(2)\ \zeta(3)\ \zeta(5)$	~~$D^{14}R^4$~~	~~$\boldsymbol{D^{12}R^5}$~~	~~$\boldsymbol{D^{10}R^6}$~~
$\alpha'^{10}\ \zeta(2)\ \zeta(5,3)$	~~$D^{14}R^4$~~	~~$\boldsymbol{D^{12}R^5}$~~	~~$\boldsymbol{D^{10}R^6}$~~
$\alpha'^{10}\ \zeta(7,3)$	~~$D^{14}R^4$~~	~~$\boldsymbol{D^{12}R^5}$~~	~~$\boldsymbol{D^{10}R^6}$~~

Note: two independent structures in $\alpha'^9 D^{12}R^4$ and $\alpha'^{10} D^{14}R^4$

$\Longrightarrow$ Constraints on higher order gravitational couplings:
very restricted sets of MZVs appear

Conjecture: Only Riemann zeta functions $\zeta(2n+1)$ of odd weight appear, but no MZVs $\zeta(n_1,\ldots,n_r)$

3. Structure of open & closed string amplitude

Question: Can we prove these results without long computations ?

$\underline{N=4}$:

$$\mathcal{A}(1,2,3,4) = A_{YM}(1,2,3,4)\ \frac{\Gamma(1-s)\ \Gamma(1-u)}{\Gamma(1-s-u)}$$

$$\mathcal{A}(1,2,3,4) \ = \ A_{YM}(1,2,3,4)\ \left\{\pi\ \frac{s\ u}{s+u}\ \frac{\sin(\pi t)}{\sin(\pi s)\ \sin(\pi u)}\right\}^{1/2}$$

$$\times \quad \exp\left\{\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}\ [\ s^{2n+1}+u^{2n+1}+t^{2n+1}\ ]\right\}$$

$$\mathcal{M}(1,2,3,4) \ = \ \sin(\pi s)\ \frac{\sin(\pi u)}{\sin(\pi t)}\ |\mathcal{A}(1,2,3,4)|^2$$

$$= \ s\ \frac{u}{t}\ e^{2\sum\limits_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})}\ |A_{YM}(1,2,3,4)|^2$$

Structure of open string amplitude

Question:

Can we cast the open string amplitude $\mathcal{A}$

into compact form

with **simple building blocks**

(taking into account our findings) ?

Yes we can !

Structure of open string amplitude

$$\mathcal{A} \;=\; P \;\Big\{1 + c_{5,3}\;\zeta(5,3)\;[M_3,M_5] + c_{7,3}\;\zeta(7,3)\;[M_3,M_7] + \ldots\Big\}$$
$$\times \quad \exp\left\{\sum_{n\geq 1}\zeta(2n+1)\;M_{2n+1}\right\}\;A_{YM}$$

$\mathcal{A}$ = $(N-3)!$ dimensional vector encoding the **string basis**

P = $(N-3)!\times(N-3)!$ matrix encoding α'–expansions with $\boldsymbol{\zeta(2)^k}$

M_{2n+1} = $(N-3)!\times(N-3)!$ matrix encoding α'–expansions with $\boldsymbol{\zeta(2n+1)}$

A_{YM} = $(N-3)!$ dimensional vector encoding the **YM-basis**

$c_{k,l}$ = rational numbers

This form is bolstered by the algebraic structure of **motivic MZVs**

work to appear

Structure of open string amplitude

E.g.:

$$N=4:\ \begin{cases} \mathcal{A}=\mathcal{A}(1,2,3,4)\ ,\ A_{YM} = A_{YM}(1,2,3,4)\ , \\ M_{2n+1} = \dfrac{\zeta(2n+1)}{2n+1}\ [\ s^{2n+1}+u^{2n+1}+t^{2n+1}\ ]\ , \\ P = \left\{\pi\ \dfrac{s\ u}{s+u}\ \dfrac{\sin(\pi t)}{\sin(\pi s)\ \sin(\pi u)}\right\}^{1/2} \end{cases}$$

$$N=5:\ \begin{cases} \mathcal{A}=\begin{pmatrix}\mathcal{A}(1,2,3,4,5)\\ \mathcal{A}(1,3,2,4,5)\end{pmatrix}\ ,\ A_{YM} = \begin{pmatrix}A_{YM}(1,2,3,4,5)\\ A_{YM}(1,3,2,4,5)\end{pmatrix}\ , \\ M_{2n+1} = \left.\begin{pmatrix}F_1 & F_2\\ \tilde F_2 & \tilde F_1\end{pmatrix}\right|_{\zeta(2n+1)}\ , \\ P = \begin{pmatrix}\sum\limits_{m\geq 0} p_{2m}\ \zeta(2m) & \sum\limits_{m\geq 0} q_{2m}\ \zeta(2m)\\ \sum\limits_{m\geq 0} \tilde q_{2m}\ \zeta(2m) & \sum\limits_{m\geq 0} \tilde p_{2m}\ \zeta(2m)\end{pmatrix} \end{cases}$$

Structure of α'–expansions

Open string α'–expansions:

E.g. weight 8:

$$\mathcal{A}\ |_{\zeta(3)\zeta(5)} = M_3 M_5\ A_{YM}$$

$$\mathcal{A}\ |_{\zeta(5,3)} = \frac{1}{5}\ [M_3, M_5]\ A_{YM}$$

$$\vdots$$

E.g. weight 10:

$$\mathcal{A}\ |_{\zeta(3)\zeta(7)} = M_3 M_7\ A_{YM}$$

$$\mathcal{A}\ |_{\zeta(7,3)} = \frac{1}{14}\ [M_3, M_7]\ A_{YM}$$

$$\mathcal{A}\ |_{\zeta(5)^2} = \left(\frac{1}{2}\ M_5\ M_5 + \frac{3}{14}\ [M_3, M_7]\right)\ A_{YM}$$

$$\vdots$$

Structure of closed string amplitude

KLT:

$$\mathcal{M}(1,\ldots,N)_{S^2} \sim \sum_{\sigma,\rho} e^{i\pi\phi(\sigma,\rho)}\ \overline{\mathcal{A}}_N(\sigma)_{D_2}\ \mathcal{A}_N(\rho)_{D_2}$$

sum over $\frac{1}{2}(N-1)! \times \frac{1}{2}(N-1)!$ open string amplitudes

$$\mathcal{M}(1,\ldots,N) = \mathcal{A}^t\ S\ \mathcal{A}$$

S = $(N-3)! \times (N-3)!$ matrix encoding KLT and monodromy relations

$\mathcal{A}$ = $(N-3)!$ dimensional vector encoding the string basis

Note:

$$\mathcal{M}(1,\ldots,N)|_{FT} = A^t_{YM}\ S_0\ A_{YM} \quad , \quad S_0 = S\ |_{FT}$$

Structure of open & closed string amplitude

The non–trivial matrix identities:

$$\begin{aligned} P^t\ S\ P &= S_0 \\ S_0\ Q + Q^t\ S_0 &= 0 \quad , \quad Q = [M_l, M_m] \end{aligned}$$

prove (or state) our conjecture !

E.g.: $Q = [M_7, M_9]$ is weight $= 16$ and fulfills $S_0\ Q + Q^t\ S_0 = 0$!

$\hookrightarrow$ Give recursion relations for M_{2n+1}.

Aspects of motivic MZVs

Introduce:

$$I_\gamma(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_\gamma \frac{dz}{z-a_1} \cdots \frac{dz}{z-a_n}\ ,$$

with γ a path in $M = \mathbf{C}/\{a_1, \ldots, a_n\}$ with endpoints $\gamma(0) = a_0 \in M,\ \gamma(1) = a_{n+1} \in M$

Note: (i) For $a_i \in \{0,1\}$ I_γ fulfills a lot of nice relations: shuffle product, duality, ...

(ii) Multiple polylogarithms $G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1}\ G(a_2, \ldots, a_n; t)$.

For the map: $\rho(n_1, \ldots, n_r) = 10^{n_1-1} \ldots 10^{n_r-1}$ we have (Kontsevich):

$$\zeta(n_1, \ldots, n_r)\ =\ (-1)^r\ I_\gamma(0; \rho(n_1 \ldots n_r); 1)$$

Aspects of motivic MZVs

Goncharov: Embedding $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$: framed mixed Tate motive over $\overline{\mathbb{Q}}$

define **motivic iterated integrals** or **symbols** $\boldsymbol{I^m}$:

$$I^m(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{H}$$

commutative graded Hopf–algebra $\mathcal{H}$

$$p_\sigma(I^m(a_i)) = I(\sigma(a_i))$$

Define **motivic MZVs**: (for $a_0, \ldots, a_{n+1} \in \{0, 1\}$)

$$\zeta^m(n_1, \ldots, n_r) \;=\; (-1)^r \; I^m(0; \rho(n_1, \ldots, n_r); 1)$$

Work of F. Brown: *On the decomposition of motivic MZVs*

Introduce map: $\phi: \ \mathcal{H} \longrightarrow \mathbf{Q}\langle f_3, f_5, \ldots \rangle \otimes_{\mathbf{Q}} \langle f_2 \rangle$

- The map ϕ sends every motivic MZV of weight less or equal to N to a non–commutative polynomial in the f_i's, *e.g.:* $\phi(\zeta^m(7,3)) = -14 f_7 f_3 - 6 f_5 f_5$
- There is an algorithm to assign to every element $\zeta^m(n_1, \ldots, n_r)$ a $\mathbf{Q}$–linear combination of monomials
- Inverting this map gives the decomposition of $\zeta^m(n_1, \ldots, n_r)$ w.r.t. to a basis B_n, with $n = \sum_{l=1}^{r} n_l$ (B_n= basis of motivic MZVs).

Aspects of motivic MZVs

Moving to motivic MZVs allows to compute the decomposition directly:

$\hookrightarrow$ hidden structure uncovered by the action of motivic derivations ∂^{ϕ}

<u>*Example:*</u> weight 10 for $\xi \in \mathcal{H}$:

$$\xi = a_0\ \zeta^m(2)^5 + a_1\ \zeta^m(2)^2\ \zeta^m(3)^2 + a_2\ \zeta^m(2)\ \zeta^m(3)\ \zeta^m(5) + a_3\ \zeta^m(5)^2$$

$$+\ a_4\ \zeta^m(2)\ \zeta^m(5,3) + a_5\ \zeta^m(3)\ \zeta^m(7) + a_6\ \zeta^m(7,3)$$

$$a_1 = \tfrac{1}{2}\ c_2^2\ \partial_3^2,\ a_2 = c_2\ \partial_5\partial_3,\ a_3 = \tfrac{1}{2}\ \partial_5^2 + \tfrac{3}{14}\ [\partial_7, \partial_3]$$

$$a_4 = \tfrac{1}{5}\ c_2\ [\partial_3, \partial_5],\ a_5 = \partial_7\partial_3,\ a_6 = \tfrac{1}{14}\ [\partial_7, \partial_3]$$

$\hookrightarrow$ Motivic multi zeta values encapsulate α'–expansion

AMPLITUDES 2013

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April 28th – May 3rd, 2013 at Ringberg Castle

$\Delta p \cdot \Delta q \geq \frac{1}{2} \hbar$

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