Amplitude Techniques for AdS/CFT

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References

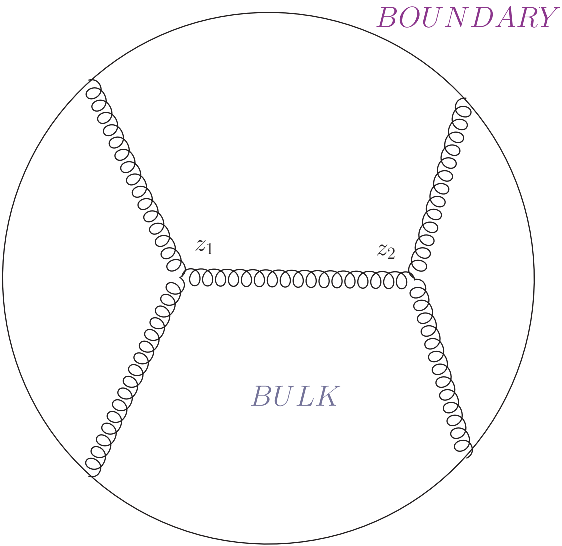
This talk is based on

- **"Four Point Functions of the Stress Tensor and Conserved Currents in AdS_4/CFT_3"**,S. Raju,arXiv:1201.6452.
- **"New Recursion Relations and a Flat Space Limit for AdS/CFT Correlators"**,S. Raju,arXiv:1201.6449.
- **"Recursion Relations for AdS/CFT Correlators"**, S. Raju, arXiv:1102.4724
- **"BCFW for Witten Diagrams"**, S. Raju, arXiv:1011.0780.

The Objective

- Can we generalize amplitude-techniques to anti-de Sitter space?
- AdS does not have S-matrices, but it has a close analogue: correlation functions in a dual CFT.
- These are computed by cousins of Feynman diagrams: Witten diagrams

Witten Diagram



The Problem

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1. Doing integrals over the bulk of AdS is very difficult.

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1. Doing integrals over the bulk of AdS is very difficult.
2. The presence of a background Riemann tensor makes interactions even more complicated.

Complicated Propagators

- Take the Poincare patch:

$$ds^2 = \frac{dz^2 + \eta_{ij} dx^i dx^j}{z^2}$$

- The bulk-bulk propagator is:

$$G(x_1, z_1, x_2, z_2) = N_\Delta \zeta^\Delta F(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}, \Delta - \frac{d}{2} + 1, \zeta)$$
$$\Delta(\Delta - d) = m^2; \quad \zeta = \frac{2 z_1 z_2}{z_1^2 + z_2^2 + (x_1 - x_2)^2}$$

- A limit of the bulk-bulk propagator gives the bulk-boundary propagator.

$$K_\Delta(x_1, x_2, z_2) = \lim_{z_1 \to 0} z_1^\Delta G(x_1, z_1, x_2, z_2) = N_\Delta \left(\frac{z_2}{z_2^2 + (x_1 - x_2)^2} \right)^\Delta$$

Difficult z-integrals

- So, even the simplest Witten diagrams are difficult to evaluate explicitly.

- For example, the four point scalar contact diagram is:

$$\int K_{\Delta_1}(x_1, z) K_{\Delta_2}(x_2, z) K_{\Delta_3}(x_3, z) K_{\Delta_4}(x_4, z) \frac{dz}{z^{d+1}} \\ = D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(x_1, x_2, x_3, x_4)$$

- This is a complicated special function essentially defined by the left hand side!

- Diagrams with propagators are even harder.

Mellin Space

- Till a few months ago, it was not known how to evaluate, say, the 6-pt diagram in the ϕ^3 theory.
- This was resolved by going to Mellin space on the boundary.
  [Mack, Penedones, Kaplan, Fitzpatrick, S.R., Van Rees, Paulos,]

Graviton Amplitudes

- However, we cannot yet use Mellin space effectively for correlators of operators with spin.
- Besides, when we have gravitational interactions in the bulk, the interaction vertices are very complicated.

Interactions in Quantum Gravity

$$
\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \longrightarrow
$$

$$
\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad -P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\Big],
\end{aligned}
$$

$$
\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \longrightarrow
$$

$$
\begin{aligned}
&\mathrm{Sym}\Big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\
&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad -P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\Big].
\end{aligned}
$$

These 3 and 4-pt vertices are written in highly condensed notation. Actually 2850 terms in 4-pt vertex. Also, an infinite number of higher vertices.

Sad Status of Knowledge

Even the four point function of the stress-tensor

- to leading order
- with just $\sqrt{-g}R$ in the bulk had not been computed

$$\langle T^{\mu_1,\nu_1}(x_1) T^{\mu_2,\nu_2}(x_2) T^{\mu_3,\nu_3}(x_3) T^{\mu_4,\nu_4}(x_4)\rangle$$

- Four point T-T correlator is dual to the tree-amplitude for 4-gravitons in AdS.
- Of interest, because it is a universal observable in CFTs with a gravity dual: doesn't care about the other matter in the theory to leading order.
- In this talk, I will compute this quantity by generalizing amplitude-techniques to anti-de Sitter space.

Overview

1. Find Recursion Relations for Graviton and Gluon Amplitudes in AdS. (dual to correlators of the stress tensor or conserved currents on the boundary.)
2. Find a New Flat Space Limit. (Extract flat space S-matrix elements from CFT correlators.)
3. Present explicit results for Current and Stress Tensor Correlators in AdS_4/CFT_3

Boundary Momentum Space

- Right language for this programme is momentum space on the boundary

$$\langle T^{i_1 j_1}(k^1) \ldots T^{i_n j_n}(k^n) \rangle \equiv \int \langle T^{i_1 j_1}(x_1) \ldots T^{i_n j_n}(x_n) \rangle e^{i \sum_{m=1}^{n} k_m \cdot x_m} d^d x_m,$$

- The Ward identity $\partial_i T^{ij}(x) = 0$ turns into

$$k^1_{i_1} \langle T^{i_1 j_1}(k^1) \ldots T^{i_n j_n}(k^n) \rangle = \text{lower-pt correlators}$$

- So, we only need to consider transverse-traceless polarization tensors

$$T(e_1, k_1, \ldots, e_n, k_n) = e_{1 i_1 j_1} \ldots e_{n i_n j_n} \langle T^{i_1 j_1}(k^1) \ldots T^{i_n j_n}(k^n) \rangle,$$

- Ward identity is manifest but special conformal transformations are not.

Modes in Momentum Space

- The equation $\Box\phi = 0$ is simple to solve.

Modes in Momentum Space

- The equation $\square \phi = 0$ is simple to solve.
- For $k^2 < 0$ (time-like), this has solutions

$$\text{normalizable:} \quad \phi(z) = (|k|z)^{\frac{d}{2}} J_{\frac{d}{2}}(|k|z) e^{ik\cdot x},$$

$$\text{non-normalizable:} \quad \phi(z) = (|k|z)^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(|k|z) e^{ik\cdot x},$$

- For $k^2 > 0$, the only solutions is

$$\text{non-normalizable:} \quad \phi(z) = (|k|z)^{\frac{d}{2}} K_{\frac{d}{2}}(|k|z) e^{ik\cdot x},$$

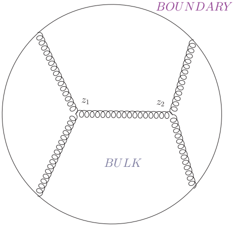
- The non-normalizable solution is the bulk to boundary propagator

Bulk-Bulk Propagator

- The bulk-bulk propagator is given by:

$$G(z_1, z_2, K) = \int \frac{z_1^{\frac{d}{2}} J_{\frac{d}{2}}(p z_1) J_{\frac{d}{2}}(p z_2) z_2^{\frac{d}{2}}}{p^2 + K^2} \frac{dp^2}{2}$$

- Witten diagrams are obtained by putting together bulk-boundary and bulk-bulk propagators.



Inefficiency of Standard Perturbation Theory

- In principle, we can use these propagators and an effective action to compute arbitrary correlators in perturbation theory.
- In practice, this programme is difficult to carry out for gravity; interaction vertices are very complicated.
- So, we need more efficient ways of computing amplitudes.

Recursion Relations in AdS/CFT

- The 4-pt correlator is a function of 4 off-shell momenta $k_1 \ldots k_4$.
- Now, consider a one-parameter deformation of the momenta by

$$k_m \to k_m + \alpha_m \epsilon_m w$$

- Here ϵ_m are polarization vectors for k_m and satisfy:

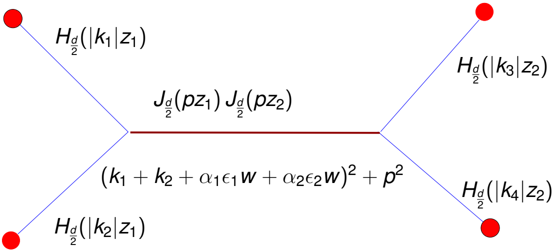
$$\epsilon_m \cdot k_m = 0, \quad \epsilon_m \cdot \epsilon_m = 0, \quad \sum_m \alpha_m \epsilon_m = 0$$

(i.e. α_m are tuned to conserve momentum.)

- Similar to Risager Extension. In $d \geq 4$, can do a BCFW-extension as well.

[S.R.,2010]

Anatomy of a Witten Diagram



- Correlator is integral of a rational function of w.

- The residue at a pole,

$$(k_1(w) + k_2(w))^2 = -p^2$$

is the product of two "on-shell" transition amplitudes.

Transition Amplitudes

- Transition amplitudes are very similar to correlators except that one bulk to boundary propagator is replaced by a "normalizable mode."
- These can be thought of as correlators of operators inserted between an excited state and the vacuum.
- Alternately, transition amplitudes can be understood as non-time-ordered correlators

Large w behaviour

- The large w behaviour is easy to determine for current and stress-tensor correlators.
- At large w, the polarization vector becomes proportional to the momentum
$$\epsilon_\mu \sim \frac{k^m_\mu(w)}{w}$$
- This means that the large w behaviour of the correlator is completely fixed by the Ward identities.
- So the residues of the rational integrand at finite w completely determine the correlators.

Recursion Relations

This allows us to write down recursion relations.

$$T(e_1, k_1, \ldots e_n, k_n) = \sum_{\{\pi\}, e'_m, \pm} \int \frac{-i\mathcal{T}^2 + B}{p^2 + (\sum_{o=1}^{m_l} k_{\pi_o})^2} \frac{dp^2}{2} \frac{w^{\mp}(p)}{w^{\pm}(p) - w^{\mp}(p)},$$

$$\mathcal{T}^2 \equiv T^*(e_{\pi_1}, k_{\pi_1}(p), \ldots e'_q, k'_q) T^*(e'_q, -k'_q, \ldots e_n, k_{\pi_n}(p)).$$

Flat Space Limit?

- Do these recursion relations reduce to flat space recursion relations in some limit?

- Related to a Longstanding question in AdS/CFT: How to extract $(d+1)$-dimensional flat space S-matrix elements from d-dimensional correlators.

  [Susskind, Polchinski, Giddings, Penedones ...,]

New Flat Space Limit

By analyzing interactions and propagators in momentum space, we can derive a new and elegant flat space limit.

Large-z is Flat Space

- Deep inside AdS, (at large-z), the momentum space wave-functions also become simple:

$$z^2 h_{\mu\nu} \underset{z\to\infty}{\longrightarrow} (|k|z)^{\frac{d-1}{2}} e^{-|k|z} + \text{subleading}$$

- The vertices also simplify:

$$R(g^{\text{ads}}_{\mu\nu}+h_{\mu\nu}) = R\left(\frac{1}{z^2}(\eta^{\mu\nu} + z^2 h_{\mu\nu})\right) = z^2 R(\eta^{\mu\nu}+z^2 h_{\mu\nu})-d(d+1).$$

- So, if we look at the coefficient of the highest power of z, where all derivatives act on the exponential inside h then this will be the same as flat space. Lower powers of z differ from flat space.

- This is actually quite intuitive. It tells us that deep inside AdS, interactions are like those of flat space.

- More precisely, consider a *n*-point contact interaction. (Gravity has vertices with arbitrary number of external legs.)

- At large z the expressions in AdS and flat space are related in a simple way:

$$A(k_i, |k_i|, z) = \frac{1}{z^{d+1}} z^2 \left(\prod |k_i|\right)^{\frac{d-1}{2}} z^{n\frac{d-1}{2}} F(\tilde{k}_i, z)$$

  where

$$\tilde{k}_i = (k_i, i|k_i|)$$

  is a “massless momentum” in $d + 1$ dimensions.

Flat Space Limit

- The other difference with flat space is that there the z-integral goes from $-\infty$ to ∞. In AdS, the integral goes from 0 to ∞.
- If we now do the z-integral, this leads to

$$M(\tilde{k}^1, \ldots \tilde{k}^n) = \lim_{E_T \to 0} \frac{(E_T)^{\alpha^0_{gr}(n)}}{\left(\prod_{m=1}^{n} |k^m|\right)^{\frac{d-1}{2}} \Gamma(\alpha^0_{gr})} T(k^1, \ldots k^n),$$

  where

$$E_T = \sum |k_m|, \quad \alpha^0_{gr}(n) = (\frac{n}{2} - 1)(d - 1) + 1$$

- The flat space amplitude is the coefficient of a pole in the AdS correlators
- This pole appears in place of a delta function for the radial momentum.

Flat Space Limit for Yang-Mills

- A similar analysis for Yang-Mills leads to the result:

$$M(\epsilon^1, \tilde{k}^1, \ldots \epsilon^n, \tilde{k}^n) = \lim_{E_T \to 0} \frac{(E_T)^{\alpha^0_{\rm gl}(n)}}{\left(\prod_{m=1}^n |k^m|\right)^{\frac{d-3}{2}} \Gamma(\alpha^0_{\rm gl})} T(\epsilon^1, k^1, \ldots \epsilon^n, k^n),$$

  with

$$\alpha^0_{\rm gl} = \left(\frac{n}{2} - 1\right)(d-3) + 1.$$

- For both gravity, and Yang-Mills, we can check that both sides have the correct scaling dimension.

Flat Space Limit at Higher Loops

- GRAVITY:

$$M(\tilde{k}^1, \ldots \tilde{k}^n) = \lim_{E_T \to 0} \frac{(E_T)^{\alpha^l_{\text{gr}}(n)} T(k^1, \ldots k^n)}{\left(\prod_{m=1}^n |k^m|\right)^{\frac{d-1}{2}} \Gamma(\alpha^l_{\text{gr}})},$$

  with

$$\alpha^l_{\text{gr}}(n) = (\frac{n}{2} - 1 + l)(d - 1) + 1,$$

- YANG-MILLS:

$$M(\tilde{k}^1, \ldots \tilde{k}^n) = \lim_{E_T \to 0} \frac{(E_T)^{\alpha^l_{\text{gl}}(n)} T(k^1, \ldots, k^n)}{\left(\prod |k^m|\right)^{\frac{d-3}{2}} \Gamma(\alpha^l_{\text{gl}})},$$

  with

$$\alpha^l_{\text{gl}}(n) = (\frac{n}{2} - 1 + l)(d - 3) + 1.$$

Flat Space Limit and the Recursion Relations

- We can show that the AdS recursion relations have the right flat space limit.
- The proof is to show that the integral over p generates a pole, and the coefficient of that pole is the flat space recursion relation
- However, the flat space limit has a larger range of validity; it is valid beyond tree level in the bulk.

Doing z Integrals

- We have, so far, not addressed the issue of difficult z-integrals
- However, in odd boundary dimension, when we are dealing with conserved currents or the stress tensor, momentum space solves this problem also!

Gauge Field in AdS_4

- For a gauge field in AdS, the modes are

$$A_i^a(x,z) = \epsilon_i^a (|k|z)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(|k|z) e^{ik\cdot x}$$
$$A_0^a = 0, \quad \text{gauge choice}$$
$$k\cdot\epsilon^a = 0 \quad \text{transversality},$$

  Here 0 refers to the z-direction.

- In $d = 3$, the mode is just:

$$A_i^a = \epsilon_i^a e^{i|k|z} e^{ik\cdot x},$$

  The same as flat space!

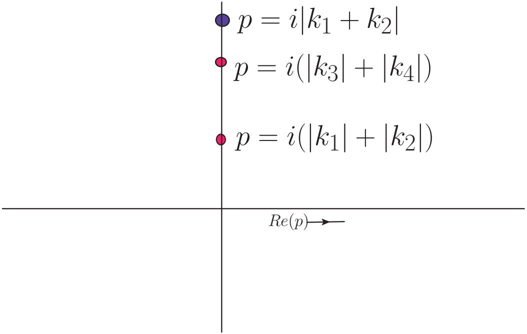
Modes of the Stress Tensor

The stress tensor also has simple modes.

$$
\begin{aligned}
h_{ij} &= \epsilon_{ij}\Big(\frac{1+|k|z}{z^2}\Big)e^{i|k|z+ik\cdot x}; \\
h_{0\mu} &= 0, \quad \text{gauge choice} \\
k_i\epsilon^{ij} &= 0, \quad \text{transversality} \\
\epsilon^i_i &= 0. \quad \text{tracelessness}
\end{aligned}
$$

So for currents, the stress tensor and scalars of special dimensions in $d = 3$, we can ameliorate the problem of doing z-integrals by going to momentum space on the boundary.

Poles of the Integrand



- For odd boundary dimensions the *p*-integral is algebraic: just collect residues.
- Some poles in *p* correspond to the “flat space” poles.
- One residue corresponds to the contribution of *T* to the OPE.

Spinor Helicity Formalism

- In AdS_4, we can also use an analogue of the spinor-helicity formalism. [Maldacena Pimentel]
- The correlators are functions of 3-momenta $k_1, \ldots k_n$ — No on-shell condition on the momenta.
- However, for each such 3-momentum we can form a null momentum in 4d:
$$\tilde{k} = (k, i|k|)$$
- We can write $\tilde{k}_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$

Spinor Products

- We can form inner products invariant under $SO(3,1)$ using

$$\langle\lambda_1, \lambda_2\rangle = \epsilon^{\alpha\beta}\lambda_{1\alpha}\lambda_{2\beta}$$

- We can also form inner products invariant under $SO(2,1)$ using

$$\left[\lambda_1, \tilde{\lambda}_2\right] = (\sigma^3)^{\alpha\dot{\beta}}(\lambda_1)_\alpha(\tilde{\lambda}_2)_{\dot{\beta}}$$

- $SO(2,1)$ invariance is all we can demand, so we should expect such mixed products.

Polarization Vectors

- We can now form polarization vectors for negative and positive helicity.
- $$\epsilon^{-}_{\alpha\dot{\alpha}} = \frac{1}{i|k|}\lambda_{\alpha}\lambda^{\beta}\sigma^{3}_{\beta\dot{\alpha}}, \quad \text{Negative Helicity}$$
- $$\epsilon^{+}_{\alpha\dot{\alpha}} = \frac{1}{i|k|}\tilde{\lambda}_{\alpha}\tilde{\lambda}^{\beta}\sigma^{3}_{\beta\dot{\alpha}}, \quad \text{Positive Helicity}$$
- graviton polarization vectors are just squares of these

$$e^{-}_{\mu\nu} = \epsilon^{-}_{\mu}\epsilon^{-}_{\nu}, \quad e^{+}_{\mu\nu} = \epsilon^{+}_{\mu}\epsilon^{+}_{\nu}$$

RESULTS

Three Point current correlators

- MHV correlator:

$$T_3^{+,+,-} = \frac{-R(|k_1|,|k_2|,|k_3|)}{2\sqrt{2}|k_1||k_2||k_3|} \times \left(|k_2| + |k_3| - |k_1|\right)\left(|k_3| + |k_1| - |k_2|\right) \times \left(|k_1| + |k_2| - |k_3|\right) \frac{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle^4}{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle \left\langle \tilde{\lambda}_2, \tilde{\lambda}_3 \right\rangle \left\langle \tilde{\lambda}_3, \tilde{\lambda}_1 \right\rangle}$$

- All plus correlator:

$$T_3^{+,+,+} = \frac{-R(|k_1|,|k_2|,|k_3|)}{2\sqrt{2}|k_1||k_2||k_3|}\left(|k_1| + |k_2| + |k_3|\right) \left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle \left\langle \tilde{\lambda}_2, \tilde{\lambda}_3 \right\rangle \left\langle \tilde{\lambda}_3, \tilde{\lambda}_1 \right\rangle.$$

- R comes from the radial integral:

$$R = \frac{1}{|k_1| + |k_2| + |k_3|},$$

[Maldacena, Pimentel (2011)]

Flat Space Limit: 3 pt current correlators

- The flat space limit is manifest. MHV Correlator gives the flat space MHV amplitude:

$$\lim_{|k_1|+|k_2|+|k_3|\to 0} \left(|k_1|+|k_2|+|k_3|\right) T^{++-} = i \frac{2\sqrt{2}\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle^4}{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle \left\langle \tilde{\lambda}_2, \tilde{\lambda}_3 \right\rangle \left\langle \tilde{\lambda}_3, \tilde{\lambda}_1 \right\rangle}.$$

- The all $+$ amplitude gives 0 in the flat space limit:

$$\lim_{|k_1|+|k_2|+|k_3|\to 0} \left(|k_1| + |k_2| + |k_3|\right) T^{+++} = 0.$$

Three Point Transition Amplitudes

Replacing a bulk-boundary propagator with a normalizable mode leads to a very similar result:

$$R(|k_1|, |k_2|, p) = \frac{\sqrt{\frac{2p}{\pi}}}{|k_1|^2 + 2|k_2||k_1| + |k_2|^2 + p^2}$$

Three Point Transition Amplitudes

Replacing a bulk-boundary propagator with a normalizable mode leads to a very similar result:

$$R(|k_1|, |k_2|, p) = \frac{\sqrt{\frac{2p}{\pi}}}{|k_1|^2 + 2|k_2||k_1| + |k_2|^2 + p^2}$$

Now

$$T_3^*(+,+,-) = \frac{-R}{2\sqrt{2}|k_1||k_2|p} \times \left(|k_2| + ip - |k_1|\right)\left(ip + |k_1| - |k_2|\right) \times \left(|k_1| + |k_2| - ip\right) \frac{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle^4}{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle \left\langle \tilde{\lambda}_2, \tilde{\lambda}_3 \right\rangle \left\langle \tilde{\lambda}_3, \tilde{\lambda}_1 \right\rangle}$$

and

$$T_3^*(+,+,+) = \frac{-R}{2\sqrt{2}|k_1||k_2|p} \left(|k_1| + |k_2| + ip\right) \left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle \left\langle \tilde{\lambda}_2, \tilde{\lambda}_3 \right\rangle \left\langle \tilde{\lambda}_3, \tilde{\lambda}_1 \right\rangle .$$

Three Point Gravity Vertex in AdS

$$
\begin{aligned}
V_3 = {} & \frac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_c h_{\rho\sigma} \nabla_d h^{\mu\nu} + \frac{1}{4} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_c h_{\rho\sigma} \nabla_d h^{\rho\sigma} \\
& - h^{\rho\sigma} \nabla_\mu h_{\rho\sigma} \nabla_\nu h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \nabla_\mu h_{\rho\sigma} \nabla_\nu h^{\rho\sigma} - \frac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_c h_{d\sigma} \nabla_\rho h^{\mu\nu} \\
& - \frac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_d h_{c\sigma} \nabla_\rho h^{\mu\nu} + h^{\rho\sigma} \nabla_\mu h_{\nu\sigma} \nabla_\rho h^{\mu\nu} - h^{\mu\nu} \nabla_\nu h_{\mu\sigma} \nabla_\rho h^{\rho\sigma} \\
& - \frac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_c h_{d\rho} \nabla^\rho h^{ab} - \frac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_d h_{c\rho} \nabla^\rho h^{ab} \\
& + \frac{1}{2} g_{ab} h^{\mu\nu} \nabla_\mu h_{\nu\rho} \nabla^\rho h^{ab} + \frac{1}{2} g_{ab} h^{\mu\nu} \nabla_\nu h_{\mu\rho} \nabla^\rho h^{ab} + \frac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_\rho h_{cd} \nabla^\rho h^{ab} \\
& - \frac{1}{2} g_{ab} h^{\mu\nu} \nabla_\rho h_{\mu\nu} \nabla^\rho h^{ab} - \frac{1}{2} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_\rho h_{c\sigma} \nabla^\rho h^{\sigma}{}_{d} + h^{\mu\nu} \nabla_\rho h_{\mu\sigma} \nabla^\rho h^{\sigma}{}_{\nu} \\
& + \frac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_\rho h^{\mu\nu} \nabla_\sigma h_{cd} + \frac{1}{2} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla^\rho h^{\sigma}{}_{d} \nabla_\sigma h_{c\rho} - h^{\rho\sigma} \nabla_\rho h^{\mu\nu} \nabla_\sigma h_{\mu\nu} \\
& - h^{\mu\nu} \nabla^\rho h^{\sigma}{}_{\nu} \nabla_\sigma h_{\mu\rho}.
\end{aligned}
$$

Three Point Transition Amplitudes

The amplitudes are much simpler:

$$T_3^{+,+,-} = \frac{R(|k_1|,|k_2|,p)}{32|k_1|^2|k_2|^2p^2}\left(|k_2|+ip-|k_1|\right)^2\left(ip+|k_1|-|k_2|\right)^2\left(|k_1|+|k_2|-ip\right)^2 \times \left(\frac{\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2\right\rangle^4}{\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2\right\rangle\left\langle\tilde{\lambda}_2,\tilde{\lambda}_3\right\rangle\left\langle\tilde{\lambda}_3,\tilde{\lambda}_1\right\rangle}\right)^2$$

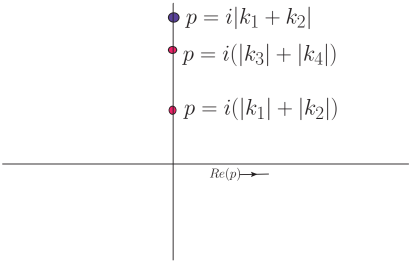
The $+++$ amplitude is given by

$$T_3^{+,+,+} = \frac{R(|k_1|,|k_2|,p)}{32|k_1|^2|k_2|^2p^2}E_p^2\left(\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2\right\rangle\left\langle\tilde{\lambda}_2,\tilde{\lambda}_3\right\rangle\left\langle\tilde{\lambda}_3,\tilde{\lambda}_1\right\rangle\right)^2.$$

where

$$R = \frac{p^{3/2}\left(|k_1|^2+4|k_2||k_1|+|k_2|^2+p^2\right)\sqrt{\frac{2}{\pi}}}{\left(|k_1|^2+2|k_2||k_1|+|k_2|^2+p^2\right)^2}$$

Four Point Computations



Use 3-pt functions to generate 4-pt functions:

$$T(h^1, k^1, \ldots h^4, k^4) = 2\pi i \sum_{\pi} \sum_{p_0 \in \mathcal{P}_\pi} \operatorname*{Res}_{p=p_0} [I_\pi(0, p)]$$

where

$$\mathcal{I}_\pi(w, p) = \frac{p}{2} \sum_{h^{\text{int}}, \pm} \frac{-i\mathcal{T}^2}{p^2 + (k_{\pi_1}(w) + k_{\pi_2}(w))^2} \frac{w - w^{\mp}(p)}{w^{\pm}(p) - w^{\mp}(p)}$$

$$\mathcal{T}^2 \equiv T^*(h^{\pi_1}, k^{\pi_1}(p), h^{\pi_2}, k^{\pi_2}(p), h^{\text{int}}, k^{\text{int}}) T^*(-h^{\text{int}}, -k^{\text{int}}, h^{\pi_3}, k^{\pi_3}(p), h^{\pi_4}, k^{\pi_4}(p)).$$

Four Point Answers: Yang-Mills

- The four point amplitude is given by

$$T^{+-+-} = \frac{\mathcal{F}}{E^T} + \mathcal{A},$$

where $\mathcal{A}$ is the product of 3-pt functions which captures the contribution of j itself running in the OPE and

$$\mathcal{F} = \left[\frac{E^{124,3}}{4|k_3||k_4||k_1|} \frac{\langle \lambda_2, \lambda_4 \rangle \left[\lambda_4, \tilde{\lambda}_3 \right] \left[\lambda_2, \tilde{\lambda}_3 \right] \left[\lambda_2, \tilde{\lambda}_1 \right]}{E^{12,34} \langle \lambda_1, \lambda_2 \rangle} + (1 \leftrightarrow \tilde{2}, 3 \leftrightarrow \tilde{4}) \right.$$
$$\left. + (1 \leftrightarrow 3, 2 \leftrightarrow 4) + (1 \leftrightarrow \tilde{4}, 2 \leftrightarrow \tilde{3}) \right] + (2 \leftrightarrow 4)$$

Flat Space Limit

- We can take the flat space limit, $E^T = 0$.
- Something remarkable happens:

$$\mathcal{F} = \frac{\langle \lambda_2, \lambda_4 \rangle^4}{\langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \langle \lambda_3, \lambda_4 \rangle \langle \lambda_4, \lambda_1 \rangle}$$

which is the famous Parke-Taylor formula for the gluon amplitude.

Four Point Answers: Gravity

Gravity answers are a little more complicated:

$$\mathcal{V}^{+}_{s_1} = \left[\frac{-i|k_2|\left(E^{34,12}(E^{T}) + 2|k_3||k_4|\right)(E^{3,124})^2}{256|k_1||k_3|^2|k_4|^2(E^{T})^2(E^{34,12})^2\,\langle\lambda_1,\lambda_2\rangle}\right]\langle\lambda_2,\lambda_4\rangle\left[\lambda_2,\,\tilde{\lambda}_1\right]^2$$

$$\times\left(\left\langle\tilde{\lambda}_1,\tilde{\lambda}_3\right\rangle\langle\lambda_2,\lambda_4\rangle E^{12,34} + E^{T}\left[\tilde{\lambda}_1,\,\lambda_4\right]\left[\lambda_2,\,\tilde{\lambda}_3\right]\right)\left[\lambda_4,\,\tilde{\lambda}_3\right]^2\left[\lambda_2,\,\tilde{\lambda}_3\right].$$

$$\mathcal{D}^{+}_{s_1} = 2i\left(|k_1| + |k_2|\right)\left(\frac{1}{E^{34,12}E^{T} + 2|k_3||k_4|} + \frac{1}{\langle\lambda_1,\lambda_2\rangle\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2\right\rangle}\right)$$

$$+ i\left(\frac{2}{E^{3,124}} + \frac{2}{E^{4,123}} + \frac{2}{E^{T}}\right) - \frac{1}{\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2(w^{+}_{s_1})\right\rangle}\left(\frac{6\left[\tilde{\lambda}_1,\,\lambda_4\right]}{\langle\lambda_4,\lambda_2\rangle} - \frac{2\left[\lambda_3(w^{+}_{s_1}),\,\tilde{\lambda}_2(w^{+}_{s_1})\right]}{\left\langle\lambda_1(w^{+}_{s_1}),\lambda_3(w^{+}_{s_1})\right\rangle}\right)$$

$$+ 2\gamma^{+}_1\left\{\frac{\beta_2\left[\tilde{\lambda}_1,\,\lambda_2\right]}{\left\langle\tilde{\lambda}_1,\tilde{\lambda}_2(w^{+}_{s_1})\right\rangle} - \frac{\beta_3\left[\tilde{\lambda}_3,\,\lambda_4\right]}{\left\langle\lambda_4,\lambda_3(w^{+}_{s_1})\right\rangle} + \frac{\beta_1\left[\tilde{\lambda}_1,\,\lambda_3(w^{+}_{s_1})\right] - \beta_3\left[\lambda_1(w^{+}_{s_1}),\,\tilde{\lambda}_3\right]}{\left\langle\lambda_1(w^{+}_{s_1}),\lambda_3(w^{+}_{s_1})\right\rangle}\right\}$$

$$- \gamma^{+}_1\frac{w^{+}_{s_1} + w^{-}_{s_1}}{w^{-}_{s_1}\left(w^{+}_{s_1} - w^{-}_{s_1}\right)}.$$

$$T_1 = \left(\mathcal{V}^{+}_{s_1}\mathcal{D}^{+}_{s_1} + 1 \leftrightarrow \tilde{2}, 3 \leftrightarrow \tilde{4}\right) + 2 \leftrightarrow 4.$$

Gravity: Term 2

There is another term

$$\mathcal{V}_{u_1}^{\pm} = \frac{|k_1||k_3| \left(E^{24,13}E^T + 2|k_2||k_4|\right) (E^{4,123})^2 (E^{2,134})^2 \left\langle \tilde{\lambda}_1, \tilde{\lambda}_3 \right\rangle \langle \lambda_4, \lambda_2 \rangle^6}{64(|k_4||k_2|)^2 (E^T)^2 \left\langle \lambda_2, \lambda_3(w_u^{\pm}) \right\rangle^2 \left\langle \lambda_4, \lambda_1(w_u^{\pm}) \right\rangle^2 \langle \lambda_1, \lambda_3 \rangle} \frac{w_{u_1}^{\mp}}{w_{u_1}^{\pm} - w_{u_1}^{\mp}}.$$

$$\mathcal{D}_{u_1}^{+} = \frac{2i(|k_1| + |k_3|)}{E^{24,13}E^T + 2|k_2||k_4|} + \frac{2i(|k_1| + |k_3|)}{\langle \lambda_1, \lambda_3 \rangle \left\langle \tilde{\lambda}_1, \tilde{\lambda}_3 \right\rangle} + \frac{2i}{E^{4,123}} + \frac{2i}{E^{2,134}} + \frac{2i}{E^T}$$

$$- \gamma_3^{+} \left(\frac{2\beta_3 \left[\lambda_2, \tilde{\lambda}_3\right]}{\left\langle \lambda_2, \lambda_3(w_u^{\pm}) \right\rangle} + \frac{2\beta_1 \left[\lambda_4, \tilde{\lambda}_1\right]}{\left\langle \lambda_4, \lambda_1(w_u^{\pm}) \right\rangle} + \frac{w_{u_1}^{+} + w_{u_1}^{-}}{w_{u_1}^{-} \left(w_{u_1}^{+} - w_{u_1}^{-}\right)} \right)$$

$$+ \frac{2}{\left\langle \tilde{\lambda}_1, \tilde{\lambda}_3 \right\rangle} \left(\frac{\left[\tilde{\lambda}_1, \lambda_2\right]}{\left\langle \lambda_2, \lambda_3(w_u^{\pm}) \right\rangle} - \frac{\left[\tilde{\lambda}_3, \lambda_4\right]}{\left\langle \lambda_4, \lambda_1(w_u^{\pm}) \right\rangle} \right).$$

$$T_{u_1} = \sum_{\pm} \mathcal{V}_{u_1}^{\pm} \mathcal{D}_{u_1}^{\pm}.$$

In addition to these terms, we get the contribution of the stress-tensor in the OPE from the products of un-deformed three point functions.

Gravity: Flat Space Limit

- In the flat space limit, this formula simplifies.
- The correlator becomes:

$$T = \frac{\mathcal{F}}{(|k_1| + |k_2| + |k_3| + |k_4|)^3} + \ldots$$

where

$$\mathcal{F} = \frac{\langle \lambda_4, \lambda_2 \rangle^8 \left\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \right\rangle}{\langle \lambda_4, \lambda_2 \rangle \langle \lambda_3, \lambda_4 \rangle^2 \langle \lambda_1, \lambda_2 \rangle \langle \lambda_2, \lambda_3 \rangle \langle \lambda_4, \lambda_1 \rangle \langle \lambda_1, \lambda_3 \rangle}$$

- This is just the flat space amplitude for 4 gravitons!

Some other consequences

- The OPE tells us that as $x \to 0$,

$$\langle \phi(x)\phi(0)\phi(y_1)\phi(y_2)\rangle = \sum \frac{C^O}{|x|^{2\Delta_\phi - \Delta_O}} \langle O(0)\phi(y_1)\phi(y_2)\rangle,$$

- In momentum space, this means:

$$\langle T(k)T(-k-p)T(p_1)T(p_2)\rangle \underset{k\to\infty}{\longrightarrow} \sum_O |k|^{d-\Delta_O} f(p_1, p_2),$$

- If a double trace operator has a small anomalous dimension: $\Delta_O = 2d + m + \frac{\delta}{N^2}$, we should get logs

$$|k|^{d-\Delta_O} \approx \frac{1 - \frac{\delta}{N^2}\log(|k|)}{|k|^{d+m}}.$$

No Logs!

- The fact that there are no logs in our answer suggests that double trace operators of the stress tensor have no anomalous dimension to leading order in $\frac{1}{N}$ in any CFT with a gravity/supergravity dual in AdS_4.
- This analytic structure is consistent with Maldacena's conformal gravity argument.
- Can, in principle, be checked by a strong coupling computation in ABJM.

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6. The results predict that double trace operators of the stress-tensor in theories with a gravity/supergravity dual in AdS_4 have no anomalous dimensions to leading order in $\frac{1}{N}$.