Multiloop Integrands & Integrals

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1. Constructing Integrands from their collinear/soft behavior

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2. A multi-Regge test of $\mathcal{N}=4$ two-loop MHV amplitudes

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2. A multi-Regge test of $\mathcal{N}=4$ two-loop MHV amplitudes

3. A natural language (?) for multiloop amplitudes in flat space

1. Bourjaily, DiRe, Shaikh, MS, Volovich (1112.6432)
   Golden, MS (to appear)

2. Prygarin, MS, Vergu, Volovich (1112.6365)

3. Paulos, MS, Volovich (to appear)

2. A multi-Regge test of $\mathcal{N}=4$ two-loop MHV amplitudes

In May 2011 S. Caron-Huot computed the symbol of the 2-loop MHV remainder function for any n.

There is a special kinematic limit in which we can simultaneously (1) find a simple function and (2) check against an independent calculation

Multi-Regge Kinematics

Consider $2 \to n-2$ scattering where the rapidities of the produced particles are "strongly ordered":

$$p_1 = (0, p_1^-, 0)$$

$$p_2 = (p_2^+, 0, 0)$$

$$\epsilon \to 0$$

$$p_3 = \mathcal{O}(\epsilon^{n-3/2}, \epsilon^{3/2-n}, 1)$$

$$p_4 = \mathcal{O}(\epsilon^{n-5/2}, \epsilon^{5/2-n}, 1)$$

$$\vdots$$

$$p_n = \mathcal{O}(\epsilon^{3/2-n}, \epsilon^{n-3/2}, 1)$$

In the Euclidean region (all Mandelstam invariants positive) the MHV remainder function vanishes in MRK (multi-Regge kinematics, i.e. $\epsilon \to 0$).

In other regions it is nonzero and in fact diverges as $\log \epsilon$.

Simplest region: change the sign of the energies of particles 4 through $n-1$.

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In this region only a single cross-ratio

$$u_{2,n-1} = \frac{s_{3\cdots n}\, s_{4\cdots n-1}}{s_{3\cdots n-1}\, s_{4\cdots n}}$$

picks up a phase $e^{-2\pi i}$.

Bartels, Kormilitzin, Lipatov & Prygarin carried out a BFKL computation of the imaginary part in this region, and we confirmed the same answer from Simon's symbol as follows:

(1) Read off all terms in the symbol with $u_{2,n-1}$ in the first entry — this gives the symbol of the imaginary part.

(2) Take the multi-Regge limit $\epsilon \to 0$ inside the symbol, terms with two or three ϵ's in them cancel as they must.

What remains is $\log \epsilon$ times the function

$$\frac{i\pi}{2} \sum_{i=4}^{n-2} \log\left[\frac{|p_3|^2 \, |p_{i+1}+\cdots+p_{n-1}|^2}{|p_3+\cdots+p_i|^2 \, |p_4+\cdots+p_{n-1}|^2}\right] \log\left[\frac{|p_3+\cdots+p_{n-1}|^2 \, |p_4+\cdots+p_i|^2}{|p_3+\cdots+p_i|^2 \, |p_4+\cdots+p_{n-1}|^2}\right]$$

which is essentially a sum of $n-5$ "shifted" copies of the previously studied $n=6$ case.

This formula agrees completely with the BFKL calculation of BKLP, who explained this structure.

3. A natural language (?) for multiloop amplitudes in flat space

Recently there has been some work on the ulility of using Mellin space for correlation functions at strong coupling (ie via AdS/CFT).

Mack, Penedones, Fitzpatrick, Kaplan, Raju, van Rees, Paulos, Nandan, Volovich, Wen

The motivation is two-fold:

3. A natural language (?) for multiloop amplitudes in flat space

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The motivation is two-fold:

It is healthy and tastes good!

Next dozen or so slides:

casual experimentation

in Mellin space.

Can we do the same for amplitudes/multi-loop integrals in flat space?

Sure, and the motivation is ... See end of talk

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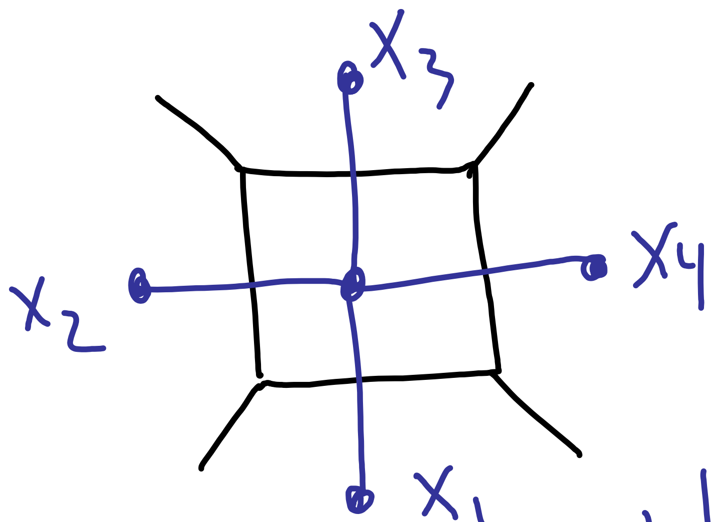
By convention the Mellin transform of an object depending on various x_{ij}^2 is

$$F(x_{ij}^2) = \int M(\delta_{ij}) \prod_{i<j} \Gamma(\delta_{ij}) (x_{ij}^2)^{-\delta_{ij}} d\delta$$

"the Mellin transform"

ubiquitous

Example 1 — One-Loop Box



$$= \frac{1}{\lambda}\left[2\,\mathrm{Li}_2(-\rho u) + 2\,\mathrm{Li}_2(-\rho v) + \log\frac{u}{v}\log\frac{1+\rho u}{1+\rho v} + \log(\rho u)\log(\rho v) + \frac{\pi^2}{3}\right]$$

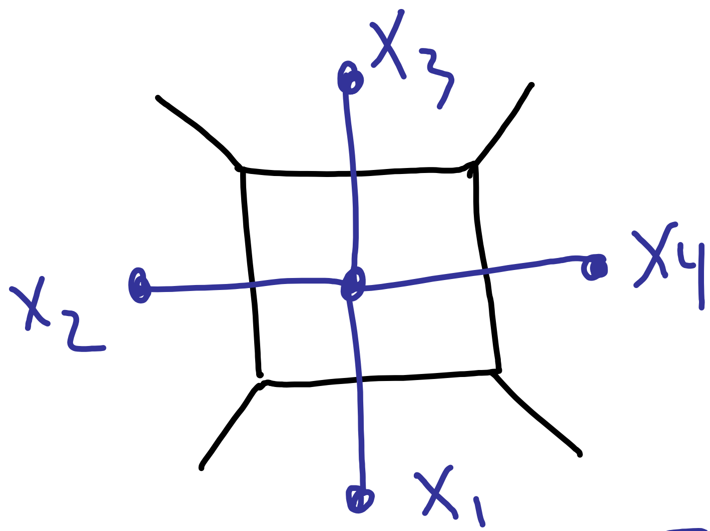
$$\lambda = \sqrt{(1-u-v)^2 - 4uv} \qquad \rho = \frac{2}{1-u-v+\lambda}$$

$$u = \frac{x_{13}^2\, x_{24}^2}{x_{14}^2\, x_{23}^2} \qquad v = \frac{x_{13}^2\, x_{24}^2}{x_{12}^2\, x_{34}^2}$$

This form from Usyukina & Davydychev

Note also A. Hodges's nice representation in terms of dog functions.

Example 1 – One-Loop Box



$$= F(u,v)$$

$$= x_{13}^2 x_{24}^2 \int \prod_{i<j} \Gamma(\delta_{ij}) \, (x_{ij}^2)^{-\delta_{ij}} \, d\delta_{ij}$$

where the δ_{ij} are constrained to satisfy

$$\sum_j \delta_{ij} = 1 \quad \forall i$$

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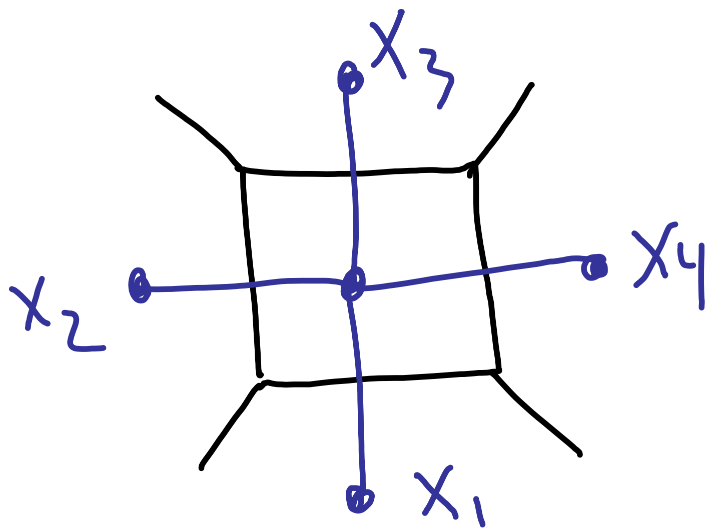

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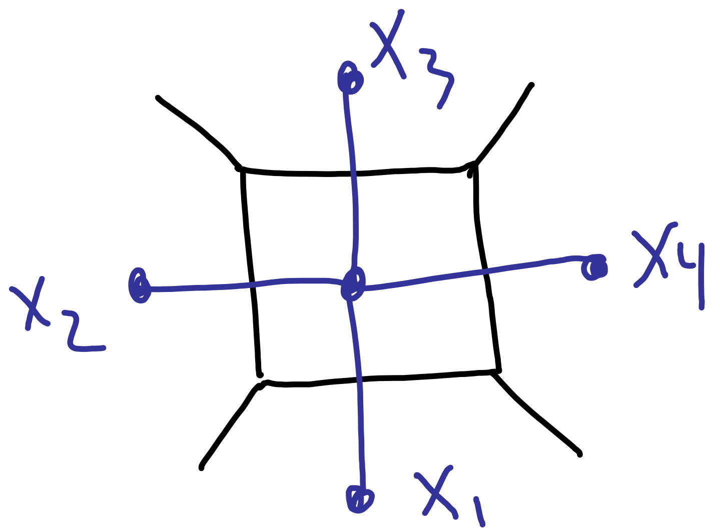
where the δ_{ij} are constrained to satisfy

$$\sum_j \delta_{ij} = 1 \quad \forall i \quad \Rightarrow$$

choose a parameterization

$$\delta_{ij} = \begin{pmatrix} 0 & 1-s & s+t-1 & 1-t \\ 1-s & 0 & 1-t & s+t-1 \\ s+t-1 & 1-t & 0 & 1-s \\ 1-t & s+t-1 & 1-s & 0 \end{pmatrix}$$

Example 1 – One-Loop Box



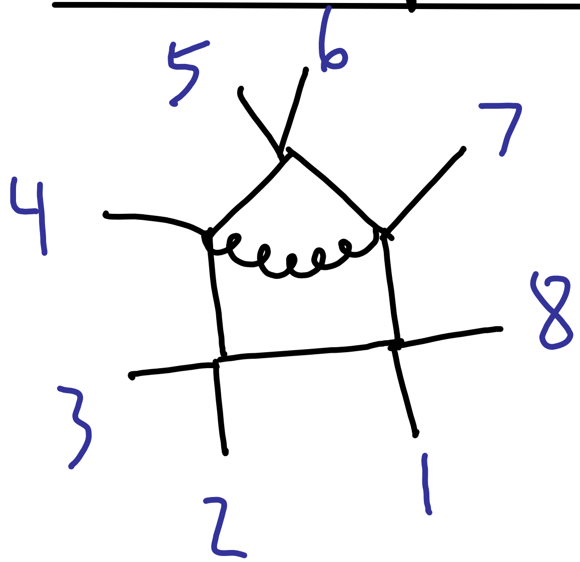
$$= f(u, v)$$

$$= \int ds\, dt \; u^{s-1} v^{t-1}\, \Gamma(1-s)^2\, \Gamma(1-t)^2\, \Gamma(s+t-1)^2$$

which is very well known!

So the "Mellin transform" of this object is just $\boxed{1}$
(or really $x_{13}^2 x_{24}^2 \times 1$)

Example 2 – A chiral pentagon

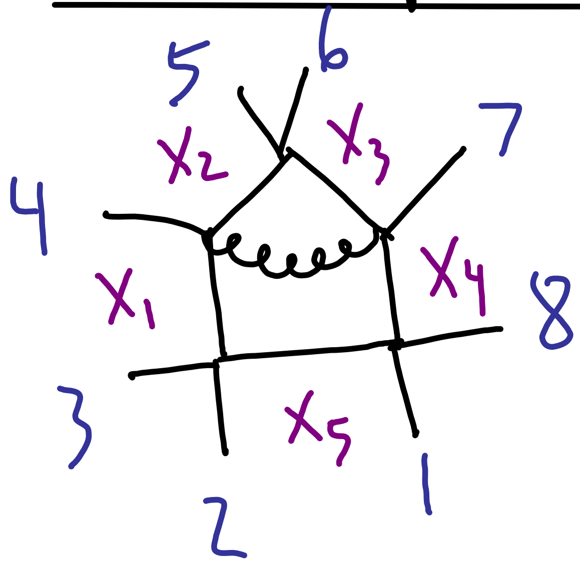


$$= \log u_3 \log u_4 + \mathrm{Li}_2(1-u_3) + \mathrm{Li}_2(1-u_4)$$
$$- \mathrm{Li}_2(1-u_1 u_3) - \mathrm{Li}_2(1-u_1 u_4)$$
$$+ \mathrm{Li}_2(1-u_1)$$

(Arkani-Hamed, Bourjaily, Cachazo, Trnka)

$$u_1 = \frac{\langle 3478\rangle\langle 4567\rangle}{\langle 3467\rangle\langle 4578\rangle} \qquad u_3 = \frac{\langle 1278\rangle\langle 3467\rangle}{\langle 1267\rangle\langle 3478\rangle} \qquad u_4 = \frac{\langle 1234\rangle\langle 4578\rangle}{\langle 1245\rangle\langle 3478\rangle}$$

Example 2 — A chiral pentagon



$$= \langle 12(345)\cap(678)\rangle\langle 1247\rangle$$

$$\times \int \prod_{i<j} \Gamma(\delta_{ij})\,(x_{ij}^2)^{-\delta_{ij}}\, d\delta$$

where the δ_{ij} satisfy $\displaystyle\sum_{i=1}^{5} \delta_{ij} = \begin{cases} 1 & j\neq 5 \\ 2 & j=5 \end{cases}$

$x_{12}^2 = 0$

$x_{13}^2 = \langle 3467\rangle$

$x_{14}^2 = \langle 3478\rangle$

$x_{15}^2 = \langle 1234\rangle$

$x_{23}^2 = \langle 4567\rangle$

$x_{24}^2 = \langle 4578\rangle$

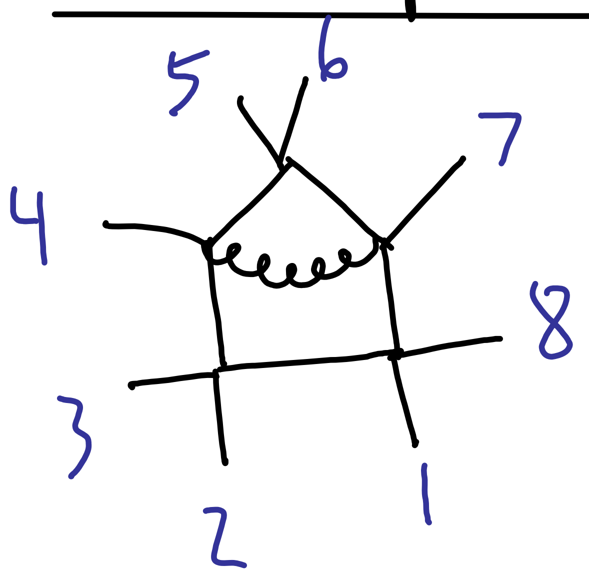
$x_{25}^2 = \langle 1245\rangle$

$x_{34}^2 = 0$

$x_{35}^2 = \langle 1267\rangle$

$x_{45}^2 = \langle 1278\rangle$

Example 2 – A chiral pentagon

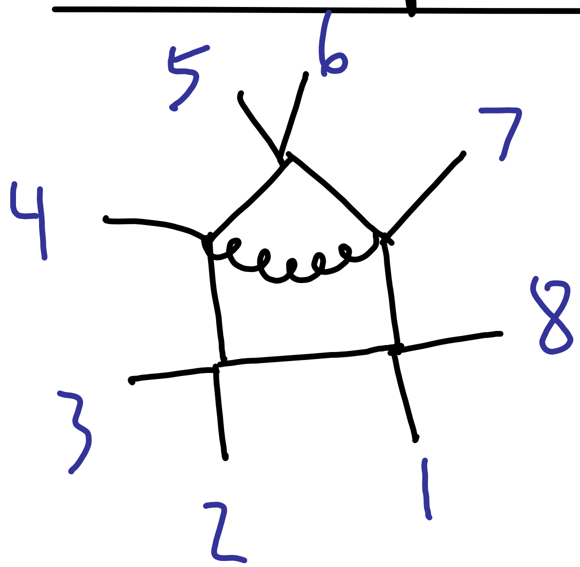


Choosing a convenient parameterization of δ_{ij} in terms of 3 free variables $\Rightarrow$

$$\int dc_1\, dc_3\, dc_4\; u_1^{c_1-1}\, u_3^{c_3-1}\, u_4^{c_4-1}\; \Gamma(1-c_1)\Gamma(1-c_3)\Gamma(1-c_4)\,\Gamma(c_1-c_3)\Gamma(c_3)\Gamma(c_1-c_4)\Gamma(c_4)\Gamma(c_3+c_4-c_1)$$

which does yield the correct answer!

Example 2 — A chiral pentagon

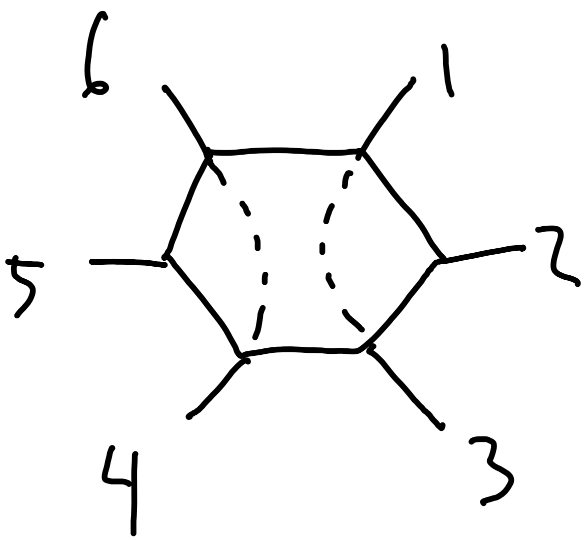


So the "Mellin transform" of this integral is just $\boxed{1}$

(or $\langle 12(345)\cap(678)\rangle\langle 1247\rangle \times 1$)

[We simply omit terms like $\Gamma(0)(0)^0$ from the product — it works in this case but requires justification; more later.]

Example 3 – chiral hexagon



$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) + \log u_2 \log u_3 - \frac{\pi^2}{3}$$

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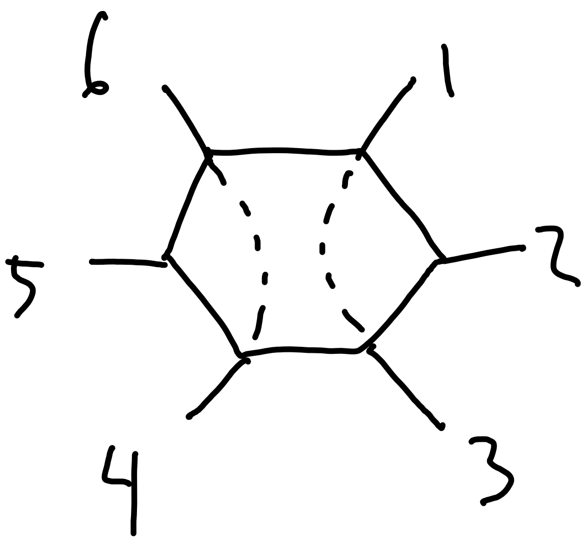


$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) + \log u_2 \log u_3 - \frac{\pi^2}{3}$$

$$= \langle 1256\rangle\langle 1346\rangle\langle 2345\rangle \int M(\delta_{ij}) \prod_{i<j} \Gamma(\delta_{ij})(X_{ij}^2)^{-\delta_{ij}}$$

$$\boxed{M = 1 - \frac{\langle 1345\rangle\langle 2346\rangle}{\langle 1346\rangle\langle 2345\rangle}\delta_{15} - (1-u_3)\delta_{25} - (1-u_1)\delta_{14} - \frac{\langle 1246\rangle\langle 1356\rangle}{\langle 1256\rangle\langle 1346\rangle}\delta_{24}}$$

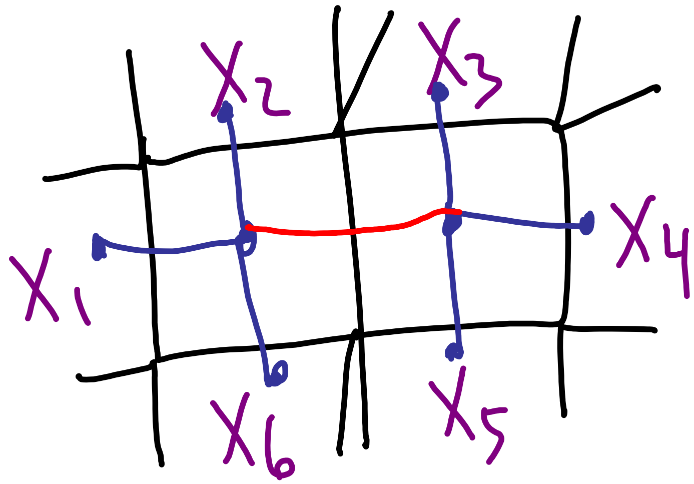
$$\sum \delta_{ij} = 1 \;\; \forall j$$

$i<j$

$$\boxed{M = 1 - \frac{\langle 1345\rangle\langle 2346\rangle}{\langle 1346\rangle\langle 2345\rangle}\delta_{15} - (1-u_3)\delta_{25} - (1-u_1)\delta_{14} - \frac{\langle 1246\rangle\langle 1356\rangle}{\langle 1256\rangle\langle 1346\rangle}\delta_{24}}$$

This particular numerator is the Mellin space manifestation of the well-known (Dixon, Drummond, Henn) differential equation relating the chiral hexagon to the scalar hexagon in $D=6$ dimensions.

Example 4 – fully massive double box



= a function of 15 (!!) cross-ratios

$$= \int \prod_{i<j} \Gamma(\delta_{ij}) (x_{ij}^2)^{-\delta_{ij}} \frac{1}{\frac{1}{2} - \delta_{61} - \delta_{12} - \delta_{62}}$$

where as usual $\sum \delta_{ij} = 1$. Such Mellin transforms are trivial to write down (graphical rules)

Interestingly the easiest integrals to write down are fully massive ones — because they resemble off-shell correlation functions in the dual space.

In a limit when some $x_{ij}^2 \to 0$,

if some cross-ratio $\to 0$, simply omit it

if a cross-ratio $\to 1$, that's more tricky

What follows is

Philosophy.

Why?

I've long been on the lookout for some kind of "partially integrated amplitudes".

We have powerful methods for generating integrands but no practical general purpose algorithm for doing integrals.

Why?

The integrand has too many integration variables — 4 per loop order.

If doing all integrals is too hard can we at least do some, as a first step, and distill the integrand to a canonical intermediate form.

integrand → ??? → amplitude

Mellin-Barnes representations are very useful if you really need to know the answer for an integral,

but they often have more than the needed number of integration variables and hence are noncanonical.

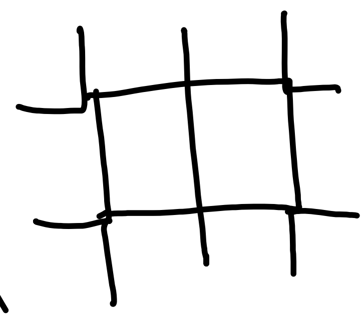
The Mellin transform has exactly the right number of integration variables — 1 per cross-ratio — at any loop order!

so it is completely canonical
(thanks to the Mellin inversion theorem)

[disappointingly useless in practice in my experience?]

I used to have high hopes on the symbol for providing a canonical, partially integrated representation of amplitudes.

But only relatively simple integrals have symbols

( does not)

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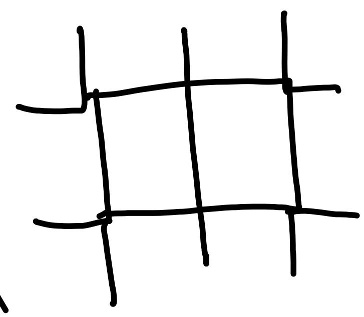
( does not)

Happily the Mellin transform works most easily exactly for these "hard" integrals.

Mellin representations have a hope of giving me what I want

$\text{integrand} \longrightarrow ??? \longrightarrow \text{amplitude}$

Provided that $\exists$ an efficient algorithm for computing them — I've given several examples but not explained the general rules

work in progress...