Where is the PdV term in the first law of black hole thermodynamics?

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Outline

- Review of black hole thermodynamics
  - First Law
  - Smarr Relation
- Enthalpy and Volume
  - Enthalpy
  - Equation of State - Van der Waals
  - Compressibility
  - de Sitter
- Summary

Black hole thermodynamics

- Entropy, $S = \frac{1}{4}\frac{A}{\ell_{Pl}^{D-2}}$: $A =$ area, ($\ell_{Pl}^{D-2} = G_N\hbar$, $c = 1$).
- Hawking temperature, $T = \frac{\kappa\hbar}{2\pi}$, $\kappa =$surface gravity.
- Internal (thermal) energy, identify $M = U(S)$,

First Law of Black Hole Thermodynamics

$$dM = dU = T\,dS = \frac{\kappa}{8\pi G_N}\,dA.$$

- Including J and Q,

$$dM = dU = T\,dS + \Omega\,dJ + \Phi\,dQ.$$

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Smarr relation

- Ordinary thermodynamics: $U(S,V,N)$ ($N =$ no. of moles) is a function of extensive variables. U is also extensive.

$$\begin{aligned} \lambda^d U(S,V,N) &= U(\lambda^d S, \lambda^d V, \lambda^d N) \\ \Rightarrow \quad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + N\frac{\partial U}{\partial N} && \text{Euler equation} \\ \Rightarrow \quad U &= ST - VP + N\mu && (\mu = \text{chemical potential}) \\ \Rightarrow \quad G &= N\mu = U - ST + VP && \text{Gibbs-Duhem relation} \end{aligned}$$

- Black hole, ADM mass M: $(G_N = 1, Q = 0)$
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First Law BPD [1008.5023].

$$dU = T\,dS + \Omega\,dJ - P\,dV$$

- $M = U + PV$ with $V := \left.\frac{\partial M}{\partial P}\right|_{S,J}$, thermodynamic volume.
- Efficiency of Penrose process modified:

  e.g. for asymptotically AdS, $D = 4$:

  $\eta_{max} = 0.5184\ldots$, for extremal black hole.

  $\eta_{max} = 1 - \frac{1}{\sqrt{2}} = 0.2929\ldots$, without the PdV term.

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Thermodynamic volume

- Asymptotically anti-de Sitter Kerr space-time,
  $\Lambda < 0 \quad \Rightarrow \quad P > 0$.
- black hole event horizon, r_h;
  "geometric volume", $V_0 = \frac{r_h A_h}{3}$. (D=4)

Thermodynamic volume

$$V = V_0 + \frac{4\pi}{3}\frac{J^2}{M} > V_0.$$

- Spherically symmetric case: $J = 0$, $V = \frac{4\pi r_h^3}{3}$.
- Reverse isoperimetric inequality
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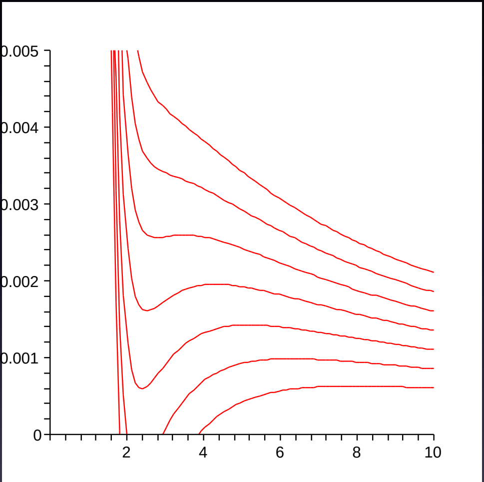
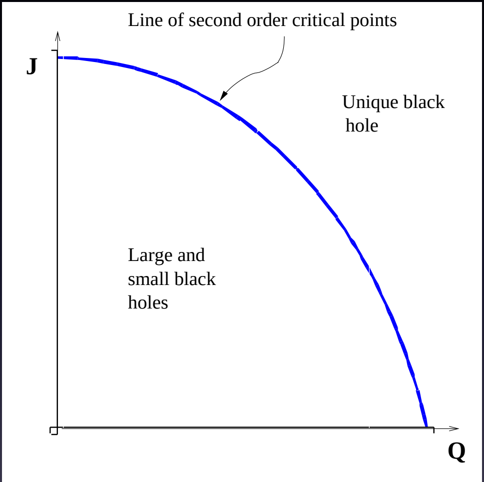
Equation of state: P–V diagram



P as a function of $\left(\frac{3V}{4\pi}\right)^{1/3}$, curves of constant T for $J = 1$.

Critical point at $T_C = 0.0413/J^{1/2}$, $P_C = 0.00280/J$ and $V_C = 12.90J^{3/2}$, Caldarelli,Gognola+Klemm (1999).

Kerr-Reissner-Nordström-AdS



- Reissner-Nordström anti-de Sitter ($J \neq 0$, $Q \neq 0$)
- Mean field exponents — same as Van der Waals gas,

  Emparan, Johnson, and Myers (1999);

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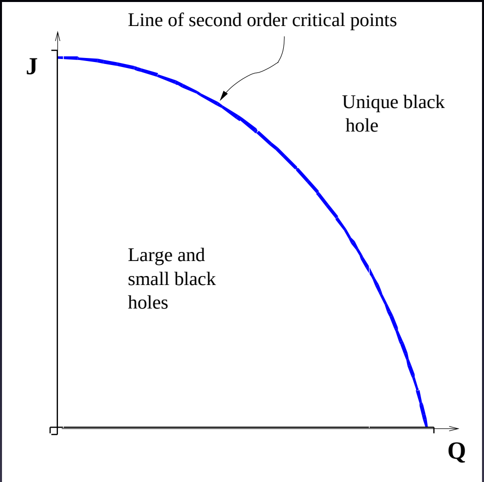

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Compressibility

BPD [arXiv:1109.0198]

- Adiabatic compressibility: $\kappa = -\frac{1}{V} \frac{\partial V}{\partial P}\Big|_{S,J} \geq 0.$
- $\kappa_{J=0} = 0.$
- Maximum for J_{max} $(T = 0)$: $\kappa_{max} = \frac{2S(1+8PS)}{(3+8PS)^2(1+4PS)}.$
- e.g. $P = 0$,

  $\kappa_{max} = \frac{2S}{9} = \frac{4\pi M^2}{9} = 2.6 \times 10^{-38} \left(\frac{M}{M_\odot}\right)^2 m\, s^2\, kg^{-1}.$

  cf. neutron star, $M \approx M_\odot$, $R \approx 10\, km$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}.$

Very stiff equation of state!

Asymptotically de Sitter

BPD, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen [1301.5926]

- $P = -\frac{\Lambda}{8\pi} < 0$.
- Two event horizons: black hole r_h; cosmological, r_c.
- Two different temperatures, $T_h \neq T_c$, in general.
- $M(S_h, P, J) = M(S_c, P, J)$.
  $V_h = \left.\frac{\partial M}{\partial P}\right|_{S_h,J}$, $V_c = \left.\frac{\partial M}{\partial P}\right|_{S_c,J}$.
- $V_h = \frac{r_h A_h}{3} + \frac{4\pi}{3}\frac{J^2}{M}$, $\qquad V_c = \frac{r_c A_c}{3} + \frac{4\pi}{3}\frac{J^2}{M}$.
- For fixed V_c the cosmological horizon entropy, $S_c = \frac{A_c}{4}$, is maximized by Schwarzschild- de Sitter space-time $(J = 0)$.
- Volume between horizons: $V = V_c - V_h = \frac{r_c A_c}{3} - \frac{r_h A_h}{3}$.
- For fixed V the total entropy, $S = S_h + S_c$, is minimized if there is no black hole.

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Summary

- Smarr relation must be modified for $\Lambda \neq 0$.
- Thermodynamical volume: $V = \frac{\partial M}{\partial P}\Big|_{S,J,Q}$ $\quad (P = -\frac{\Lambda}{8\pi})$.
- Reverse isoperimetric inequality, $V > V_0$, for AdS.
- Including pressure, first law becomes:

$$dU = TdS + \Omega dJ + \Phi dQ - PdV\,.$$

- Compressibility, $\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_{S,J} \geq 0$.
- Efficiency of Penrose process modified:
  e.g. $Q = 0$: $\eta_{max} \approx 0.5184$ for an extremal black hole
  ($\eta \approx 0.2929$ without the PdV term).
- Generalises to higher dimensions.