Transport Coefficients for Non-Newtonian Fluids and Causal Dissipative Hydrodynamics

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Abstract

We discuss the roles of viscosity in relativistic fluid dynamics from the point of view of memory effects. We show that, depending on what quantity the memory effect is applied, different terms appear in higher order corrections. We generalize the application of the Green-Kubo-Nakano (GKN) to calculate transport coefficients when the memory effects are present.

1 Why should relativistic fluid be non-Newtonian?

The effect of dissipation in relativistic fluids is one of current topics in the physics of relativistic heavy-ion collisions [1]. Here, we discuss this problem focusing on the memory effect on irreversible current. First let us illustrate the basic idea of memory effect as the solution for the problem of relativistic hydrodynamics using the example of diffusion equation.

In the usual derivation of diffusion equation, we assume that the irreversible current $J(t)$ is simply proportional to the corresponding thermodynamic force $F(t)$,

$$J(t) = DF(t), \tag{1}$$

where D is a transport coefficient. The fluid with this current is referred to as a Newtonian fluid, and the evolution is described by, for example, the Navier-Stokes equation. However, exactly speaking, there should exist a time retardation effect (remember the linear response theory) and the expression (1) is justified only when there is a clear separation of microscopic and macroscopic scales. This assumption is obviously satisfied in fluids around us, where the velocity of molecules is around $10^2 - 10^3$ m/s but the speed of a fluid diffusion is much slower.

However, it is not clear if this is still true in relativistic fluids, because the fluid is accelerated up to the speed of light. Then it is natural to consider the retardation effect in the definition of relativistic irreversible currents,

$$J(t) = \int_{-\infty}^{t} G(t-s)F(s), \tag{2}$$

where $G(t)$ is a memory function which represents the retardation effect.

This is a natural extension of hydrodynamics to the relativistic region and, as a matter of fact, this extension is related to the problem of acausality. Let us consider a diffusion process [3].

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Then the thermodynamic force is given by the spatial derivative of a conserved density n and the linear relation (1) is called Fick's law. By substituting Fick's law into the equation of continuity, we obtain the diffusion equation. As is well known, the diffusion equation has a problem of infinite propagation speed. On the other hand, when we use Eq. (2) with a simple exponential memory function,

$$G(t) = \frac{D}{\tau_R} e^{-t/\tau_R}, \tag{3}$$

where τ_R is the relaxation time and D is the diffusion constant, we obtain a telegraph equation and then the maximum propagation speed is given by $\sqrt{D/\tau_R}$. It is also worth mentioning that what we can obtain from microscopic dynamics is not the diffusion equation but the telegraphic equation [3].

The same discussion is applicable to relativistic dissipative hydrodynamics [2]. In the relativistic Navier-Stokes (Landau-Lifshitz) equation, the viscous flows are defined by assuming Eq. (1). For example, in the bulk viscosity, the corresponding thermodynamic force is $\partial_\mu u^\mu$, where u^μ is the fluid velocity. Thus the bulk viscosity is given by

$$\Pi = -\eta \partial_\mu u^\mu. \tag{4}$$

On the other hand, in the hydrodynamics consistent with causality, it is given by

$$\pi(\tau) = \int_{-\infty}^{\tau} G(\tau - s) \partial_\mu u^\mu(s), \tag{5}$$

where τ is a proper time. As is shown in Ref. [4, 5], the propagation speed of signal exceeds the sped of light in the case of Eq. (4), and this acausality makes the hydrodynamic evolution unstable. On the other hand, causal dissipative hydrodynamics (5) is causal and stable. In this sense, it is impossible to solve the Landau-Lifshitz theory without using artificial tricks.

2 Memory Effects on Extensive Measures

Another important mechanism is the finite volume effect of fluid cells [6]. In the derivation of hydrodynamics, it is assumed that the local equilibrium is achieved in each fluid cell which has finite spatial extension. Then, the second law of thermodynamics and the memory effect should be applied on extensive measures associated with this finite volume. To introduce an extensive measure for the density of an additive quantity, let us consider a fluid cell of proper volume V. Due to the fluid flow, this volume changes in time and its time rate of change is given by

$$\frac{1}{V}\frac{dV}{dt} = \nabla \cdot \vec{v},$$

where $\vec{v}$ is the fluid velocity field. This equation can be written in a covariant form as

$$\partial_\mu(\sigma(\mathbf{r}, t) u^\mu(\mathbf{r}, t)) = 0, \tag{6}$$

where $\sigma(\mathbf{r}, t)$ is the inverse of the volume of the fluid cell at $\mathbf{r}$, and u^μ is the fluid velocity. Then the irreversible current is given by

$$\frac{J(t)}{\sigma(t)} = \int_{t_0}^{t} ds G(t-s) \frac{F(s)}{\sigma(s)} \tag{7}$$

This is equivalent to the solution of the following differential equation,

$$\tau_R \frac{d}{dt} J(t) + J(t) = DF(t) - \tau_R J(t) \partial_\mu u^\mu(t). \tag{8}$$

Noted that this result does not depend on the choice of the volume of the fluid cell $\sigma(t)$.

The last non-linear term is important not only for the physical concept but also for stability of relativistic fluids. There are two different instabilities in relativistic dissipative hydrodynamics. One is the instability induced by acausality. It is sometimes claimed that the problems of acausality and instability are not correlated and the relativistic Navier-Stokes equation is still available when we carefully remove unphysical signals caused by acausality. However, as is shown in Ref. [5], the instability of relativistic fluids is induced by acausality. Thus, the causal dissipative hydrodynamics will be a more appropriate theory to describe relativistic fluids.

The other instability appears in ultra-relativistic phenomena. In such an extreme situation, the bulk viscosity becomes very small and we can analytically show that the causal dissipative hydrodynamics is instable for such a small bulk viscosity. On the other hand, this instability is solved by considering the non-linear term, because the non-linear term prevents the bulk viscosity to have very small values and the minimum is given by [6]

$$\Pi_{min} = -\frac{\zeta}{\tau_R}, \tag{9}$$

where ζ is the bulk viscosity coefficient and τ_R is the corresponding relaxation time. Thus the causal dissipative hydrodynamics with finite size effect is stable even for ultra-relativistic phenomena. Another important fact is that, the so-called full Israel-Stewart theory can also be expressed in terms of a memory function, but in this case, a very peculiar form of thermodynamical quantity should be taken to apply the memory effect [6].

In Fig. 1, we show the shock formation with an initial velocity $\gamma = 5$. As is shown in the left panel, in this ultra-relativistic initial condition, the causal dissipative hydrodynamics without the finite size effect becomes unstable. On the other hand, if the finite size effect is taken into account, the instability disappears.

A very different scenario where relativistic fluid dynamics can be applied is found in astrophysics. The speed of flow becomes $\gamma \sim 100$ in the Gamma-raybursts. Thus, the construction of the dissipative hydrodynamics applicable to such an extreme situation is important for some astrophysical processes, too.

3 GKN formula: applicable to non-Newtonian fluids ?

As we have shown, it is more natural to consider that the relativistic fluid is a non-Newtonian fluid. It means that we cannot use various techniques which are known to analyse the Newtonian fluids. The Green-Kubo-Nakano (GKN) formula is one of them. The GKN formula has been used to calculate the viscosity coefficients and the heat conductivity of relativistic fluids. However, it should be noted that the assumption of Newtonian fluids is implicitly used in the derivation of the expressions. Thus the usual GKN formula is not applicable for calculation of relativistic fluids.

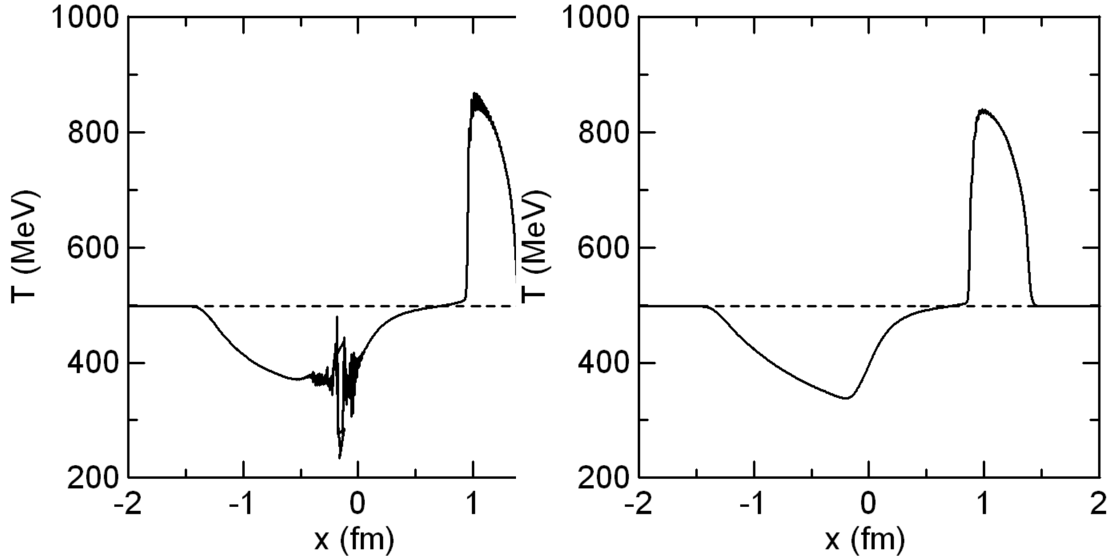

Fig. 1: The temperature in the shock formation calculated without the finite size effect (left) and with the finite size effect (right), starting from the homogeneous initial condition (dotted line) [6].

To show this, we consider the system whose Hamiltonian is given by H. By applying an external force, the total Hamiltonian is changed from H to $H + H_{ex}(t)$, with

$$H_{ex}(t) = -AF(t), \tag{10}$$

where A is an operator and $F(t)$ is the c-number external force.

We consider the current J induced by the external force. From the linear response theory, we obtain

$$\langle J \rangle = \int_{-\infty}^{t} ds \Psi(t-s) F(s), \tag{11}$$

where the response function is given by

$$\Psi(t) = \int_0^\beta d\lambda \langle \dot{A}(-i\lambda) J(t) \rangle_{eq}, \tag{12}$$

where β is the inverse of temperature. This is the exact result in the sense of the linear approximation. This formula is one of the expressions of the GKN formula. However, in particular, when we define transport coefficients of hydrodynamics, we do not use this expression.

In deriving the GKN formula for hydrodynamic transport coefficients, we assume a linear relation between currents and the external force, $J(t) = D_{GKN} F(t)$ with the transport coefficient D_{GKN}. One can easily see that this is nothing but the assumption of Newtonian fluids. And the formula to calculate this D_{GKN} is usually called the GKN formula of the shear viscosity, the bulk viscosity, heat conduction and so on. To derive the expression, we have to ignore the memory effect (time-convolution integral) in Eq. (11),

$$\langle J \rangle \approx \int_0^\infty ds \Psi(s) F(t). \tag{13}$$

Then the GKN formula is

$$D_{GKN} = \int_0^\infty ds \Psi(s). \tag{14}$$

In the shear viscosity, this is given by the time correlation function of the energy-momentum tensor.

In principle, we can derive the formula for non-Newtonian transport coefficients by assuming Eq. (8) instead of Eq. (1). From the exact result (11), we can derive the following equation,

$$\partial_t J(t) = \Psi(0) F(t) + \int_0^\infty ds \partial_s \Psi(s) F(t). \tag{15}$$

In the second term, we ignore the time-convolution integral. We further assume the usual GKN formula to reexpress the first term. Then we finally obtain

$$\partial_t J(t) = \frac{\Psi(0)}{D_{GKN}} J(t) + \int_0^\infty ds \partial_s \Psi(s) F(t). \tag{16}$$

By comparing this equation with Eq. (8) ignoring the non-linear term, we can derive the expressions for D and τ_R.

Exactly speaking, the shear viscosity is induced not by the external force but by the difference of the boundary conditions. Actually, the GKN formula of the shear viscosity is derived by using the nonequilibrium statistical operator method proposed by Zubarev. Thus the discussion that we developed here is not applicable to the problems discussed in this paper. The exact expression of the non-Newtonian transport coefficients are given in [7, 8].

We have investigated the viscous fluid dynamics emphasizing the memory effect. When a fluid posesses a non-Newtonian behavior, the use of GKN formula should be cautious.

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