

Production amplitudes in $N = 4$ SUSY and Mandelstam cuts

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Abstract

Pomeron in QCD is a composite state of reggeized gluons. The BDS ansatz for production amplitudes in the planar approximation for $N = 4$ SUSY is not valid beyond one loop due to the presence of the Mandelstam cuts. The hamiltonian for the corresponding composite states in the adjoint representation coincides with the hamiltonian of an integrable open Heisenberg spin chain.

1 BFKL Pomeron and anomalous dimensions

In the leading and next-to-leading approximations for QCD and the supersymmetric gauge theories the high energy production amplitude in the planar approximation has the multi-Regge form [1–3], which gives a possibility to write a Bethe-Salpeter-type equation for the total cross-section σ_t . The corresponding Pomeron wave function satisfies the equation of Balitsky, Fadin, Kuraev and Lipatov (BFKL) [1]

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E, \quad \sigma_t \sim s^{\Delta_{max}}. \quad (1)$$

It is important, that the BFKL Hamiltonian H_{12} in the coordinate representation ρ is invariant under the Möbius transformations [4, 5].

One can write the Bartels-Kwiecinski-Praszalowicz (BKP) equation [6] for colorless composite states of several reggeized gluons and the corresponding hamiltonian in the large- N_c -limit has the separable form [7]

$$E \Psi = \frac{1}{2}(h + h^*) \Psi, \quad h = \sum_{k < l} h_{kl}, \quad (2)$$

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \ln(\rho_{12}) p_1 + \frac{1}{p_2} \ln(\rho_{12}) p_2 - 2\psi(1). \quad (3)$$

Apart from the Möbius invariance h has the duality symmetry [8]

$$p_k \rightarrow \rho_{k,k+1} \rightarrow p_{k+1} \quad (4)$$

and n integrals of motion q_r, q_r^* [9]. The operators h and h^* are local hamiltonians of the integrable Heisenberg spin model [10]. As usual, for the integrable system one can introduce the transfer (T) and monodromy (t) matrices according to the definitions [9]

$$T(u) = \text{Tr} t(u) = \sum_{r=0}^n u^{n-r} q_r, \quad t(u) = L_1(u) L_2(u) \dots L_n(u), \quad (5)$$

[†] speaker

$$L_k(u) = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix}, \quad t(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (6)$$

The matrix elements $A(u), B(u), C(u), D(u)$ satisfy some bilinear commutation relations following from the Yang-Baxter equation [9] which can be solved with the use of the Bethe ansatz and the Baxter-Sklyanin approach [11, 12].

One can calculate next-to-leading corrections to the BFKL equation [13]. The eigenvalue of its kernel $\Delta(n, \gamma)$ does not contain the non-analytic terms $\delta_{|n|,0}$ and $\delta_{|n|,2}$ only in $N = 4$ SUSY [3]. Further, all functions entering this expression have the property of the maximal transcendentality [14]. This property is valid also for the anomalous dimensions of twist-2 -operators in $N = 4$ SUSY [15, 16] contrary to the case of QCD [17]. One can calculate the higher loop corrections with the use of the effective action [18, 19].

The leading Pomeron singularity in $N = 4$ SUSY should be situated in the strong coupling regime near the point $j = 2$ coinciding with the graviton Regge pole. This conclusion is related to the AdS/CFT correspondence, formulated in the framework of the Maldacena hypothesis claiming, that $N = 4$ SUSY is equivalent to the superstring model living on the 10-dimensional anti-de-Sitter space [20–22]. In this case according to Ref. [21] the eigenvalue for the BFKL hamiltonian in the diffusion approximation coincides with the expression for the graviton Regge trajectory [16]. From the knowledge of this trajectory at large \hat{a} and j [23] one can calculate the explicit expression for the Pomeron intercept at large coupling constants $j = 2 - \hat{a}^{-1/2}/2\pi$ [16, 24].

More than ten years ago it was argued [25], that for $N = 4$ SUSY the evolution equations for anomalous dimensions of quasi-partonic operators are integrable in LLA. Later such integrability was generalized to other operators [26] and to higher loops [27]. With the use of the maximal transcendentality and integrability the equation for the cusp anomalous dimension was constructed in all orders of perturbation theory [28, 29]. Later the anomalous dimension of twist-2 operators in four loops was calculated [30]. After taking into account the wrapping effects [31] the obtained expressions agree with the BFKL predictions [3, 14].

2 Two gluon production amplitudes and the Mandelstam cuts

For the case of the maximal helicity violation in $N = 4$ SUSY Bern, Dixon and Smirnov suggested a simple ansatz for the multi-gluon scattering amplitude in the planar limit $\alpha N_c \sim 1$ [32]. It can be expressed as a product of an infraredly divergent factor and some special functions.

Recently the BDS ansatz was investigated in the multi-Regge kinematics [33] (see also Ref. [34]). It turns out, that the elastic amplitude has the Regge asymptotics and the amplitude for one gluon production has the multi-Regge form in an agreement with the Steinman relations [33]. However, for two gluon production in the physical kinematical region, where $s, s_2 > 0$ but $s_1, s_3 < 0$ the Regge factorization for the BDS amplitude is broken [33]. A similar situation is valid for the BDS amplitude describing the transition $3 \rightarrow 3$ in the region, where $s, s_2 = t_2 > 0$ but $s_1, s_3 < 0$. The reason for the breakdown of the Regge factorization is that the production amplitudes in these regions should contain the Mandelstam cuts in the j -plane of the t_2 -channel [33]. It means, that the BDS amplitudes are not correct beyond 1 loop.

In the elastic amplitude the cut in the j -plane appears only in the non-planar diagrams

because the integrals over the Sudakov variables $\alpha = 2kP_A/s$ and $\beta = 2kp_B$ for the reggeon momenta k and $q-k$ should have the singularities above and below the corresponding integration contours. For the case of planar diagrams this condition is fulfilled only for the multi-particle amplitudes starting from six external particles in the region $s, s_2 > 0$ and $s_1, s_3 < 0$. The imaginary part of the amplitude $A_{2 \rightarrow 4}$ in the s_2 -channel can be written in terms of the BFKL-like equation for the state in the adjoint representation [33]. One can obtain the exact solution of this equation [35]

$$\Im M_{2 \rightarrow 4} \sim s_2^{\omega(t_2)} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} \left(\frac{q_3^* k_1^*}{k_2^* q_1^*} \right)^{i\nu - \frac{n}{2}} \left(\frac{q_3 k_1}{k_2 q_1} \right)^{i\nu + \frac{n}{2}} s_2^{\omega(\nu, n)}. \quad (7)$$

The eigenvalue of the reduced BFKL kernel is

$$\omega(\nu, n) = -a \left(\psi(i\nu + \frac{|n|}{2}) + \psi(-i\nu + \frac{|n|}{2}) - 2\psi(1) \right). \quad (8)$$

This result is in a disagreement with the BDS ansatz starting from two loops. The leading singularity of the t_2 -partial wave corresponds to $n = 1$ and is situated at $j - 1 = \omega(t_2) + a(4 \ln 2 - 2)$.

3 Integrability for multi-gluon composite states

Here we shall discuss the Mandelstam cuts constructed from several reggeons [36]. The non-vanishing contribution from the exchange of $n + 1$ reggeons appears in the planar diagrams only if the number of external lines is $r \geq 2n + 4$. The Green function describing the interaction of n reggeized gluons in the adjoint representation satisfies the BFKL equation with the integral kernel [36]

$$K = \omega(t) - \frac{g^2 N_c}{16\pi^2} H, \quad \omega(t) = a \left(\frac{1}{\epsilon} - \ln \frac{-t}{\mu_2} \right), \quad t = -|q|^2. \quad (9)$$

The reduced Hamiltonian H has the property of the holomorphic separability [36]

$$H = h + h^*, \quad h = \ln \frac{p_1 p_{n+1}}{q^2} + \sum_{l=1}^n h_{l, l+1}. \quad (10)$$

With the use of the duality transformations (cf. [8])

$$p_1 = z_{0,1}, \quad p_r = z_{r-1,r}, \quad q = z_{0,n}, \quad \rho_{r,r+1} = i \frac{\partial}{\partial z_r} = i \partial_r \quad (11)$$

one can present the holomorphic hamiltonian h in the form invariant under the Möbius transformations, which gives a possibility to put $z_0 = 0$, $z_n = \infty$. For this choice of these coordinates one can present h as follows [36]

$$\ln(z_1^2 \partial_1) + \ln(\partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h_{r, r+1}, \quad (12)$$

where $h_{r, r+1}$ coincides in fact with the expression (3) after the substitution $\rho_r \rightarrow z_r$.

One can verify the commutativity of h with the matrix element $D(u)$ of the monodromy matrix (6) introduced for the description of integrability of the BKP equations in the multi-color QCD [36]

$$[D(u), h] = 0. \quad (13)$$

Moreover, h coincides with the local hamiltonian of the open integrable Heisenberg model in which spins are generators of the Möbius group.

To solve this model we can use the algebraic Bethe ansatz. For this purpose it is convenient to go to the transposed space, where the pseudo-vacuum state Ψ_0 can be written in the simple form

$$\Psi_0 = \prod_{r=1}^{n-1} z_r^{-2}, \quad C^t(u)\Psi_0 = 0. \quad (14)$$

Here the operator $C^t(u)$ is the transposed matrix element $C(t)$ of the monodromy matrix (6). The eigenfunctions of h and $D(u)$ are constructed in the framework of the Baxter-Sklyanin approach by applying the product of the Baxter functions $Q(u)$ to the pseudovacuum state

$$\Psi = \prod_{r=1}^n Q^t(\hat{u}_r) \Psi_0, \quad (15)$$

where the operators \hat{u}_r are zeroes of the matrix element $B(u)$ of the monodromy matrix $t(u)$. The Baxter function satisfies the Baxter equation which is reduced to the simple recurrent relation

$$\Lambda(u) Q(u) = (u + i)^{n-1} Q(u + i). \quad (16)$$

The function $\Lambda(u)$ is an eigenvalue of the integral of motion $D(u)$ and can be written in terms of its roots

$$D(u) \Psi_{a_1, a_2, \dots, a_{n-1}} = \Lambda(u) \Psi_{a_1, a_2, \dots, a_{n-1}}, \quad \Lambda(u) = \prod_{r=1}^{n-1} (u - ia_r). \quad (17)$$

As a result, one can present the solution of the Baxter equation in the form [36]

$$Q(u) = \prod_{r=1}^{n-1} \frac{\Gamma(-iu - a_r)}{\Gamma(-iu + 1)}. \quad (18)$$

The composite state Regge trajectory has the additivity property

$$\omega_n(t) = \omega(t) - \frac{a}{2} E, \quad E = \sum_{r=1}^{n-1} \epsilon(a_r) + \sum_{r=1}^{n-1} \epsilon(\tilde{a}_r), \quad (19)$$

where

$$\epsilon(a) = \psi(a) + \psi(1 - a) - 2\psi(1), \quad a_r = i\nu_r + \frac{n_r}{2}. \quad (20)$$

For three gluon composite state in the adjoint representation one can find an explicit solution of these equations in the momentum space [36]

$$\Psi^t(\vec{p}_1, \vec{p}_2) = (p_1 + p_2)^{-a_1 - a_2} (p_1^* + p_2^*)^{-\tilde{a}_1 - \tilde{a}_2} \int d^2 u \phi(u, \tilde{u}) \left(\frac{p_1}{p_2}\right)^{-iu} \left(\frac{p_1^*}{p_2^*}\right)^{-i\tilde{u}}, \quad (21)$$

where

$$-iu = i\nu_u + \frac{N_u}{2}, \quad -i\tilde{u} = i\nu_u - \frac{N_u}{2}, \quad \int d^2u \equiv \int_{-\infty}^{\infty} d\nu_u \sum_{N_u=-\infty}^{\infty}. \quad (22)$$

and

$$\phi(u, \tilde{u}) = u \tilde{u} Q(u, \tilde{u}), \quad (23)$$

where

$$Q(u, \tilde{u}) \sim \frac{\Gamma(iu)\Gamma(i\tilde{u})}{\Gamma(1-iu)\Gamma(1-i\tilde{u})} \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma(1+i\tilde{u}+\tilde{a}_1)\Gamma(1+i\tilde{u}+\tilde{a}_2)} \quad (24)$$

in an accordance with the Baxter-Sklyanin representation [11].

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