Production amplitudes in $N = 4$ SUSY and integrability

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1 Gluon reggeization

Regge kinematics

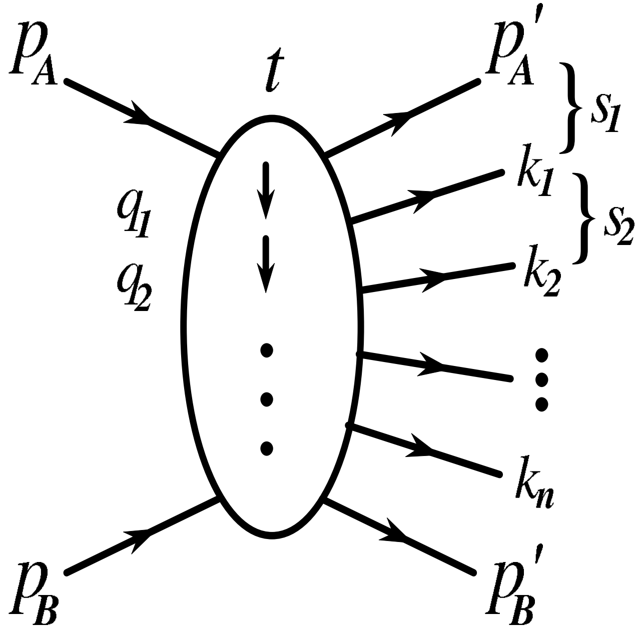
$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2$$

Amplitude for the colored particle scattering

$$M_{AB}^{A'B'}(s,t)|_{LLA} = 2\,g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{s^{1+\omega(t)}}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$



Multi-Regge amplitudes (F.,K.,L. (1975))

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, g T^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} \ldots C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2},$$

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \ \sigma_t = \sum_n \int d\Gamma_n \left|M_{2\to 1+n}\right|^2$$

2 Effective action approach

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x)\,,\;\; A_\pm(x) = -iT^a A_\pm^a(x)$$

Local gauge transformations

$$\delta v_\mu(x) = \frac{1}{g}[D_\mu, \chi(x)],\; \delta\psi(x) = -\chi(x)\,\psi(x),\; \delta A_\pm(x) = 0$$

Effective action for reggeized gluons (L., 1995)

$$S = \int d^4x \left(L_0 + L_{ind}^{GR}\right)\,,\;\; L_0 = i\bar{\psi}\hat{D}\psi + \frac{1}{2}\,Tr\,G_{\mu\nu}^2$$

$$L_{ind}^{GR} = -\frac{1}{g}\partial_+\,P\exp\left(-g\frac{1}{2}\int_{-\infty}^{x^+} v_+(x')d(x')^+\right)\,\partial_\sigma^2 A_- + (+ \to -)$$

3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\;\; \sigma_t \sim s^{\Delta}\,,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\,,\;\; \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu$$

4 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\ ,\ \ H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = h + h^*\ ,\ \ h_{12} = \ln p_1 + \ln p_2 + \frac{1}{p_1}(\ln \rho_{12})\, p_1 + \frac{1}{p_2}(\ln \rho_{12})\, p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

5 Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u),$$

$$L_k = \begin{pmatrix} u + \rho_k \, p_k & p_k \\ -\rho_k^2 \, p_k & u - \rho_k \, p_k \end{pmatrix}, \; [T(u), T(v)] = [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u) \, t_{r_2'}^{s_2}(v) \, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u) \, t_{r_2}^{s_2'}(v) \, t_{r_1}^{s_1'}(u), \; \hat{l} = u \, \hat{1} + i \, \hat{P}$$

Duality symmetry (L. (1999))

$$p_r \longrightarrow \rho_{r+1,r} \longrightarrow p_{r+1}$$

Heisenberg spin model (L. (1994); F., K.(1995))

6 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\;\beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

7 Maximal helicity violation

BDS amplitudes in $N = 4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,\ldots,a_n} = \sum_{\{i_1,\ldots,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} \ldots T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, \ldots, p_{i_n})\,, \;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right),$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}, \;\; C^{(1)} = 0\,, \;\; C^{(2)} = -\zeta_2^2/2\,, \;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4} \gamma_K^{(l)}\,, \;\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + \ldots$$

8 Steinman relations

Overlapping channels

$$(s_1, s_2)\ (2 \to 3)\,;\ \ (s_1, s_2),\ (s_2, s_3),\ (s_{012}, s_2),\ (s_{123}, s)\ (2 \to 4)$$

Dispersion representation for $M_{2\to 3}$ in the Regge anzatz

$$M_{2\to 3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Violation of the dispersion representation for $M_{2\to 4}$

$$M_{2\to 4} \neq d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2}+d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

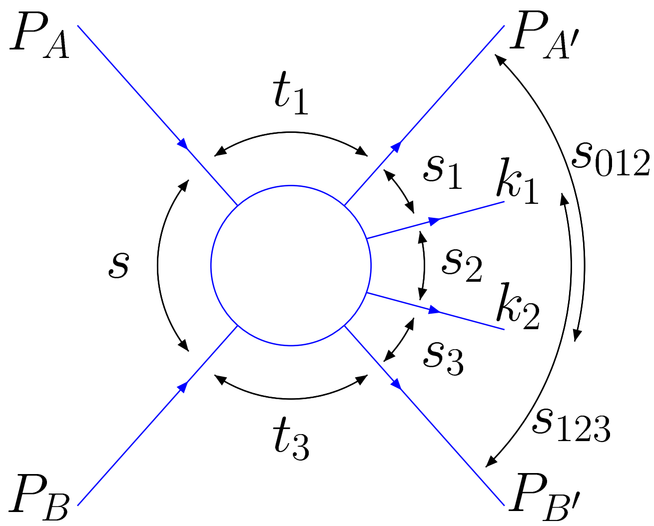
$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\ \ j_r = j(t_r)$$

Important relations

$$\Phi \equiv \frac{(-s)(-s_2)}{(-s_{012})(-s_{123})}\,,\ \ Li_2(1-\Phi)_{\Phi\to\exp(2\pi i)} = \ln(1-\Phi) \approx \ln\frac{(\vec{k}_1+\vec{k}_2)^2}{s_2}$$

9 Regge factorization violation



$$M_{2\to 4}|_{s_2>0;\, s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi\, \left(\ln\frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}\Gamma(t_3)$$

10 Mandelstam cuts in j_2-plane

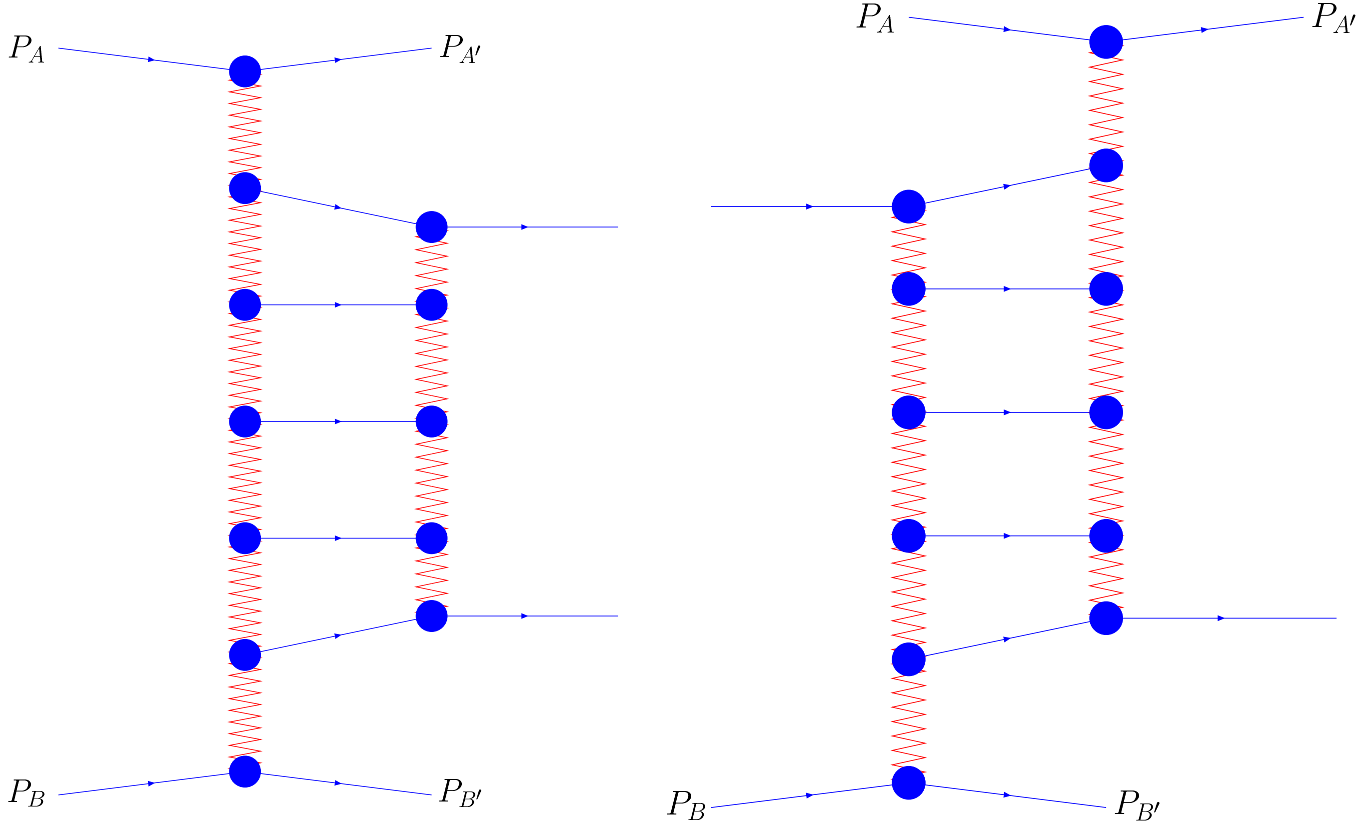


Figure 1: BFKL ladders in $M_{2\to4}$ and $M_{3\to3}$

11 BFKL equation in octet channels

Factorization of infrared divergencies in LLA

$$\lim_{\epsilon\to 0} M_{2\to 4}^{LLA} = f_{2\to 4}^{LLA} \lim_{\epsilon\to 0} M_{2\to 4}^{BDS} \,,$$

Renormalization of the intercept in the s_2-channel

$$\Delta_2 = -a\left(E + \ln\frac{t_2}{\mu^2} - \frac{1}{\epsilon}\right)$$

BFKL hamiltonian for the partial wave f_{j_2}

$$H = \ln\frac{|p_1 p_2|^2}{|p_1+p_2|^2} + \frac{1}{2}\, p_1 p_2^* \ln|\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2}\, p_1^* p_2 \ln|\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2} \left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2} \,,\ E_{n,\nu} = 2Re\,\psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

12 Multi-gluon states in octet channels

Holomorphic hamiltonian for n-gluon composite states

$$h = \ln(z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,,\ p_k = z_{k-1,k}\,,\ z_0 = 0\,,\ z_n = \infty$$

Pair hamiltonian of the spin chain

$$h_{k,k+1} = \ln(z_{k,k+1}^2\partial_k) + \ln(z_{k-1,k}^2\partial_k) - \ln(z_{k-1,k+1}^2) + 2\gamma$$

Monodromy matrix

$$t(u) = L_1(u)...L_{n-1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}\,,\ L_k = \begin{pmatrix} u + z_k\,p_k & p_k \\ -z_k^2\,p_k & u - z_k\,p_k \end{pmatrix}$$

Integrals of motion and Baxter equation for the open spin chain

$$[D(u), h] = 0\,,\ D(u)Q(u) = (u-i)^{n-1}Q(u-i)$$

13 Three-gluon composite state

Wave function in the coordinate representation

$$\Psi = z_2^{a_1+a_2}\,(z_2^*)^{\widetilde{a}_1+\widetilde{a}_2}\int \frac{d^2y}{|y|^2}\, y^{-a_2}\,(y^*)^{\widetilde{a}_2}\left(\frac{y-1}{y-z_2/z_1}\right)^{a_1}\left(\frac{y^*-1}{y^*-z_2^*/z_1^*}\right)^{\widetilde{a}_1}$$

Fourier transformation

$$\Psi = \int d^2p_1\, d^2p_2\, \exp(i\vec{p}_1\vec{z}_1)\,\exp(i\vec{p}_2\vec{z}_2)\,\Psi(\vec{p}_1,\vec{p}_2)\,,\;\; E = E(a_1)+E(a_2)$$

Baxter-Sklyanin representation

$$\Psi^t(\vec{p}_1,\vec{p}_2) = P^{-a_1-a_2}\,(P^*)^{-\widetilde{a}_1-\widetilde{a}_2}\int d^2u\, u\,\widetilde{u}\, Q(u,\widetilde{u})\left(\frac{p_1}{p_2}\right)^{u}\left(\frac{p_1^*}{p_2^*}\right)^{u^*}$$

Baxter function

$$Q(u,\widetilde{u}) = \frac{\Gamma(-u)\,\Gamma(-\widetilde{u})}{\Gamma(1+u)\,\Gamma(1+\widetilde{u})}\,\frac{\Gamma(u-a_1)\,\Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1)\,\Gamma(1-\widetilde{u}+\widetilde{a}_2)}\,,\;\int d^2u = \int d\nu \sum_n$$

14 Discussion

1. Reggeized gluons and BFKL equation.
2. Effective action for reggeized gluons.
3. Integrability of BKP equations at $N_c \to \infty$.
4. Remarkable properties of the BFKL kernel in $N = 4$ SUSY.
7. Breakdown of the Regge factorization for BDS amplitudes.
8. Mandelstam cuts in the planar diagrams.
9. Integrable open spin chain for gluon scattering amplitudes.